Anomalies!

Note Title 11/3/2009

Currents in Path Integrals

J_μ = current $\qquad J_\mu = \bar{\psi}\gamma^\mu\psi$

$v_\mu(x)$ = external source — tool

$$Z[J, v_\mu] = \int [d\phi]\, e^{i\int d^4x\, [\mathcal{L} - J\phi - v_\mu J^\mu]}$$

Define

$$\bar{J}_\mu(x) = \frac{i}{Z[J]} \frac{\delta Z[J, v_\mu]}{\delta v_\mu}\Big|_{v=0}$$

Further matrix element

$$\langle 0 | T\, J_\mu(x)\, \phi(y_1)\, \phi(y_2) | 0\rangle = i^2 \frac{\delta}{\delta J(y_1)} \frac{\delta}{\delta J(y_2)} \bar{J}_\mu(x)\Big|_{J=0}$$

Example $J_\mu = e\bar{\psi}\gamma^\mu\psi$

$$\mathcal{L} - A_\mu J^\mu = \bar{\psi}\, i(\partial_\mu + eA_\mu)\gamma^\mu\psi - m\bar{\psi}\psi$$

↖ just like photon field

$$\langle 0|T(J_\mu(x)\,\psi(y_1)\,\bar{\psi}(y_2))|0\rangle = i S_F(y_1 - x)\,\underbrace{e\gamma_\mu}_{\times}\,S_F(x - y_2)$$

drop propagators

Path integrals + symmetries

Symmetry $\quad \phi_i \to \phi_i' = \phi_i + \underset{\uparrow\ \varepsilon(x)}{\varepsilon} f_i(\phi)$

Try $\quad \varepsilon \to \varepsilon(x)$

Define $\quad J^\mu = \dfrac{\partial \mathcal{L}(\phi', \partial_\mu \phi')}{\partial(\partial_\mu \varepsilon(x))}$ $\quad \Leftarrow$ Noether current

or $\mathcal{L}(\phi', \partial_\mu \phi') = \mathcal{L}(\phi, \partial_\mu \phi) + J^\mu (\partial_\mu \varepsilon(x))$ $\quad$ Invariance of $\mathcal{L}$

Usually $\quad$ Eq of motion $\quad \partial^\mu J_\mu = 0$

P.I version — add $v_\mu J^\mu$

From $\bar{J}^\mu(x) = i \dfrac{\delta \ln Z[v_\mu]}{\delta v_\mu}$

Change to integral form

$$\ln Z[v_\mu + \delta v_\mu] - \ln Z[v_\mu] = -i \int d^4x' \, \bar{J}_\mu(x') \, \delta v_\mu(x')$$

from $\dfrac{\delta}{\delta v_\mu(x)} v_\nu(x') = g_{\mu\nu} \, \delta^4(x - x')$

Now we choose $\delta v_\mu = -\partial_\mu \varepsilon(x)$

$$\ln Z[v_\mu - \partial_\mu \varepsilon] - \ln Z[v_\mu] = +i \int d^4x' \, \bar{J}_\mu(x') \, \partial_\mu \varepsilon(x)$$

$$= -i \int d^4x \, \partial^\mu \bar{J}_\mu(x) \, \varepsilon(x)$$

Test of symmetry

$$\underbrace{Z[v_\mu - \partial_\mu \varepsilon] = Z[v_\mu]}_{K} \Longrightarrow \partial^\mu J_\mu(x) = 0$$ for all matrix elements

Apply test

$$Z[v_\mu - \partial_\mu \varepsilon] = \int [d\phi]\, e^{i\int d^4x \left[\mathcal{L}(\phi, \partial_\mu \phi) - (v_\mu - \partial_\mu \varepsilon) J^\mu \right]}$$

$$= \int [d\phi]\, e^{i\int d^4x \left[\mathcal{L}(\phi', \partial_\mu \phi') - v_\mu J^\mu \right]}$$ if $\mathcal{L}$ is invariant

$$= \int [d\phi']\, e^{i\int d^4x \left[\mathcal{L}(\phi', \partial_\mu \phi') - v_\mu J^\mu \right]}$$ if $[d\phi'] = [d\phi]$

$$= Z[v_\mu]$$

Two conditions

1) Invariance of $\mathcal{L}$ (Noether)

2) $\int [d\phi] = \int [d\phi']$ (P.I measure)

Quantum theories require both

Anomaly #1 satisfied
#2 is not

$\Bigg\}$ Noether's thm lied
Not a quantum symmetry

Trace Anomaly - QCD

- integrate out heavy quark $\sim 1/M_H^2$

$$\mathcal{L} = -\frac{1}{4}F^2 + \bar{\psi}\, i\not{D}\psi - \bar{\psi}\underbrace{m}\psi$$

$m_u \sim 4$ MeV, $m_d \sim 7$, $m_s \sim 100$ MeV

$M_p = 940$ MeV

If $m = 0 \Rightarrow$ no scale in $\mathcal{L}_{QCD}$

Scale Invariance

$x = \lambda x' \Rightarrow d^4x = \lambda^4 d^4x'$ $\qquad \partial_\mu = \frac{1}{\lambda}\partial'_\mu$

$A_\mu(x) = \lambda^{-1} A_\mu(\lambda x')$, $\psi(x) = \lambda^{-3/2}\psi(\lambda x')$

$$S = \int d^4x \left[- \qquad \right]$$

$$= \int d^4x' \lambda^4 \left[\frac{1}{\lambda^4} \frac{-1}{4} F^2(\lambda x') + \frac{1}{\lambda^4} \bar\psi i \not\partial' \psi - \frac{1}{\lambda^3} \bar\psi m \psi \right]$$

$m \equiv 0 \Rightarrow$ scale invariant

Dilaton current

$$J^\mu_D = T^{\mu\nu} x_\nu$$

$$\partial^\mu J_{D\mu} = (\overbrace{\partial_\mu T^{\mu\nu}}^{0}) x_\nu + T^{\mu\nu} \underbrace{\partial_\mu x_\nu}_{g^{\mu\nu}} = T^\mu{}_\mu$$

Construct $T^{\mu\nu}$

$$T^{\mu\nu} = F^{\mu\lambda} F_\lambda{}^\nu - \frac{1}{4} g^{\mu\nu} F^{\lambda\sigma} F_{\lambda\sigma} + \frac{i}{2} \bar{\psi} (\gamma^\mu D^\nu + \gamma^\nu D^\mu) \psi - g^{\mu\nu} \underbrace{\bar{\psi} (i \not{D} - m) \psi}_{\text{Drop}}$$

$$T^\mu{}_\mu = \left(1 - \frac{d}{4}\right) F^{\sigma\lambda} F_{\sigma\lambda} + i \bar{\psi} \not{D} \psi$$

$$= m \bar{\psi} \psi$$

$$= 0 \quad \text{if } m = 0$$

The problem

Non rel norm $= \frac{2}{2m}$ $\Rightarrow \langle p | T^{00} | p \rangle = m$

often $N = \frac{2}{2E}$

$$\langle p | T^{\mu\nu} | p \rangle = N p^\mu p^\nu$$

Then

$$\langle p | T^\mu_\mu | p \rangle = N m_p^2 = 0 \quad \text{if } T_\mu^\mu = 0$$

$\Rightarrow$ all particles are massless

But Noether's Theorem wrong

QCD does not have scale invariance

$$T^\mu_\mu = \frac{\beta(g)}{2g} F^a_{\mu\nu} F^{a\mu\nu} + \sum_i m_i \bar{\psi}_i \psi_i$$

Path integral - rescaling the fields - fermions

$$\psi(x) = e^{-\alpha(x)/2}\,\psi'(x)$$

$$\int dx_1 \cdots dx_n = \int dy_1 \cdots dy_n \,\det\left[\frac{\partial x}{\partial y}\right]$$

$$\int [d\psi(x)][d\bar\psi] = \int [d\psi'][d\bar\psi'] \,\det\left[e^{-\alpha(x)}\right]$$

Fujikawa - gauge invariant regularization

$$\not{D} = \left(\not\partial + i g \frac{A^a \lambda^a}{2}\right)$$

$$\det\left[e^{-\alpha(x)}\right] = \lim_{M\to\infty} \det\left[e^{-\left\{\alpha(x)\, e^{-\not{D}\not{D}/M^2}\right\}}\right]$$

removes high momentum modes

$$= e^{\int d^4x \,\mathrm{Tr}\langle x|\,\alpha(x)\, e^{-\not{D}\not{D}/M^2}|x\rangle}$$

$$H(x,t) = \langle x | e^{-\tau \mathcal{D}} | x \rangle \qquad \swarrow d_\mu = \partial_\mu + \Gamma$$

$$\mathcal{D} = \not{D}\not{D} = \left(\not{\partial} + ig \not{A}^a \frac{\lambda^a}{2}\right)\left(\not{\partial} + ig \not{A}^b \frac{\lambda^b}{2}\right) = d_\mu d^\mu + \sigma$$

$$\Gamma_\mu = ig \frac{\lambda^a}{2} A^a_\mu$$

$$\sigma = \frac{1}{2} \sigma^{\mu\nu} ig \frac{\lambda^a}{2} F^a_{\mu\nu}$$

$m = 0$

$$H(x,t) = \frac{1\, e^{-m^2 \tau}}{(4\pi)^{d/2} \tau^{d/2}} \left[1 - \tau \sigma + \tau^2 \left\{ \frac{1}{12} [d_\mu, d_\nu] d^\mu d^\nu + \cdots \right\} \right]$$

$1/M^2$

$$\mathrm{Tr} \langle x | e^{-\not{D}\not{D}/M^2} | x \rangle = \frac{3 i M^4}{4\pi^2} + \frac{i g^2}{48\pi^2} F^a_{\mu\nu} F^{a\mu\nu}$$

$\times\times!$

anomaly