Math 331: Representative Problems

These problems are the kind you should be able to solve at the end of Math 331. They indicate the scope and level of the course. They are *not* meant to constitute a sample exam!

1. Consider the equation
$$2(x^3+1)\,\frac{dy}{dx} = -3\,x^2\,y - \frac{2}{y}.$$
   (a) Is it linear? separable? exact? Explain why or why not for each answer.
   (b) Multiply the equation by the integrating factor $\mu = y$, and then find the general solution.

2. Find the general solution of the differential equation
$$y' - 2y = t^2 e^{2t}.$$

3. Find the solution of the differential equation
$$y' = (1+t)(1+y) \quad \text{with} \quad y(0) = 1.$$

4. If you drop a stone from the leaning tower of Pisa, whose height is 56 meters, how long does it take for the stone to reach the ground? Assume no air drag and constant gravitational acceleration $g = 9.81 m/sec^2$.

5. A tank contains 1 kg of a solvent in 200 liters of water. A pulsating flow through the tank begins, with the inflow at time t given by $1 - \cos t$ liters per minute. The outflow equals inflow, so that the volume of solution in the tank remains constant. The concentration of solvent in the inflow pipe is 0.07 kg per liter. Assume that the tank is well mixed.
   (a) Write down a differential equation for the amount of solvent in the tank. Be sure to explain the terms used and include an initial condition.
   (b) *Without* solving the equation, decide whether the amount of solvent settles down to a certain value after a long time, and if so, find this value. Explain your reasoning.

6. Consider the differential equation
$$y' = y(y-1).$$
   (a) Find all the equilibrium solutions of this equation and characterize their stability.
   (b) Determine the solution $y(t)$ with initial condition $y(0) = -1$. Is this solution defined for all values of $t \in \mathbf{R}$?

7. Explain how to solve the equation
$$\frac{dy}{dt} = 2\,t + y, \quad y(0) = 0$$
numerically using Euler's method with a general stepsize h. Now set $h = 0.1$ and carry out three steps of the method.

8. Find the solution to the differential equation
$$y'' + 4y' + 5y = 0$$
with initial conditions $y(0) = 0$ and $y(0) = 1$.

9. Find the general solution to
$$y'' + 4y' + 4y = 0.$$

10. Find the general solution to the differential equation
$$y'' + 3y' + 2y = 3e^{-2t} + t.$$

11. Find the general solution of the inhomogeneous system
$$\frac{d\mathbf{y}}{dt} = \begin{pmatrix} 0 & 1 \\ -2 & -2 \end{pmatrix} \mathbf{y} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$
Characterize whether the origin is a sink, source, saddle or spiral for the homogeneous system.

12. For the system
$$\frac{d\mathbf{y}}{dt} = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \mathbf{y},$$
draw the phase plane and a few trajectories. Classify the equilibrium at the origin and describe the behavior of solutions as $t \to \infty$.

13. A spring-dashpot system has spring constant 1 and damping constant 2 (in dimensionless units); we hang a mass m from the end of the spring. What should the mass m be so that the system can undergo sustained (damped) oscillations?

14. Solve the linear ODE
$$y'' + 2y' + 2y = \delta(t - \frac{\pi}{2}) - u_{2\pi}(t)$$
with initial conditions $y(0) = y'(0) = 0$. Here $u_c(t)$ is the step function which is 0 for $t < c$ and 1 for $t \geq c$.

15. Consider the second order nonlinear equation
$$y'' + y'\,y = 1 + y.$$
   (a) Write it as a first order system, and find *all* equilibria of the system.
   (b) Write down the linearized system near each equilibrium point.

16. Solve the system
$$\frac{d\mathbf{y}}{dt} = \begin{pmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & 1 \end{pmatrix} \mathbf{y}, \qquad \mathbf{y}(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$