

# GR as a Quantum Effective Field Theory

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Note Title

Summary from last time :

E.P.  $\Rightarrow \bar{T}_{\mu\nu}$  as source  $\Rightarrow$  gauge spacetime translation  
 $\Rightarrow$  metric as field or tetrad and Lorentz connection as fields  
 $\Rightarrow t_a' = \frac{\partial x^a}{\partial x^{\mu}} t_b^{\mu} \Lambda^b_{\nu}(s)$   
 $A_{\mu}' = S A_{\mu} S^{-1} + (\partial_{\mu} S) S^{-1}$

$$[D_\mu, D_\nu] = \underline{R}_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$$

$$S = \int d^4x \sqrt{-g} - \frac{1}{k^2} R$$

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G \bar{T}_{\mu\nu}$$

$$\lambda^2 = 32\pi G$$

Equations (with sources included)

$$S = S_g + S_m$$

$$R_{\mu}^{\alpha} - t_{\mu}^{\alpha} R = \frac{K^2}{4} T_{\mu}^{\alpha}$$

$$T_{\mu}^{\alpha} = \frac{1}{v_g} \frac{\delta S_m}{\delta t_{\mu}^{\alpha}}$$

$$\Rightarrow \text{usual } R_{\mu\nu} - g_{\mu\nu} R = \frac{K^2}{4} T_{\mu\nu}$$

And extra "torsion" term

$$t_{\mu}^{\alpha} C_{ij}^k - t_{\mu}^{\alpha} C_{ik}^j - t_{\mu}^{\alpha} C_{ij}^k = \frac{K^2}{16} \frac{\delta S_m}{\delta A_{\mu}^{ij}}$$

$$C_{ab}^c = t_a^{\mu} t_b^{\nu} \frac{1}{2} (\partial_{\mu} t_{\nu}^c - \partial_{\nu} t_{\mu}^c)$$

$$\frac{1}{2} t_r^c \bar{\psi} \gamma_c S^{\alpha b} \psi \sim \text{Spin source}$$

General decomposition of Lorentz connections

$$A_{\mu}^{ab} = \gamma_{\mu}^{ab} + K_{\mu}^{ab}$$

$$\gamma_{\mu}^{ab} = t^c_{\mu} (C_c{}^{ab} - C^{ab}{}_c - C^a{}_c{}^b) \quad \leftarrow \text{gives traditional GR}$$

$$K_{\mu}^{ab} = \text{"contorsion"} \quad \leftarrow \text{non-traditional - only with spin source}$$

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## Comments on non-traditional aspects

(not focus)

With torsion, two new invariants (at this order)

$$\tilde{\tau}^\lambda_{\sigma\nu} = D_\mu K^\lambda_{\sigma\nu} - D_\nu K^\lambda_{\sigma\mu} + K^\lambda_{\sigma\mu} K^\sigma_{\nu\rho} - K^\lambda_{\sigma\nu} K^\sigma_{\mu\rho} \quad \leftarrow$$

$$\tilde{\tau} = g^{\sigma\nu} \tilde{\tau}^\lambda_{\sigma\nu} \text{ invariant}$$

$$\tilde{R} = t_a^\mu t_b^\nu \epsilon^{abcd} \tilde{R}_{\mu\nu c d} \text{ invariant} \quad \leftarrow \text{Holst}$$

$$S = \int d^4x \Gamma_g \left\{ -\frac{2}{\kappa^2} R_+ + \beta \tilde{\tau} - \frac{1}{\kappa^2 \gamma} \tilde{R} \right\}$$

$\equiv \equiv \underbrace{\qquad \qquad \qquad}_{\text{Lagrange parameters}}$

Torsion in general behaves as a massive field sourced by spin density

## Effective field theory of GR

Here is [Feynman's 1963 "Quantum theory of gravitation"](#). It is fun to read, and historically very important.

['t Hooft and Veltman's](#) paper on the quantization and renormalization of gravity is a classic.

[Appendix B](#) from Dynamics of the Standard Model has an explanation of the heat kernel method, although it is in flat space.

Andrei Barvinsky has a [Scholarpedia article](#) on the heat kernel in gravity.

I have put together a whole [page](#) of references on the effective field theory treatment.

## Some basic facts

1) Curvature involves two derivatives

(power counting EFT)

$$R_{\mu\nu}^\lambda = \frac{1}{2} g^{\lambda\sigma} [\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}]$$

$$R_{\mu\nu\alpha}^\beta = \partial_\mu R_{\nu\alpha}^\beta - \partial_\nu R_{\mu\alpha}^\beta + R_{\mu\rho}^\sigma R_{\nu\alpha}^\rho - R_{\nu\rho}^\sigma R_{\mu\alpha}^\rho$$

2) Weak field limit

$$g_{\mu\nu} = \eta_{\mu\nu} + K h_{\mu\nu}$$

↙ no inverse

$$[\partial_\mu \partial_\lambda h_\nu^\lambda + \partial_\nu \partial_\lambda h_\mu^\lambda - \partial_\lambda \partial_\nu h_\mu^\lambda - \square h_{\mu\nu} - \eta_{\mu\nu} (\partial_\lambda \partial^\lambda h^\mu_\nu - \square h^\lambda_\lambda)] = 16\pi G \bar{f}_{\mu\nu}$$

3) Gauge invariance

$$h'_{\mu\nu} = h_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu$$

(weak field)

4) Gauge fixing

$$\partial_\mu h^{\mu\nu} - \frac{1}{2} \partial_\nu h^\lambda_\lambda = 0$$

Harmonic or de Donder gauge

$$\Rightarrow \Box (h_{\mu\nu} - \frac{1}{2} \gamma_{\mu\nu} h^\lambda_\lambda) = -16\pi G T_{\mu\nu}$$



5) Flat space propagator

(or PI.)

$$D_{\alpha\beta\gamma\delta} = \frac{i}{\bar{g}^2 + i\varepsilon} \frac{1}{2} [\gamma_{\mu\alpha} \gamma_{\nu\delta} + \gamma_{\mu\delta} \gamma_{\nu\alpha} - \gamma_{\mu\nu} \gamma_{\alpha\delta}]$$

6) Point mass

$$h_{\mu\nu} = \begin{pmatrix} 2\phi_1 & & & \\ & 2\phi_2 & & \\ & & 2\phi_3 & \\ & & & 2\phi_4 \end{pmatrix}$$

$$\phi_3 = -\frac{GM}{r}$$

7) Schrödinger eq in grav field

$$\mathcal{L} = \frac{1}{2m} \left( \bar{g}^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - m^2 \phi^2 \right) \Rightarrow (\Box_g + m^2) \phi = 0$$

$$\Box = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu}) \partial_\nu$$

$$\text{Let } \phi(x, t) = e^{-imt} \psi(x, t) \Rightarrow i \frac{\partial}{\partial t} \psi = \left[ -\frac{\nabla^2}{2m} + m\phi_3 \right] \psi$$

$$V(r) = -\frac{GMm}{r}$$

## E.F.T.

1) Only Low E D.O.F.  $\Rightarrow g_{\mu\nu}(x)$

- H.E.  $\Rightarrow$  Local L

2) Most general L

- consistent with symmetry  $\Rightarrow R, \tilde{g}^{\mu\nu}$

3) Order by energy expansion

$$R \sim \partial^2 g$$

$$\text{expansion } \left( \frac{E^2}{M_P^2} \right)^n$$

4) Start with lowest order  $\Rightarrow R$

- rest as pert.

- quantize, renormalize

5) Match or measure renorm param.  
known               $\tau$  unknown

6) Residual prediction

Most general L

$$S = \int d^4x F_2 \left[ -1 - \frac{2}{k^2} R + c_1 R^2 + c_2 R_{\mu\nu} R^{\mu\nu} + \dots \right]$$

Measure:

$$1) \Lambda \lesssim (10^{-3}, V)^4 \sim 10^{-122} M_{Pl}$$

$\Rightarrow$  drop

$$2) k^2 = 32\pi G$$

$$(K) \sim \frac{1}{M_p}$$

$$3) C_1, C_2 \quad (\text{Stable}) \Rightarrow (\square + \frac{C_2}{M_p^2} B^2) h = GT$$

$$GT \xrightarrow{\square} \frac{1}{g^2 + \frac{C_2}{M_p^2} g^4} = \frac{1}{g^2} - \frac{1}{(g^2 + \frac{M_p^2}{C_2})}$$

$$\checkmark \text{Yukawa } m = \frac{M_p}{C_2}$$

no need for  $R_{\mu\nu\rho} R^{\mu\nu\rho}$   
 $= 4R_{\mu\nu} R^{\mu\nu} - R^2$   
+ total deriv

$$\Rightarrow \text{mm scale tests of gravity} \Rightarrow c_i < 10^{65} \quad (\text{coulomb})$$

$\sqrt{\frac{1}{g^2} + \frac{1}{\tilde{g}^2}} \frac{g^4}{\tilde{g}^2}$

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# Quantization

('t Hooft Veltman)

- 1) Background field
- 2) Ghosts
- 3) Heat Kernel

$$g_{\mu\nu}(x) = \bar{g}_{\mu\nu}(x) + kh_{\mu\nu}, \quad g^{\mu\nu} = \bar{g}^{\mu\nu} - kh^{\mu\nu} + k^2 g^{\alpha}{}_{\alpha} + \dots$$

$$\begin{aligned} \mathcal{L} = & \sqrt{g} \left[ R - \frac{2}{k} h^{\mu\nu} \left[ R_{\mu\nu} - \frac{1}{2} \bar{g}_{\mu\nu} R \right] \right. \\ & \left. + \frac{1}{2} \bar{D}_\lambda h_{\mu\nu} \bar{D}^\lambda h^{\mu\nu} - \dots + 2 h^{\mu\nu} h_\alpha^\beta R^\alpha_\nu \right] \end{aligned}$$

$\Rightarrow$  generally covariant w.r.t  $\bar{g}$  !!  
preserves symmetry!

Gauge fixing

$$\partial_\nu h_\mu^\nu - \frac{1}{2} \partial_\mu h^\lambda_\lambda = 0$$

$$C_\mu = (D_\nu h_\mu^\nu - \frac{1}{2} D_\mu h^\lambda_\lambda)$$

$$\delta(C_\mu - F_\mu(r)) \xrightarrow{\text{exp}} \int [d^4x] \delta(C_\mu - F_\mu(r)) \frac{-i}{2} \int d^4x \delta F_\alpha F^\alpha \\ = \frac{i}{2} \int d^4x F_\mu C^\mu$$

$$\mathcal{L}_{gf} = \frac{1}{2} [D_\nu h_\mu^\nu - \frac{1}{2} D_\mu h^\lambda_\lambda]^2$$

Simplify  $\mathcal{L}$

$$\mathcal{L} = \frac{1}{2} D_\mu h_{\alpha\beta} D^\mu h^{\alpha\beta} - \frac{1}{2} D_\mu h^\lambda_\lambda D^\mu h^\sigma_\sigma + R_-$$

Ghosts

Recall

$$A_\mu^\theta = A + \partial_\mu \theta$$

$$I = S[d\theta] \Delta(A) \delta(f(A_\mu^\theta))$$

$$\Delta = \det \frac{\partial f}{\partial \theta} = \int d\bar{c} [d\bar{c}]^T$$

$\equiv \int d\bar{c} \bar{c} \frac{\partial f}{\partial \theta} c$

Gauge change

$$h_{\mu\nu} \rightarrow h_{\mu\nu} + g_{\mu\alpha} D_\nu \xi^\alpha + g_{\nu\alpha} D_\mu \xi^\alpha$$

$$\delta C_m = \underbrace{\bar{D}_\lambda (D^\nu \xi_\nu + D_\mu \xi_\nu)}_{D^2} - \underbrace{\frac{1}{2} D_\mu D_\nu \xi^\nu}_{R_{\mu\nu}}$$

$$\mathcal{L}_{\text{ghost}} = \Gamma_S \overline{\eta}^\mu (\bar{D}^2 g_{\mu\alpha} + R_{\mu\alpha}) \eta^\alpha$$

# Feynman rules:

## A.1 Scalar propagator

The massive scalar propagator is:

$$\begin{array}{c} \text{---} \xrightarrow{\text{q}} \text{---} \\ \text{q} \end{array} = \frac{i}{q^2 - m^2 + i\epsilon}$$

## A.2 Graviton propagator

The graviton propagator in harmonic gauge can be written in the form:

$$a^\beta \text{---} \xrightarrow{q} \gamma^\delta = \frac{i \mathcal{D}^{\alpha\beta\gamma\delta}}{q^2 + i\epsilon}$$

where

$$\mathcal{D}^{\alpha\beta\gamma\delta} = \frac{1}{2} [\eta^{\alpha\gamma}\eta^{\beta\delta} + \eta^{\beta\gamma}\eta^{\alpha\delta} - \eta^{\alpha\beta}\eta^{\gamma\delta}]$$

## A.3 2-scalar-1-graviton vertex

The 2-scalar-1-graviton vertex is discussed in the literature. We write it as:

$$\begin{array}{c} \text{---} \xrightarrow{\text{p}} \text{---} \xrightarrow{\text{p}'} \gamma^\delta \\ \text{---} \xrightarrow{\text{q}} \end{array} = \tau_1^{\mu\nu}(p, p', m)$$

where

$$\tau_1^{\mu\nu}(p, p', m) = -\frac{i\kappa}{2} [p^\mu p'^\nu + p'^\mu p^\nu - \eta^{\mu\nu} ((p \cdot p') - m^2)]$$

## A.4 2-scalar-2-graviton vertex

The 2-scalar-2-graviton vertex is also discussed in the literature. We write it here with the full symmetry of the two gravitons:

$$\begin{array}{c} \text{---} \xrightarrow{\text{p}} \text{---} \xrightarrow{\text{p}'} \text{---} \xrightarrow{\text{q}} \gamma^\delta \\ \text{---} \xrightarrow{\text{p}'} \text{---} \xrightarrow{\text{p}} \end{array} = \tau_2^{\mu\nu\sigma\tau}(p, p', m)$$

$$\begin{aligned} \tau_2^{\mu\nu\sigma\tau}(p, p') &= i\kappa^2 \left[ \left\{ I^{\lambda\alpha\delta} I^{\mu\sigma\beta}_\delta - \frac{1}{4} \left\{ \eta^{\eta\lambda} I^{\mu\sigma\alpha\beta} + \eta^{\beta\sigma} I^{\mu\lambda\alpha\beta} \right\} \right\} (p_\alpha p'_\beta + p'_\alpha p_\beta) \right. \\ &\quad \left. - \frac{1}{2} \left\{ I^{\eta\mu\sigma} - \frac{1}{2} \eta^{\eta\lambda} \eta^{\mu\sigma} \right\} [(p \cdot p') - m^2] \right] \end{aligned} \quad (61)$$

with

$$I_{\alpha\beta\gamma\delta} = \frac{1}{2} (\eta_{\alpha\gamma} \eta_{\beta\delta} + \eta_{\alpha\delta} \eta_{\beta\gamma}).$$

## A.5 3-graviton vertex

The 3-graviton vertex can be derived via the background field method and has the form [9],[10]

$$\begin{array}{c} \text{---} \xrightarrow{\text{p}} \text{---} \xrightarrow{\text{p}'} \text{---} \xrightarrow{\text{q}} \gamma^\delta \\ \text{---} \xrightarrow{\text{p}'} \text{---} \xrightarrow{\text{p}} \gamma^\nu \\ \text{---} \xrightarrow{\text{q}} \end{array} = \tau_3^{\mu\nu\lambda}(k, q)$$

where

$$\begin{aligned} \tau_3^{\mu\nu\lambda}(k, q) &= -\frac{i\kappa}{2} \times \left( \mathcal{P}_{\alpha\beta\gamma\delta} \left[ k^\nu k^\mu + (k-q)^\mu (k-q)^\nu + q^\mu q^\nu - \frac{3}{2} \eta^{\mu\nu} q^2 \right] \right. \\ &\quad + 2\mathfrak{Q}_\lambda q_\sigma \left[ I_{\alpha\beta}^{\sigma\lambda} I_{\gamma\delta}^{\mu\nu} + I_{\gamma\delta}^{\sigma\lambda} I_{\alpha\beta}^{\mu\nu} - I_{\alpha\beta}^{\mu\sigma} I_{\gamma\delta}^{\nu\lambda} - I_{\gamma\delta}^{\mu\sigma} I_{\alpha\beta}^{\nu\lambda} \right] \\ &\quad + \left[ \mathfrak{Q}_\lambda q^\nu \left( \eta_{\alpha\beta} I_{\gamma\delta}^{\nu\lambda} + \eta_{\gamma\delta} I_{\alpha\beta}^{\nu\lambda} \right) + \mathfrak{Q}_\lambda q^\mu \left( \eta_{\alpha\beta} I_{\gamma\delta}^{\mu\lambda} + \eta_{\gamma\delta} I_{\alpha\beta}^{\mu\lambda} \right) \right. \\ &\quad \left. - q^2 \left( \eta_{\alpha\beta} I_{\gamma\delta}^{\mu\nu} - \eta_{\gamma\delta} I_{\alpha\beta}^{\mu\nu} \right) - \eta^{\mu\nu} \mathfrak{Q}_\sigma \mathfrak{Q}_\lambda \left( \eta_{\alpha\beta} I_{\gamma\delta}^{\sigma\lambda} + \eta_{\gamma\delta} I_{\alpha\beta}^{\sigma\lambda} \right) \right] \\ &\quad + \left[ 2\mathfrak{Q}_\lambda \left( I_{\alpha\beta}^{\lambda\sigma} I_{\gamma\delta}^{\mu\nu} (k-q)^\mu + I_{\alpha\beta}^{\lambda\sigma} I_{\gamma\delta}^{\mu\nu} (k-q)^\nu - I_{\gamma\delta}^{\lambda\sigma} I_{\alpha\beta}^{\mu\nu} k^\mu - I_{\gamma\delta}^{\lambda\sigma} \right. \right. \\ &\quad \left. \left. + q^2 \left( I_{\alpha\beta}^{\mu\lambda} I_{\gamma\delta}^{\nu\sigma} + I_{\alpha\beta}^{\nu\sigma} I_{\gamma\delta}^{\mu\lambda} \right) + \eta^{\mu\nu} \mathfrak{Q}_\sigma \mathfrak{Q}_\lambda \left( I_{\alpha\beta}^{\lambda\mu} I_{\gamma\delta}^{\nu\sigma} + I_{\gamma\delta}^{\lambda\mu} I_{\alpha\beta}^{\nu\sigma} \right) \right] \right. \\ &\quad \left. + \left\{ (k^2 + (k-q)^2) \left[ I_{\alpha\beta}^{\mu\sigma} I_{\gamma\delta}^{\nu\lambda} + I_{\gamma\delta}^{\mu\sigma} I_{\alpha\beta}^{\nu\lambda} - \frac{1}{2} \eta^{\mu\nu} \mathcal{P}_{\alpha\beta\gamma\delta} \right] \right. \right. \\ &\quad \left. \left. - \left( I_{\gamma\delta}^{\mu\nu} \eta_{\alpha\beta} k^2 + I_{\alpha\beta}^{\mu\nu} \eta_{\gamma\delta} (k-q)^2 \right) \right\} \right) \end{aligned}$$

One loop

$$S[h_{\mu\nu}] [d_{\bar{\eta}} \bar{D}^{\bar{\eta}}] e^{i S d^4 \left[ \frac{1}{2} h_{\mu\nu} \underbrace{D^{\mu\nu\alpha\beta}}_{D^2} h_{\alpha\beta} + \eta(\ ) \eta \right]}$$

Make  $\det \underline{D^{\mu\nu\alpha\beta}}$

Fancy way - heat kernel

$$\text{Tr } \ln \mathcal{O} = \int dx \int_0^\infty d\tau \langle x | e^{-\tau \mathcal{O}} | x \rangle$$

$\underbrace{\qquad\qquad\qquad}_{K(x, \tau)}$

$$K(x, \tau) = \frac{i}{(4\pi)^n} \frac{e^{-m^2 \tau}}{\tau^n} \left[ a_0(x) + \tau a_1(x) + \tau^2 a_2(x) + \dots \right]$$

Seeley De Witt  
Gilkey  
Birrell + Davies)

$$\text{Tr } \ln \mathcal{O} = \dots \star \underline{\underline{\Gamma(2-d_n) a_2(x)}}$$

$$\mathcal{O} = \underline{\underline{d_m d^n}} + \sigma \qquad d_m = \partial_m + \Gamma_m$$

$$a_2 = \frac{1}{2} \sigma^2 + \frac{1}{2} [\underline{d_m}, \underline{d_n}] [\underline{d_m}, \underline{d_n}] + \frac{1}{6} [\underline{d_m}, [\underline{d^n}, \sigma]]$$

Here  $\frac{t}{\epsilon} \neq 1$

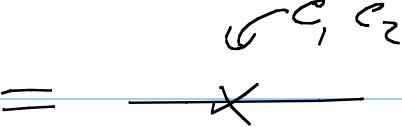
$$\Delta L = \int d^4x F_\mu^2 - \frac{1}{16\pi^2} \alpha_2 R^2 = \int d^4x F_\mu^2 - \frac{1}{16\pi^2} \left[ \frac{1}{120} R^2 + \frac{7}{20} R_{\mu\nu} R^{\mu\nu} \right] \left[ \frac{1}{\epsilon} - \dots \right]$$

local  $\checkmark$   
general covariant  $\checkmark$

Renorm.

$$C_i^{\text{ren}} = C_i^{\text{bare}} + \frac{1}{120} \frac{1}{16\pi^2} \left[ \frac{1}{\epsilon} + \dots \right] \quad \leftarrow \star$$

Poor man's way



$$\pi^{\mu\nu\alpha\beta} =$$

$$\Delta I = c_1 R^2 + c_2 R_{\mu\nu} R^{\mu\nu}$$

} ungrav

## Power counting

More loops  $\Rightarrow$  more power of energy

$\mathcal{O}(E^2) \Rightarrow$  Tree R

$\mathcal{O}(E^4) \Rightarrow$  One loop R + Tree R<sup>2</sup>

$\mathcal{O}(E^6) \Rightarrow$  2 loop - 1

Poor man's way Power of  $\epsilon^\alpha$

$$\text{action} = \epsilon^2 \left( \underbrace{g^m g^n g^k g^l}_{\sim \frac{1}{\epsilon}} \dots \right) \pi(\epsilon^\alpha) \sim \mathcal{O}(E^2)$$

Pure gravity is one loop finite (HV)

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0$$

$$\bar{\tau}_0^1$$

OK to use eq of motion in one loop  $\mathcal{L}$   $\leftarrow$

(But #1 matter exists)

(But #2 2 loops

(see Ferrer Sagnotti)

$$\Delta \mathcal{L} = \frac{209}{2880} \frac{1}{(1/\pi)^2} \underbrace{\frac{1}{\varepsilon} R_{\mu\nu\rho\sigma} R^{\alpha\beta\gamma\delta}}_{\mathcal{O}(E^6)} R_{\gamma\delta}^{\mu\nu}$$