

Vienna Schrödinger Lecture 2

Oct 20, 2015

Note Title

2/11/2014

Overall theme: General Relativity as a quantum field theory like our other fundamental interactions

Today: Constructing GR as a gauge theory

Next week: Treating quantum GR as an effective field theory

Nov 3: Low energy theorems of quantum gravity and frontier directions

Web page: JFD – Teaching – Vienna Schrödinger lectures - Also GRQFT pages

Today: Gauge theory construction

References:

A great early reference along the lines that I presented the material is [Kibble on fermions and gauge structure](#)

The other primary reference is [Utiyama on Gravity as a gauge theory](#).

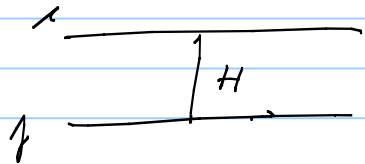
I have an appreciation of a paper by [Hayashi and Shirafuji](#) – a remarkably good early paper in the spirit of effective field theory (although I would do some things differently).

Weinberg's GR textbook has a related development, but without fermions, in Chapter 12.

I highly recommend Maurizio Gasperini's book *Theory of Gravitational Interactions* (Springer) as a resource if you are interested in following up on this material. Even though he does not construct GR in this fashion, his treatment is more general than most and he derives many useful relations.

Particle Physics gravity

Higgs $\mathcal{L} = -\frac{m_i}{\sqrt{2}v} \phi \bar{\psi} \psi$



$$V(r) = - \underbrace{\left(\frac{1}{8\pi v^2} \right)}_{G?} \frac{m_1 m_2}{r} e^{-\frac{m_H r}{\hbar}} \quad \uparrow m_H \rightarrow 0$$

Fails:

- 1) $m_g \neq m_H$
 - 2) B.E. not accounted for
 - 3) No combo of scalar couplings work
- } \Rightarrow Path

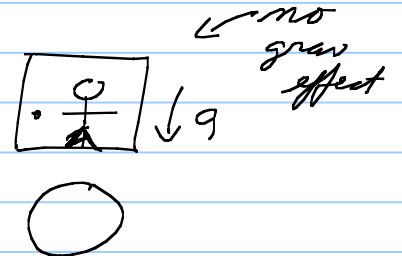
Role of equivalence principle

1) Role of Energy $E = mc^2$

2) Coord.



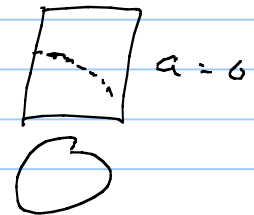
Free fall



Light:



\Rightarrow



E + Mom as source

$$T_{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi - g_{\mu\nu} \mathcal{L}$$

Noether

Scalar field

$$T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} \eta_{\mu\nu} (\partial_\alpha \phi \partial^\alpha \phi - m^2 \phi^2)$$

$$\frac{T_{\mu\nu}}{g} \leftarrow \begin{matrix} M=0 \\ \sim \\ \text{N.R} \end{matrix}$$

$$G \frac{M M}{r}$$

$T_{\mu\nu}$ as source?

Gauge theories: currents as sources

Global invariance $\psi \rightarrow e^{-i\theta} \psi$, $\psi \rightarrow U\psi$ $\swarrow \text{SU}(N)$

\Rightarrow current

Local invariance $\psi \rightarrow \psi' = U(x) \psi$

$$D_\mu \psi \rightarrow D'_\mu \psi' = U(x) D_\mu \psi$$

$$D_\mu = \partial_\mu + i \underline{A}_\mu = \partial_\mu + i \frac{\lambda^a}{2} A_\mu^a$$

$$\Rightarrow \mathcal{L} = \bar{\psi} (\not{D} - m) \psi = \dots - \underbrace{\lambda^a \bar{\psi} \gamma^\mu \frac{\lambda^a}{2} \psi}_{\mathcal{L}_A}$$

$$[D_\mu, D_\nu] \psi = i \underline{F_{\mu\nu}} \psi$$

$$\underline{F_{\mu\nu}} = \partial_\mu \underline{A_\nu} - \partial_\nu \underline{A_\mu} + i [\underline{A_\mu}, \underline{A_\nu}]$$

$$F_{\mu\nu} \Rightarrow U \underline{F} U^{-1}$$

$$\mathcal{L} = -\frac{1}{2g^2} \text{Tr}(\underline{F_{\mu\nu}} \underline{F^{\mu\nu}}) \quad \checkmark$$

H, \vec{p} assoc. with \swarrow global time, space translation

$T_{\mu\nu}$ as source \Rightarrow gauge spacetime transformations
 \Rightarrow general coord invariance

Coordinates

Global: $x'^{\mu} = L^{\mu}_{\nu} x^{\nu} + a^{\mu}$

Local $dx'^{\mu} = \underline{\Lambda^{\mu}_{\nu}(x)} dx^{\nu}$, $\Lambda^{\mu}_{\nu}(x) = \frac{\partial x'^{\mu}}{\partial x^{\nu}}$

New field $g_{\mu\nu}(x)$

$$d\tilde{s}^2 = g_{\mu\nu}(x) dx^{\mu} dx^{\nu}$$

Invariance $g'_{\alpha\beta} \Lambda^{\alpha}_{\mu} \Lambda^{\beta}_{\nu} = g_{\mu\nu}$

Inverse $\Lambda^{\nu}_{\mu}(x) = [\Lambda^{\mu}_{\nu}(x)]^{-1}$

$$g'_{\alpha\beta} = \Lambda^{\mu}_{\alpha} \Lambda^{\nu}_{\beta} g_{\mu\nu}$$

$$g_{\mu\nu} g^{\nu\sigma} = \delta^{\sigma}_{\mu}$$

Tetrad/vierbein

$$g_{\mu\nu}(x) = \eta_{ab} t_{\mu}^a(x) t_{\nu}^b(x)$$

EP

$$t_{\mu}^a(x) = \Lambda_{\mu}^{\nu}(x) t_{\nu}^a(x)$$

Immediate success!

Invariant volume $\int d^4x \sqrt{-g}$ $\swarrow g = \text{Det } g_{\mu\nu}$

$$S_m = \int d^4x \sqrt{-g} \frac{1}{2} \left[\overline{\overline{g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - m^2 \phi^2}} \right] \quad \text{invariant}$$

$$\frac{\delta S_m}{\delta g^{\mu\nu}} = \frac{\sqrt{-g}}{2} \left[\partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g_{\mu\nu} (\partial_\lambda \phi \partial^\lambda \phi - m^2 \phi^2) \right]$$
$$= \frac{\sqrt{-g}}{2} T_{\mu\nu} \quad \longleftarrow \text{source!}$$

$$S = \boxed{S_{\text{grav}}} + \underline{\underline{S_m}}, \quad \frac{\delta S}{\delta g^{\mu\nu}} = \underline{\underline{(-) T_{\mu\nu}}}$$

Prelude: Lorentz + Fermions

$$x'^{\mu} = L^{\mu}_{\nu} x^{\nu} \quad \leftarrow$$

$$\psi'(x') = S \psi(x)$$

$\nearrow 4 \times 4$

$$L_D = \psi (i \not{\partial} - m) \psi \quad \text{invariant}$$

$$\gamma_0 S^\dagger \gamma_0 = S^{-1} \quad (1)$$

$$S^{-1} \gamma^\mu L^{\nu}_{\mu} S = \gamma^\nu \quad (2)$$

$$L^{\mu}_{\nu} = \delta^{\mu}_{\nu} + \omega^{\mu}_{\nu}$$

$$S = 1 - \frac{i}{4} \sigma_{\mu\nu} \omega^{\mu\nu} \rightarrow \exp\left(-\frac{i}{4} \sigma_{\mu\nu} \omega^{\mu\nu}\right)$$

$\nearrow \frac{1}{2} [\gamma^\mu, \gamma^\nu]$

$$S_{\mu\nu} = \frac{i}{2} \sigma_{\mu\nu}$$

$$[S_{\mu\nu}, S_{\alpha\beta}] = \eta_{\nu\alpha} S_{\mu\beta} - \eta_{\mu\alpha} S_{\nu\beta} - \eta_{\nu\beta} S_{\mu\alpha} + \eta_{\mu\beta} S_{\nu\alpha} \quad SO(1,3)$$

Gauging Poincare

Global $x'^{\mu} = L^{\mu}_{\nu} x^{\nu} + a^{\mu}$
 $\psi'(x') = S \psi(x)$

Two invariances

1) Translations
 $x'^{\mu} = x^{\mu} + a^{\mu}(x) \quad \leftarrow \quad dx'^{\mu} = \Lambda^{\mu}_{\nu}(x) dx^{\nu}$
 $\psi' = \psi$

2) Lorentz or spin transformations

$$\psi'(x) = S(x) \psi(x)$$

$$\approx \left(1 + \frac{1}{2} \omega_{ab}^{(x)} S_{ab} \right) \psi$$

could add $dx' = \Lambda^{\mu}_{\nu}(x) dx^{\nu}$

General covariance

1) Translation tetrad \checkmark $\mu = 0, 1, 2, 3$ spacetime indices (coord trans)

$$\mathcal{L} = \bar{\Psi} (i \gamma^\mu \partial_\mu - m) \Psi$$

\uparrow $a = 0, 1, 2, 3 \leftarrow$ Lorentz like
makes invariant

2) Lorentz trans. covariant derivative

$$\Psi'(x) = S(x) \Psi(x)$$

$$\mathcal{L} = \bar{\Psi} (i \gamma^\mu \partial_\mu - m) \Psi \rightarrow \bar{\Psi} S^{-1} (i \gamma^\mu \partial'_\mu - m) S \Psi = \mathcal{L}$$

$$D_\mu = \partial_\mu + \underline{A}_\mu = \partial_\mu + \frac{1}{2} S_{ab} A_\mu^{ab}$$

Invariance $\underline{A}'_\mu = S \underline{A}_\mu S^{-1} + (\partial_\mu S) S^{-1}$

Also need $S^{-1} \gamma^a S = \gamma^b \Lambda^a_b(w(x))$

$$t'^a_\mu(x) = \Lambda^b_a(w(x)) t^a_\mu$$

Invariant

$$D_\mu = \partial_\mu + \underline{A}_\mu$$

$$[D_\mu, D_\nu] = \underline{R}_{\mu\nu} = \frac{1}{2} S_{ab} R_{\mu\nu}^{ab}$$

$$\underline{R}_{\mu\nu} = \partial_\mu \underline{A}_\nu - \partial_\nu \underline{A}_\mu + [\underline{A}_\mu, \underline{A}_\nu]$$

$$R_{\mu\nu}^{ab} = \partial_\mu A_\nu^{ab} - \partial_\nu A_\mu^{ab} + A_\mu^a A_\nu^{db} - A_\nu^a A_\mu^{db}$$

Action — Diff from usual gauge theory

Simplest invariant $R = \underline{\underline{t_a^m t_v^b R_{mv}^{ab}(A)}}$

$$S_{\text{grav}} = \int d^4x \sqrt{-g} \quad \frac{-2}{\kappa^2} R(t, A)$$

Variation

$$1) \text{ w.r.t. } A \quad \overline{D}_\gamma t_v^b - \overline{D}_v t_\gamma^b = 0 \quad \overline{D}_\gamma t_v^b = \partial_\gamma t_v^b + A_{\gamma c}^b t_v^c$$

$$\Rightarrow \text{solve for } A_v^{\mu c} = (\quad)$$

$$2) \text{ w.r.t. } t \quad t_m^a \Rightarrow R_m^a - t_m^a R = 0$$

$\times t_{av}$

$$\boxed{R_{mv} - g_{mv} R = 0}$$

Einstein

Relation to the usual formulation

If no fermions, can use spacetime indices only

$$D_\mu V^\lambda = \partial_\mu V^\lambda + \Gamma_{\mu\nu}^\lambda V^\nu$$

$$R_{\mu\nu\alpha}{}^\beta = \partial_\mu \Gamma_{\nu\alpha}^\beta - \partial_\nu \Gamma_{\mu\alpha}^\beta + \Gamma_{\mu\rho}^\beta \Gamma_{\nu\alpha}^\rho - \Gamma_{\nu\rho}^\beta \Gamma_{\mu\alpha}^\rho$$

$$R = g^{\mu\alpha} R_{\mu\beta\alpha}{}^\beta$$

But $\Gamma_{\mu\nu}^\lambda$ is ambiguous: \leftarrow "contorsion tensor"

$$\Gamma_{\mu\nu}^\lambda = \{_{\mu\nu}^\lambda\} - K_{\mu\nu}^\lambda$$

$$\leftarrow \text{Christoffel connection} = \frac{1}{2} g^{\lambda\sigma} \{ \partial_\mu g_{\sigma\nu} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu} \}$$

$$\text{"Torsion"} = \Gamma_{\mu\nu}^\lambda - \Gamma_{\nu\mu}^\lambda \quad * \quad \leftarrow$$

$$\Gamma_{\mu\nu}^\lambda = t^\lambda_a \partial_\mu t^\alpha_\nu + t^\lambda_a A^a_b t^\alpha_\nu \quad * \quad \text{not automatically symmetric}$$