Name (Last, First) ______________________ **ID #** __________

Signature ______________________________________

Lecturer ____________________ **Section (01, 02, 03, etc.)** ______

UNIVERSITY OF MASSACHUSETTS AMHERST
DEPARTMENT OF MATHEMATICS AND STATISTICS

Math 233 **Exam 1** **October 15, 2018**
7:00-9:00 p.m.

Instructions

- **Turn off all cell phones!** Put away all electronic devices such as iPods, iPads, laptops, etc.
- There are five (5) questions.
- Do all work in this exam booklet. You may continue work to the backs of pages and the blank page at the end, but if you do so indicate clearly where your work is for the grader.
- Calculators are **not** allowed, nor are formula sheets or any other external materials.
- Organize your work in an unambiguous order. Show all necessary steps.
- **Unless indicated otherwise, you must show work to obtain credit for your answers.**
- Be ready to show your UMass ID card when you hand in your exam booklet.

QUESTION	POINTS	SCORE
1	20	
2	20	
3	20	
4	20	
5	20	
TOTAL	100	

1. (20 points) For each question, please select the best response. Please clearly indicate your choice; ambiguous answers will not receive credit. In this problem, there is *no partial credit* awarded and it is *not necessary to show your work.*

(a) (4 points) Find the area of the triangle with vertices $(2, 1, 1)$, $(1, 2, 1)$, $(1, 1, 2)$. **ANSWER:** (iv). This is given by 1/2 the magnitude of $\vec{\boldsymbol{a}} \times \vec{\boldsymbol{b}}$, where $\vec{\boldsymbol{a}}$ and $\vec{\boldsymbol{b}}$ are vectors running along the sides of the triangle.

(i) 1	(ii) 2	(iii) $\sqrt{3}$
(iv) $\sqrt{3}/2$	(v) $\sqrt{3}/4$	(vi) $\sqrt{6}/2$

(b) (4 points) Find the **cosine** of the angle between the two planes $x + 2y = 0$ and $x + 2z = 3$. **ANSWER:** (iv) This is given by taking the two normal vectors $\vec{\boldsymbol{n}}_1$, $\vec{\boldsymbol{n}}_2$ to the planes and then computing $(\vec{\boldsymbol{n}}_1 \cdot \vec{\boldsymbol{n}}_2)/(|\vec{\boldsymbol{n}}_1||\vec{\boldsymbol{n}}_2|)$.

(i) $2/3$	(ii) $3/4$	(iii) $\sqrt{3}/2$
(iv) $1/5$	(v) $1/\sqrt{2}$	(vi) $1/2$

(c) (4 points) Find the **unit tangent vector** to the parametric curve $\vec{\boldsymbol{r}}(t) = \langle \sin t, 2t, t^2 \rangle$ at $t = 0$. **ANSWER:** (i) First find $\vec{\boldsymbol{r}}'(t)$, plug in $t = 0$, and then divide the result by its magnitude.

(i) $\langle 1/\sqrt{5}, 2/\sqrt{5}, 0 \rangle$	(ii) $\langle 1/\sqrt{5}, -2/\sqrt{5}, 0 \rangle$	(iii) $\langle -2/\sqrt{5}, 1/\sqrt{5}, 0 \rangle$
(iv) $\langle 2/\sqrt{5}, 1/\sqrt{5}, 0 \rangle$	(v) $\langle -1/\sqrt{5}, -2/\sqrt{5}, 0 \rangle$	(vi) $\langle 2/\sqrt{5}, -1/\sqrt{5}, 0 \rangle$

(d) (4 points) Describe the **level curves** of the function $f(x, y) = \sqrt{1 - x^2 - 2y^2}$. **ANSWER:** (ii) They are determined by setting $1 - x^2 - 2y^2$ to a constant, which means $x^2 + 2y^2$ is a constant. These are concentric ellipses.

(i) concentric circles	(ii) concentric ellipses (not circles)
(iii) parabolas with the same vertex	(iv) parabolas with different vertices
(v) hyperbolas with the same vertex	(vi) hyperbolas with different vertices

(e) (4 points) The function $f(x, y) = x^2 + y^2 + 3xy$ has one critical point. Determine its location and type. **ANSWER:** (i) We must solve $\nabla f = \langle 0, 0 \rangle$ and then use the 2nd derivative test (Hessian) to classify the solutions. There is only one solution at the origin, and the determinant of the Hessian is < 0, which means a saddle point.

(i) $(0, 0)$, saddle point	(ii) $(0, 0)$, maximum point
(iii) $(0, 0)$, minimum point	(iv) $(2, 1)$, saddle point
(v) $(2, 1)$, maximum point	(vi) $(2, 1)$, minimum point

2. (20 points)

(a) (6 points) Let P be the plane through the points $(1,0,0)$, $(0,2,0)$, and $(0,0,3)$. Find an equation for P.

ANSWER: Let $p=(1,0,0)$, $q=(0,2,0)$, $r=(0,0,3)$. We need to take two vectors lying in the plane, such as $\vec{\boldsymbol{v}}=pq$, $\vec{\boldsymbol{w}}=pr$, and then take the cross product $\vec{\boldsymbol{n}}=\vec{\boldsymbol{v}}\times\vec{\boldsymbol{w}}$ to make a normal vector to P. We find $\vec{\boldsymbol{v}}=\langle -1,2,0\rangle$, $\vec{\boldsymbol{w}}=\langle -1,0,3\rangle$, and then $\vec{\boldsymbol{n}}=\langle 6,3,2\rangle$. Thus the equation has the form $6x+3y+2z=D$ for a constant D. Plugging in p we find $D=6$. Therefore an equation for P is $6x+3y+2z=6$.

(b) (6 points) Let L be the line through the origin in the direction of $\vec{\boldsymbol{r}}=\langle 2,-2,-3\rangle$. Find parametric equations for L.

ANSWER: The direction vector is $\langle 2,-2,-3\rangle$ and a point on the line L is $(0,0,0)$. Thus parametric equations are $x=2t$, $y=-2t$, and $z=-3t$ where t ranges over all real numbers.

(c) (8 points) Does L intersect P? If yes, find the point of intersection. If not, find the distance between L and P.

ANSWER: The dot product $\vec{\boldsymbol{n}}\cdot\vec{\boldsymbol{r}}$ is zero. This means the direction vector of L is perpendicular to the normal vector of P, which implies either L is contained in P or is parallel to P. The origin is on L but not on P, so L must be parallel to P. Thus we have to find the distance. We can compute it as $d=|\vec{\boldsymbol{a}}\cdot\hat{\boldsymbol{n}}|$, where $\vec{\boldsymbol{a}}$ is a vector running from L to P and $\hat{\boldsymbol{n}}$ is a unit vector in the direction of the normal vector $\vec{\boldsymbol{n}}$ (this is the direction along which the distance is measured). We can take $\vec{\boldsymbol{a}}$ to run from the origin to p, i.e. $\vec{\boldsymbol{a}}=\langle 1,0,0\rangle$. The length of $\vec{\boldsymbol{n}}=\langle 6,3,2\rangle$ is $\sqrt{36+9+4}=7$, so $\hat{\boldsymbol{n}}=\langle 6/7,3/7,2/7\rangle$ and $d=|\vec{\boldsymbol{a}}\cdot\hat{\boldsymbol{n}}|=6/7$.

3. (20 points) Suppose a particle moves with position function $\vec{\boldsymbol{r}}(t) = \langle t^2, t\sqrt{2}, (\ln t)/2 \rangle$, where $t > 0$.

(a) (8 points) Find the velocity and acceleration of the particle.

ANSWER:

$$\vec{\boldsymbol{v}}(t) = \vec{\boldsymbol{r}}'(t) = \left\langle 2t, \sqrt{2}, \frac{1}{2t} \right\rangle \text{ is the velocity of the particle, and}$$

$$\vec{\boldsymbol{a}}(t) = \vec{\boldsymbol{v}}'(t) = \vec{\boldsymbol{r}}''(t) = \left\langle 2, 0, -\frac{1}{2t^2} \right\rangle \text{ is the acceleration of the particle.}$$

(b) (12 points) Find the distance traveled by the particle from $t = 1$ to $t = e$.

ANSWER: Arclength for a curve $r(t)$ between $t = a$ and $t = b$ is

$$Arc(a, b) = \int_a^b \left|\vec{\boldsymbol{r}}'(t)\right| dt = \int_a^b \left|\vec{\boldsymbol{v}}(t)\right| dt.$$

Plugging in from (a) we have

$$Arc(1, e) = \int_1^e \left|\left\langle 2t, \sqrt{2}, \frac{1}{2t} \right\rangle\right| dt = \int_1^e \sqrt{(2t)^2 + (\sqrt{2})^2 + \left(\frac{1}{2t}\right)^2} dt$$

$$= \int_1^e \sqrt{4t^2 + 2 + \frac{1}{4t^2}} dt = \int_1^e \sqrt{\frac{1}{4t^2}(16t^4 + 8t^2 + 1)} dt = \int_1^e \frac{1}{2t}\sqrt{(4t^2+1)^2} dt$$

$$= \int_1^e \frac{1}{2t}(4t^2+1) dt = \int_1^e \left(2t + \frac{1}{2t}\right) dt = \left(t^2 + \frac{1}{2}\ln t\right)\Big|_1^e = e^2 + \frac{1}{2}\ln e - 1 - \frac{1}{2}\ln 1$$

$$= e^2 + \frac{1}{2}\cdot 1 - 1 - \frac{1}{2}\cdot 0 = e^2 - \frac{1}{2}.$$

4. (20 points) Let $f(x, y) = 2x^2y + x + y^3$.

(a) (6 points) Compute ∇f.

ANSWER:

$$\nabla f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle = \langle 4xy + 1,\ 2x^2 + 3y^2 \rangle.$$

(b) (7 points) Compute the directional derivative of f at the point $(1, 2)$ in the direction that forms an angle (measured clockwise) of $\pi/3$ with the x-axis.

ANSWER: The angle $\pi/3$ in the clockwise direction is $-\pi/3$ when measured in the usual way, which takes the positive angle directions to be counter-clockwise. Then $\vec{u} = \langle \cos\left(-\frac{\pi}{3}\right), \sin\left(-\frac{\pi}{3}\right) \rangle = \langle \frac{1}{2}, -\frac{\sqrt{3}}{2} \rangle$, and $\nabla f(1, 2) = \langle 4 \cdot 1 \cdot 2 + 1,\ 2 \cdot 1^2 + 3 \cdot 2^2 \rangle = \langle 9,\ 14 \rangle$. Then the directional derivative is

$$\nabla f \cdot \vec{u} = \langle 9, 14 \rangle \cdot \langle 1/2, -\sqrt{3}/2 \rangle = 9/2 - 7\sqrt{3}.$$

(c) (7 points) Find an equation of the tangent plane to the graph of f at the point $(1, 2, f(1, 2))$.

ANSWER: The tangent plane to the surface $z = f(x, y)$ at the point $P = (x_0, y_0, z_0)$ is

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

We have $(x_0, y_0, z_0) = (1, 2, f(1, 2)) = (1, 2, 13)$, and from part b, $f_x(1, 2) = 9$ and $f_y(1, 2) = 14$, so the tangent plane is

$$z - 13 = 9(x - 1) + 14(y - 2)$$

5. (20 points) Let $f(x, y) = x^2 - 4x + y^2 - 2y + 2$. Find the absolute maximum and the absolute minimum of f on the closed rectangle with vertices $(0, 0)$, $(4, 0)$, $(0, 4)$, $(4, 4)$.

ANSWER: Let $f(x, y) = x^2 - 4x + y^2 - 2y + 2$. This is a max/min problem on a closed and bounded region (the rectangle R), so we need to look for candidate max/mins among (i) the critical points of f in the interior of R, (ii) any critical points gotten by restricting f to the edges of R, and (iii) the vertices of R. Note that we do not need to use the 2nd derivative test (Hessian). Note also that multiple points in the domain may give the same maximum *value* of f, and similarly for the minimum.

We first do (i) and find the critical points in the interior by solving the equation $\nabla f = \langle 2x - 4, 2y - 2\rangle = \langle 0, 0\rangle$. The only solution is $(x, y) = (2, 1)$. This point is inside the rectangular domain, so it is a candidate for a max/min.

Now we do (ii). There are 4 boundary edges:

(I) $\{y = 0, 0 \leq x \leq 4\}$.

(II) $\{x = 4, 0 \leq y \leq 4\}$.

(III) $\{y = 4, 0 \leq x \leq 4\}$.

(IV) $\{x = 0, 0 \leq y \leq 4\}$.

We look at each in turn. On (I), the function f becomes $g_1(x) = f(x, 0) = x^2 - 4x + 2$ defined on the closed interval $[0, 4]$. The critical point of g_1 in the interior is comes from $g_1'(x) = 0$, which is $x = 2$. We treat the other edges similarly and find the candidates $(4, 1)$, $(2, 4)$, and $(0, 1)$.

Finally we have (iii) the four vertices of the rectangle: $(0, 0)$, $(0, 4)$, $(4, 4)$, $(4, 0)$.

Altogether, we have found eight candidates for the absolute max/min:

$$(0, 0), (2, 0), (4, 0), (4, 1), (4, 4), (2, 4), (0, 4), (0, 1).$$

To find the max and min, we evaluate the function $f(x, y)$ at these points. The largest value is 10 at two points $(0, 4)$ and $(4, 4)$; the smallest value is -3 at the point $(2, 1)$.

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