Dynamic and Distributed Probing for Covert Cognitive Mobile Edge Computing Networks

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APPENDIX

A. Deriving Equation (7)

Based on (8), we calculate the derivative of $\mathcal{D}\left(p_1 \| p_0\right)(t)$ with respect to each d_{iw}:

$$\frac{\partial \mathcal{D}\left(p_1 \| p_0\right)(t)}{\partial d_{iw}} = -\frac{BP_i\chi}{4\eta_w^2} \cdot \frac{1}{d_{iw}^{(\chi/2)+1}} \left(\frac{\frac{1}{\eta_w^2}\sum_{w=1}^{W} \frac{P_i}{d_{iw}^{\chi/2}}}{1+\frac{1}{\eta_w^2}\sum_{w=1}^{W} \frac{P_i}{d_{iw}^{\chi/2}}} \right) \tag{43}$$

Note that this derivative is negative, indicating that as d_{iw} increases, the divergence decreases. In other words, as Alice gets farther away from a given Willie, the divergence decreases, making it more challenging to determine if she is transmitting. Also note that the magnitude of the derivative is inversely proportional to the distance to the given Willie. This suggests that the impact on covertness of being farther (or closer) from a Willie is more significant than being closer (or farther) to a more distant Willie, as expected. Therefore, we are actually seeking the gradient with respect to $\underline{x}$, and as such we must perform another step of the multivariate chain rule, as follows:

$$\begin{aligned}\frac{\partial \mathcal{D}\left(p_1 \| p_0\right)(t)}{\partial x_i} &= \frac{\partial \mathcal{D}(p_1\|p_0)(t)}{\partial d_{i1}}\frac{\partial d_{i1}}{\partial x_i} + \frac{\partial \mathcal{D}(p_1\|p_0)(t)}{\partial d_{i2}}\frac{\partial d_{i2}}{\partial x_i} + \\ &\quad \dots + \frac{\partial \mathcal{D}(p_1\|p_0)(t)}{\partial d_{iw}}\frac{\partial d_{iw}}{\partial x_i} \\ &= -\frac{BP_i\chi}{2\eta_w^2}\left(\frac{\frac{1}{\eta_w^2}\sum_{w=1}^{W} \frac{P_i}{d_{iw}^{\chi/2}}}{1+\frac{1}{\eta_w^2}\sum_{w=1}^{W} \frac{P_i}{d_{iw}^{\chi/2}}} \right) \cdot \sum_{w=1}^{W} \frac{1}{d_{iw}^{(\chi/2)+1}} \cdot \frac{\partial d_{iw}}{\partial x_i}\end{aligned} \tag{44}$$

Now, if we decompose $\Delta\underline{x}$ into derivatives along two directions, we can rewrite equation (7) as follows:

$$\begin{aligned}&\mathcal{D}\left(p_1 \| p_0\right)(t+1) \approx \\ &\quad \mathcal{D}\left(p_1 \| p_0\right)(t) + \frac{\partial \mathcal{D}(p_1\|p_0)(t)}{\partial x}\Delta x + \frac{\partial \mathcal{D}(p_1\|p_0)(t)}{\partial y}\Delta y\end{aligned} \tag{45}$$

where Δx and Δy are two Gaussian distributions that reflect the degree of motion in two orthogonal directions when updating the position of Alice i under the assumption of a two-dimensional Brownian motion model described in Section III. We redefine the covertness interruption event $\{\mathcal{D}\left(p_1 \| p_0\right)(t+1) > \epsilon\}$ as follows:

$$\mathcal{D}\left(p_1 \| p_0\right)(t) + \frac{\partial \mathcal{D}(p_1\|p_0)(t)}{\partial x}\Delta x + \frac{\partial \mathcal{D}(p_1\|p_0)(t)}{\partial y}\Delta y > \epsilon \tag{46}$$

B. Deriving Equation (13)

The integral region $\mathcal{D}_r$ is defined as the intersection of circles $C_{\bar{R}_m}$ and $C_{\bar{R}_j}$, which can be expressed as:

$$\mathcal{D}_r = \left(C_{\bar{R}_m} \cap C_{\bar{R}_j}\right) \tag{47}$$

where $C_{\bar{R}_m}$ is the circle representing the random movement range with Alice as the center, given by $C_{\bar{R}_m} = \{(x,y) : (x-x_i)^2 + (y-y_i)^2 \le \bar{R}_m^2\}$, where (x_i, y_i) is Alice's position at time t. $C_{\bar{R}_j}$ is the circle representing the communication range with Bob as the center: $C_{\bar{R}_j} = \{(x,y) : (x-x_j)^2+(y-y_j)^2 \le \bar{R}_j^2\}$, where (x_j, y_j) is Bob's position at time t. In order to represent the integration region $\mathcal{D}_{\mathrm{r}}$ more intuitively, we can transform it into polar coordinates with Bob as the origin. Alice i's angle and radius in polar coordinates is be expressed as (r_i, θ_i). The polar form of $C_{\bar{R}_m}$ and $C_{\bar{R}_j}$ can be organized as:

$$C_{\bar{R}_m} = \{(r,\theta) : \sqrt{r^2 + r_i^2 - 2rr_i\cos(\theta-\theta_i)} \le \bar{R}_m\} \tag{48}$$

$$C_{\bar{R}_j} = \{(r,\theta) : r \le \bar{R}_j\} \tag{49}$$

In this case, the lower limit of the polar angle of integration should be set as follows:

$$\theta_1^{'} = \max\left(\theta_i - \arccos\left(\frac{\bar{R}_j^2 + r_i^2 - \bar{R}_m^2}{2\bar{R}_j r_i}\right)\right) \tag{50}$$

The upper limit of the polar angle of integration should be set to:

$$\theta_2^{'} = \min\left(\theta_i + \arccos\left(\frac{\bar{R}_j^2 + r_i^2 - \bar{R}_m^2}{2\bar{R}_j r_i}\right)\right) \tag{51}$$

Since $C_{\bar{R}_j}$ is defined as $r \le \bar{R}_j$ in polar coordinates with Bob as the origin, it is easy to see that the upper limit of the radial distance of the integral should be $\bar{R}_j$. The lower limit of the radial distance of the integral should be defined as:

$$\begin{aligned}&r_1^{'}(\theta) = \\ &\quad \min\left\{r \mid \left(r^2 + r_i^2 - 2rr_i\cos\left(\theta-\theta_i\right)\right)^{1/2} = \bar{R}_m, \theta_1^{'} \le \theta \le \theta_2^{'}\right\}\end{aligned} \tag{52}$$

The polar form of the double integral formula (12) should be represented as:

$$\begin{aligned}&r_{ij}^{b,P_i}(t+1) = \\ &\int_{\theta_1^{'}}^{\theta_2^{'}}\int_{r_1^{'}(\theta)}^{\bar{R}_j} \frac{r}{2\pi\sigma_m^2}\exp\left(-\frac{1}{2}\left[\frac{(r\cos(\theta)-\mu_m)^2+(r\sin(\theta)-\mu_m)^2}{\sigma_m^2}\right]\right)drd\theta\end{aligned} \tag{53}$$

C. Deriving Equation (14)

The power threshold P_w can be obtained by solving the below equation:

$$\frac{P_w}{d_{iw}^{\chi/2}\eta_w^2} - \ln\left(1 + \frac{P_w}{d_{iw}^{\chi/2}\eta_w^2}\right) - \frac{2\epsilon}{B} = 0 \tag{54}$$

where, d_{iw} represents the Euclidean distance between Alice i and Willie w at time t. η_w^2 represents the noise power at Willie's position. ϵ is the threshold for the relative entropy described in Section III. B. B represents the number of channels. To obtain the close-form expression of P_w, we can perform the following Taylor series expansion of the $\ln(\cdot)$ term in (54):

$$\ln\left(1+\frac{P_w}{d_{iw}^{\chi/2}\eta_w^2}\right)\approx \frac{P_w}{d_{iw}^{\chi/2}\eta_w^2}-\frac{1}{2}\left(\frac{P_w}{d_{iw}^{\chi/2}\eta_w^2}\right)^2+\frac{1}{3}\left(\frac{P_w}{d_{iw}^{\chi/2}\eta_w^2}\right)^3-\cdots \tag{55}$$

The above (55) can be further simplified as:

$$\frac{P_w}{d_{iw}^{\chi/2}\eta_w^2}-\left(\frac{P_w}{d_{iw}^{\chi/2}\eta_w^2}-\frac{1}{2}\left(\frac{P_w}{d_{iw}^{\chi/2}\eta_w^2}\right)^2\right)-\frac{2\epsilon}{B}=0 \tag{56}$$

Therefore, the close-form expression for the maximum power P_w of Alice i at time t not detected by Willie should be:

$$P_w=\varrho_i\cdot\sqrt{\frac{4\epsilon}{B}}\cdot d_{iw}^{\chi/2}\eta_w^2 \tag{57}$$

where ϱ_i is a correction factor employed to compensate for the numerical discrepancy in calculating P_w arising from the truncation of higher-order terms in the process of simplifying (55).

D. Deriving Equation (15)

The integration region $\mathcal{D}_s(P_i)$ can be defined as the relative complement of the intersection of sets $C_{\bar{R}_m}$ and $C_{\bar{R}_j}$ with the intersection of set $C_{\bar{R}_w}$, expressed as:

$$\mathcal{D}_s(P_i)=\left(C_{\bar{R}_m}\cap C_{\bar{R}_j}\right)\setminus\left(C_{\bar{R}_m}\cap C_{\bar{R}_j}\cap C_{\bar{R}_w}\right) \tag{58}$$

Here, $C_{\bar{R}_m}$ represents the circle of the random movement range centered at Alice and can be expressed as $C_{\bar{R}_m}=\{(x,y):(x-x_i)^2+(y-y_i)^2\le\bar{R}_m^2\}$, where (x_i,y_i) is the position of Alice at time t. $C_{\bar{R}_j}$ represents the circle of the communication range centered at Bob: $C_{\bar{R}_j}=\{(x,y):(x-x_j)^2+(y-y_j)^2\le\bar{R}_j^2\}$, where (x_j,y_j) is the position of Bob at time t. $C_{\bar{R}_w}$ represents the circle of the surveillance range centered at Willie: $C_{\bar{R}_w}=\{(x,y):(x-x_w)^2+(y-y_w)^2\le\bar{R}_w^2\}$, where (x_w,y_w) is the position of Willie at time t. To simplify the integration calculation, we will consider Bob as the origin of the polar coordinate system. Then, Alice's and Willie's polar coordinates with respect to Bob should be (r_i,θ_i) and (r_w,θ_w), respectively. The polar coordinate expressions for $C_{\bar{R}_m}$, $C_{\bar{R}_j}$, and $C_{\bar{R}_w}$ should be:

$$C_{\bar{R}_m}=\{(r,\theta):\sqrt{r^2+r_i^2-2rr_i\cos(\theta-\theta_i)}\le\bar{R}_m\} \tag{59}$$

$$C_{\bar{R}_j}=\{(r,\theta):r\le\bar{R}_j\} \tag{60}$$

$$C_{\bar{R}_w}=\{(r,\theta):\sqrt{r^2+r_w^2-2rr_w\cos(\theta-\theta_w)}\le\bar{R}_w\} \tag{61}$$

The integral expression for the conditional probability of secure communication can be further derived as:

$$p_s(P_i)=\int_{\theta_1}^{\theta_2}\int_{r_1(\theta)}^{r_2(\theta)}f_{t+1}(r\cos(\theta),r\sin(\theta))rdrd\theta \tag{62}$$

When the conditions $-1\le\arccos\left(\frac{\bar{R}_j^2+r_i^2-\bar{R}_m^2}{2\bar{R}_jr_i}\right)\le1$ and $-1\le\arccos\left(\frac{\bar{R}_j^2+r_w^2-\bar{R}_w^2}{2\bar{R}_jr_w}\right)\le1$ are met, we can determine the values of θ_1 and θ_2 using the following formulas:

$$\theta_2=\min\left(\theta_i+\arccos\left(\frac{\bar{R}_j^2+r_i^2-\bar{R}_m^2}{2\bar{R}_jr_i}\right),\theta_w-\arccos\left(\frac{\bar{R}_j^2+r_w^2-\bar{R}_w^2}{2\bar{R}_jr_w}\right)\right) \tag{63}$$

$$\theta_1=\max\left(\theta_i-\arccos\left(\frac{\bar{R}_j^2+r_i^2-\bar{R}_m^2}{2\bar{R}_jr_i}\right),\theta_w+\arccos\left(\frac{\bar{R}_j^2+r_w^2-\bar{R}_w^2}{2\bar{R}_jr_w}\right)\right) \tag{64}$$

The solutions for $r_1(\theta)$ and $r_2(\theta)$ can be derived using the following equations:

$$r_1(\theta)=\min\left\{r\mid\sqrt{r^2+r_i^2-2rr_i\cos(\theta-\theta_i)}=\bar{R}_m,\theta_1\le\theta\le\theta_2\right\} \tag{65}$$

$$r_2(\theta)=\max\left\{r\mid\sqrt{r^2+r_w^2-2rr_w\cos(\theta-\theta_w)}=\bar{R}_w,\theta_1\le\theta\le\theta_2\right\} \tag{66}$$

In other conditions where the argument of the arccos function exceeds its domain of [-1, 1], the condition for secure communication is always not met, and therefore, we have $p_s(P_i)\approx0$.

E. Deriving Equation (20)

Inspired by the fact that each Alice's power is directly related to the Euclidean distance between itself and Bob, we can establish a formula to describe the relationship between Euclidean distance and power based on previous historical records. We use a second-order polynomial regression model to establish this formula:

$$P_{i'j'}^{(t)}=g\left(d_{i'j'}^{(t)}\right)=a_0+a_1\cdot d_{i'j'}^{(t)}+a_2\cdot\left(d_{i'j'}^{(t)}\right)^2 \tag{67}$$

When given a set of historical data $\left\{\left(d_{i'j'}^{(\kappa)},P_{i'j'}^{(\kappa)}\right)\right\}_{\kappa=1}^{t-1}$, where $d_{i'j'}^{(\kappa)}$ is the distance at time κ, and $P_{i'j'}^{(\kappa)}$ is the corresponding power output, we can minimize the difference between predicted power obtained from formula and actual power by adjusting a set of coefficients a_0,a_1,a_2. Adjusting the coefficients from historical records can be achieved by solving the least squares error minimization problem:

$$\sum_{\kappa=1}^{t-1}\min_{a_0,a_1,a_2}\left(P_{i'j'}^{(\kappa)}-\left(a_0+a_1d_{i'j'}^{(\kappa)}+a_2\left(d_{i'j'}^{(\kappa)}\right)^2\right)\right)^2 \tag{68}$$

The coefficient vector $a=[a_0,a_1,a_2]^T$ can be estimated using the least squares method:

$$a=\left(X^TX\right)^{-1}X^Ty \tag{69}$$

where the matrix X contains all historical distance values:

$$X=\begin{bmatrix}1 & d_{i'j'}^{(1)} & \left(d_{i'j'}^{(1)}\right)^2\\ 1 & d_{i'j'}^{(2)} & \left(d_{i'j'}^{(2)}\right)^2\\ \vdots & \vdots & \vdots\\ 1 & d_{i'j'}^{(t-1)} & \left(d_{i'j'}^{(t-1)}\right)^2\end{bmatrix} \tag{70}$$

and the vector y contains all historical power values:

$$y = \begin{bmatrix} P_{i'j'}^{(1)} \\ P_{i'j'}^{(2)} \\ \vdots \\ P_{i'j'}^{(t-1)} \end{bmatrix} \tag{71}$$

After obtaining the relationship formula (67), we can convert the PDF of $d_{i'j'}^{(t)}$ to the PDF of $P_{i'j'}^{(t)}$. In (19), we directly obtained the expected distance $E\left[d_{i'j}(t)\right]$ between Alice $i^{'}$ and the target Bob j (subject to interference) by a double integral involving the movement step l, the direction θ, and geometric relationships, without providing a specific PDF expression directly related to the distance. Here, we reconstruct the PDF of $d_{i'j'}^{(t)}$ using a Dirac delta function $\delta(\cdot)$,

$$f_{d_{i'j'}^{(t)}}(d) = \int_0^{2\pi} \int_0^{\bar{R}_m} f_l\left(l;\sigma_m\right) f_\theta(\theta) \delta\left(d - \sqrt{\left(d_{i'j'}^{(t-1)}\right)^2 + l^2 - 2d_{i'j'}^{(t-1)} l \cos(\theta)}\right) dl d\theta \tag{72}$$

which ensures that the probability distribution of the Euclidean distance d contributes to the integral value only when d is equal to $\sqrt{\left(d_{i'j'}^{(t-1)}\right)^2 + l^2 - 2d_{i'j'}^{(t-1)} l \cos(\theta)}$. Therefore, based on (72), we obtain the PDF of $P_{i'j'}^{(t)}$ as:

$$f_{P_{i'j'}^{(t)}}\left(P_{i'j'}\right) = f_{d_{i'j'}^{(t)}}\left(g^{-1}\left(P_{i'j'}\right)\right) \left| \frac{\partial}{\partial P_{i'j'}} g^{-1}\left(P_{i'j'}\right) \right| \tag{73}$$

If we know the Euclidean distance between Alice $i^{'}$ and Bob $j^{'}$ at the previous time $d_{i'j'}^{(t-1)}$, then according to Fig. **??**, we can determine the maximum and minimum range $\left[\min\left(d_{i'j'}^{(t)}\right), \max\left(d_{i'j'}^{(t)}\right)\right]$ for $d_{i'j'}^{(t)}$ to be $(d_{i'j'}^{(t-1)} - \bar{R}_m, d_{i'j'}^{(t-1)} + \bar{R}_m)$, where $\bar{R}_m$ is the maximum movement distance of Alice $i^{'}$. Based on the relationship formula $P_{i'j'}^{(t)} = g\left(d_{i'j'}^{(t)}\right)$, we can obtain the power range $\left[\min\left(P_{i'j'}^{(t)}\right), \max\left(P_{i'j'}^{(t)}\right)\right]$ for Alice $i^{'}$ at time t. Therefore, the expected power of Alice $i^{'}$ relative to Bob $j^{'}$ is:

$$E\left[P_{i'j'}\right] = \int_{g\left(\left(d_{i'j'}^{(t-1)} - \bar{R}_m\right)\right)}^{g\left(d_{i'j'}^{(t-1)} + \bar{R}_m\right)} P_{i'j'} \cdot f_{P_{i'j'}^{(t)}}\left(P_{i'j'}\right) \, d\left(P_{i'j'}\right) \tag{74}$$

F. Deriving Equation (40)

To establish the indexability of an arm and obtain a closed-form expression for the Whittle index, we consider the following cases:

1) When $\tau_r = 1$, assuming periodic data packet transmissions by Alice, the transition probability $p(s'|s,y) = 1$, regardless of the action y. Alice will transition to $s' = (Q^{max}, \tau^{max}, \Theta)$, initializing the amount of data and maximum time required for a new transmission in the next time slot. If $Q_r = 0$, we obtain:

$$\begin{cases} V^0(0,1,0) = \nu + \beta V(Q^{max}, \tau^{max}, 0), & \text{for } y = 0, \\ V^1(0,1,0) = \beta V(Q^{max}, \tau^{max}, 0), & \text{for } y = 1. \end{cases} \tag{75}$$

Consequently, the Whittle index as in (39) is given by $\nu(0,1,0) = 0$. If $Q_r > 0$, $\Theta = 1$, we find :

$$\begin{cases} V^0(Q_r,1,1) = \nu + \rho_2' + \beta V(Q^{max}, \tau^{max}, 0), \\ \qquad \text{for } y = 0, \\ V^1(Q_r,1,1) = \omega r - (1-\omega)\rho_1' + \beta V((Q^{max}, \tau^{max}, 0), \\ \qquad \text{for } y = 1. \end{cases} \tag{76}$$

Hence, the Whittle index is $\nu(Q_r,1,1) = \omega r - (1-\omega)\rho_1' - \rho_2'$. Furthermore, when $Q_r > 0$, $\Theta = 0$, we have:

$$\begin{cases} V^0(Q_r,1,0) = \nu + \rho_2 + \beta V(Q^{max}, \tau^{max}, 0), \\ \qquad \text{for } y = 0, \\ V^1(Q_r,1,0) = \omega r - (1-\omega)\rho_1 + \beta V(Q^{max}, \tau^{max}, 0), \\ \qquad \text{for } y = 1. \end{cases} \tag{77}$$

Thus, the Whittle index becomes $\nu(Q_r,1,0) = \omega r - (1-\omega)\rho_1 - \rho_2$.

2) When $\tau_r > 1$ and $Q_r = 0$, irrespective of the action y, the transition probability is $p(s'|s,y) = 1$ leading to the state $s' = (0, \tau_r - 1, 0)$, we obtain:

$$\begin{cases} V^0\left(0,\tau_r,0\right) = \nu + \beta V\left(0,\tau_r - 1, 0\right), & \text{for } y = 0, \\ V^1\left(0,\tau_r,0\right) = \beta V\left(0,\tau_r - 1, 0\right), & \text{for } y = 1. \end{cases} \tag{78}$$

Consequently, the Whittle index is $\nu\left(0,\tau_r,0\right) = 0$. If $Q_r > 0$, $\Theta = 1$, and $y = 0$, the transition probability is $p(s'|s,y) = 1$ and the next state is $s' = (Q_r, \tau_r - 1, 1)$. The expected reward in this state is:

$$V^0(Q_r,\tau_r,1) = \nu + \rho_2' + \beta V(Q_r, \tau_r - 1, 1) \tag{79}$$

On the other hand, for $y = 1$, the user transitions to state $s' = (Q_r - Q_i, \tau_r - 1, 0)$ with probability ω, and to state $s' = (Q_r, \tau_r - 1, 1)$ with probability $1 - \omega$. Therefore, we have:

$$V^1(Q_r,\tau_r,1) = \omega r - (1-\omega)\rho_1' + \beta\omega V(Q_r - Q_i, \tau_r - 1, 0) + \beta(1-\omega)V(Q_r, \tau_r - 1, 1) \tag{80}$$

Similarly, when $Q_r > 0$, $\Theta = 0$, and $y = 0$, the transition probability is $p(s'|s,y) = 1$, leading to the state $s' = (Q_r, \tau_r - 1, 0)$. The expected reward in this state is:

$$V^0(Q_r,\tau_r,0) = \nu + \rho_2 + \beta V(Q_r, \tau_r - 1, 0) \tag{81}$$

Furthermore, for $y = 1$, the user transitions to state $s' = (Q_r - Q_i, \tau_r - 1, 0)$ with probability ω, and to state $s' = (Q_r, \tau_r - 1, 1)$ with probability $1 - \omega$. Hence, we obtain:

$$V^1(Q_r,\tau_r,0) = \omega r - (1-\omega)\rho_1 + \beta\omega V(Q_r - Q_i, \tau_r - 1, 0) + \beta(1-\omega)V(Q_r, \tau_r - 1, 1) \tag{82}$$

Subsequently, we will analyze the indexability when $\tau_r > 1$. We establish the following formulas:

1) For $\tau_r > 1, Q_r > 0$ and $\Theta = 1$,

$$\begin{aligned} h(Q_r,\tau_r,1) &= V^0(Q_r,\tau_r,1) - V^1(Q_r,\tau_r,1) \\ &= \nu - \omega r + (1-\omega)c_1^{'} + c_2^{'} + \beta\omega f_2(\tau_r - 1) \end{aligned} \tag{83}$$

2) For $\tau_r > 1, Q_r > 0$ and $\Theta = 0$,

$$g(Q_r, \tau_r, 0) = V^0(Q_r, \tau_r, 0) - V^1(Q_r, \tau_r, 0) \\ = (\nu + c_2 - \omega r + (1-\omega)c_1) + \beta f_1(\tau_r - 1) + \beta\omega f_2(\tau_r - 1) \tag{84}$$

where we define the following:

$$f_1(\tau_r - 1) = V(Q_r, \tau_r - 1, 0) - V(Q_r, \tau_r - 1, 1) \tag{85}$$

$$f_2(\tau_r - 1) = V(Q_r, \tau_r - 1, 1) - V(Q_r - Q_i, \tau_r - 1, 0) \tag{86}$$

Taking the derivative of $h(Q_r, \tau_r, 1)$ and $g(Q_r, \tau_r, 0)$ with respect to ν, we obtain:

$$\frac{\partial h(Q_r, \tau_r, 1)}{\partial \nu} = 1 + \beta\omega \frac{\partial f_2(\tau_r - 1)}{\partial \nu} \tag{87}$$

$$\frac{\partial g(Q_r, \tau_r, 0)}{\partial \nu} = 1 + \beta \frac{\partial f_1(\tau_r - 1)}{\partial \nu} + \beta\omega \frac{\partial f_2(\tau_r - 1)}{\partial \nu} \tag{88}$$

We need to prove $min\{\frac{\partial f_1(\tau_r-1)}{\partial \nu}\} \geq 0$ for the condition $min\{\frac{\partial f_1(\tau_r-1)}{\partial \nu}\} + \omega * min\{\frac{\partial f_2(\tau_r-1)}{\partial \nu}\} \geq -\frac{1}{\beta}$.

$$f_1(\tau_r) = \begin{cases} (1-\omega)(c_1^{'} - c_1); & \text{if } \nu < 0 \\ \omega r - (1-\omega)c_1 - c_2^{'} - \nu - \beta\omega f_2(\tau_r - 1); & \text{if } 0 \leq \nu < \nu(Q_r, \tau_r, 0) \\ c_2 - c_2^{'} + \beta f_1(\tau_r - 1); & \text{if } \nu \geq \nu(Q_r, \tau_r, 0) \end{cases} \tag{89}$$

Thus, the partial derivative of $f_1(\tau_r - 1)$ is

$$\frac{\partial f_1(\tau_r - 1)}{\partial \nu} = \begin{cases} 0; & \text{if } \nu < 0 \\ -1 - \beta\omega \frac{\partial f_2(\tau_r-2)}{\partial \nu}; & \text{if } 0 \leq \nu < \nu(Q, \tau_r, 0) \\ \beta \frac{\partial f_1(\tau_r-2)}{\partial \nu}; & \text{if } \nu \geq \nu(Q, \tau_r, 0) \end{cases} \tag{90}$$

We know that $1 - \beta + (1-\beta)\beta\omega \frac{\partial f_2(\tau_r-1)}{\partial \nu} = 0$, thus $\frac{\partial g(Q, \tau_r, 0)}{\partial \nu} = 1 + \beta \frac{\partial f_1(\tau_r-1)}{\partial \nu} + \beta\omega \frac{\partial f_2(\tau_r-1)}{\partial \nu} \geq 0$ and therefore $\frac{\partial g(Q, \tau_r, 0)}{\partial \nu} \geq 0$.

For $0 < \beta < 1; 0 < \omega < 1, -\frac{1}{\beta\omega} < -1$, we have the following for $\frac{\partial f_2(\tau_r-1)}{\partial \nu}$:

$$f_2(\tau_r) = \begin{cases} (1-\omega)(c_1 - c_1^{'}); & \text{if } \nu < 0 \\ \omega r - (1-\omega)c_1^{'} - c_2 - \nu + \beta(1-\omega) f_2(\tau_r - 1); & \text{if } 0 \leq \nu < \nu(Q_r, \tau_r, 1) \\ c_2^{'} - c_2 + \beta f_2(\tau_r - 1); & \text{if } \nu \geq \nu(Q_r, \tau_r, 1) \end{cases} \tag{91}$$

Thus, the partial derivative of $f_2(\tau_r - 1)$ is

$$\frac{\partial f_2(\tau - 1)}{\partial \nu} = \begin{cases} 0; & \text{if } \nu < 0 \\ -1 + \beta(1-\omega) \frac{\partial f_2(\tau-2)}{\partial \nu}; & \text{if } 0 \leq \nu < \nu(Q, \tau, 1) \\ \beta \frac{\partial f_2(\tau-2)}{\partial \nu}; & \text{if } \nu \geq \nu(Q, \tau, 1) \end{cases} \tag{92}$$

From this, $-1 + \beta(1-\omega)\frac{\partial f_2(\tau_r-2)}{\partial \nu} > -1$ and thus, $\frac{\partial h(Q_r, \tau_r, 0)}{\partial \nu} \geq 0$. Hence, we can conclude the indexability in the state (Q_r, τ_r, Θ) where $Q_r > 0$ and $\tau_r > 1$.