# Anomalies 2

Note Title 11/5/2009

$$\psi \doteq e^{-\alpha/2}\psi'$$

$$\int d\psi\, d\bar\psi = \int d\psi'\, d\bar\psi'\, J$$

$$J = e^{i\int d^4x\, \alpha(x)\left[\frac{3M^4}{4\pi^2} + \frac{g^2}{48\pi^2} F^a_{\mu\nu}F^{a\mu\nu}\right]}$$ Anomaly!

Goal

$$T^\mu_\mu = \Theta^\mu_\mu = \frac{\alpha_s}{12\pi} F^a_{\mu\nu}F^{a\mu\nu} + m\bar\psi\psi$$

# Anomaly relation

$$T^{\mu\nu} \doteq \frac{i}{2}\bar{\Psi}\gamma^{\mu}\overleftrightarrow{D}\Psi$$

add with source

$$Z[h] = \int d\Psi d\bar{\Psi}\, e^{i\int d^4x\,[\mathcal{L}_{QCD} + h\,T^{\mu\nu}]}$$

Test

$$\Psi \to e^{-\alpha/2}\Psi$$

$$Z[h+\alpha] = Z[h]$$

$$\bar{\Psi} i \not{D} \Psi = \bar{\Psi}' i \not{D} \Psi' - \alpha\, \underbrace{\bar{\Psi}' i \not{D} \Psi'}_{T^{\mu}_{\mu}}$$

$$\bar{\Psi} m \Psi = \bar{\Psi}' m \Psi' - \alpha\, \bar{\Psi} m \Psi$$

$$Z[h+\alpha] = \int d\psi\, d\bar\psi\; e^{\,i\int d^4x\,[\bar\psi(i\not{\partial}-m)\psi + (h+\alpha)T^\mu{}_\mu]}$$

$$= \quad '' \quad e^{\,i\int d^4x\,[\bar\psi'(i\not{\partial}-m)\psi' + h\,T^\mu{}_\mu + \alpha m\bar\psi'\psi']}$$

$$= \int d\psi'\, d\bar\psi'\, \underbrace{J}_{e^{\ln J}}\; e^{\,i\,---\;----\;-}$$

Comparing first & last forms

$$i\int d^4x\,\alpha\, T^\mu{}_\mu = \underbrace{\ln J}_{i\int d^4x\,\alpha\left[\frac{3M^4}{4\pi^2} + \frac{g^2}{48\pi^2}F^2\right]} + i\int d^4x\;\alpha\, m\bar\psi\psi$$

$$\Rightarrow T^\mu{}_\mu = \underbrace{\frac{3M^4}{4\pi^2}}_{\text{drop}} + \underbrace{\frac{g^2}{48\pi^2}F^2}_{\text{anomaly}} + \underbrace{m\bar\psi\psi}_{\text{explicit}} \quad \checkmark$$

## Comments:

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \partial_\nu \phi - g^{\mu\nu} \mathcal{L}$$

constant in $\mathcal{L}$ $\Rightarrow \Lambda$

$$= \Lambda g_{\mu\nu}$$

cosmological constant

$$T^{00} = \Lambda \leftarrow \text{Zero point energy}$$

2) This was fermion

$\int [dA_\mu]$ also

$$\Rightarrow T^\mu_\mu = \frac{\beta(g)}{2g} F^2 + \sum_i m_i \bar{\psi}_i \psi_i$$

fermion contrib to $\beta$ function

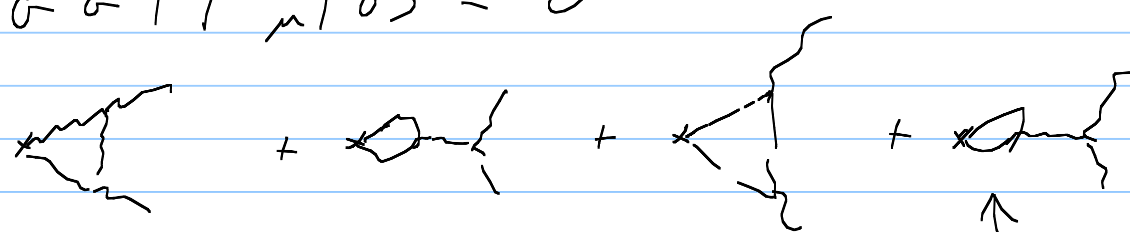
## 3) Feynman diagrams

- dim reg

$$T^\mu{}_\mu = \frac{4-d}{4} F^2 + (1-d)\,\bar\psi(i\not{\partial} - m)\psi + m\bar\psi\psi$$

$d=4$ eq of motion

$$\langle G G | T^\mu{}_\mu | 0\rangle \stackrel{?}{=} 0$$



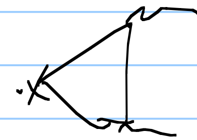
$$\frac{4-d}{4} \times g^2, \quad \frac{1}{4-d} = \text{finite}$$

finite

– or Pauli Villars

$$T^\mu{}_\mu = \ldots \bar{\Psi}\, i(\not{D} - m)\Psi \;-\; \underbrace{\bar{\Psi}'(i\not{D} - M)\Psi'}_{}$$

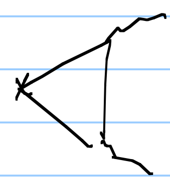
P.V. $\quad M \to \infty$

 $\sim \dfrac{M}{M} \to$ finite

$$\frac{1}{\not{p} - m} - \frac{1}{\not{p} - M}$$

4) Heavy quarks

$$T^\mu{}_\mu = \frac{\beta}{2g} G F^2 + \quad \ldots \quad + m_t \bar{t} t$$

Integrate out  $= \dfrac{g^3}{48\pi^2} F^2$ not $1/m_t^2$ ?

$$T^\mu_\mu = \frac{\beta_5}{2g^2} F^2 + \ldots + m\,\bar{q}q$$

5) $\langle p | T^\mu_\mu | p \rangle = m_p \, \bar{u} u$

$$= \langle p | \underset{\uparrow\ 890\,\mathrm{MeV}}{\frac{\beta}{2g} F^2} + \underbrace{m_u \bar{u} u + m_d \bar{d} d}_{\approx 40\,\mathrm{MeV}} + \underset{\uparrow\ ?}{m_s \bar{s} s} | p \rangle$$

## 1) Mass due to gluon?

- Gluons in Vacuum $\qquad \langle 0 | F^2 | 0 \rangle \neq 0$
-  $\Delta E = \Delta\rho_v V$

$$\langle P | T^{\mu\nu} | P \rangle = (\Delta\rho V)\, g^{\mu\nu} + \underbrace{\rho_{matter}}_{\text{massless}} \begin{pmatrix} 1 & & & \\ & 1/3 & & \\ & & 1/3 & \\ & & & 1/3 \end{pmatrix}$$

$$= m_p\, \delta_{\mu 0}\, \delta_{\nu 0}$$

$$\Rightarrow \rho_{matter} = 3(\Delta\rho_v V) \qquad \Rightarrow \quad m_p = 4(\Delta\rho V)$$

$$T^\mu_{\ \mu} = 4(\Delta\rho V) \Leftarrow \text{Gluon in vacuum}$$

# Non rigorous derivation

$$gA = A'$$

$$\not{D}\psi = \left(\partial_\mu + i\frac{\lambda^a}{2}A^a_\mu\right)\psi \qquad \text{no coupling}$$

$$F_{\mu\nu} = \partial_\mu A + gA^2 = \frac{1}{g}\left[g\partial_\mu A + g^2 A\right] = \frac{1}{g}F'_{\mu\nu}$$

$$S = \int d^4x \; -\frac{1}{4g^2} F'_{\mu\nu} F'^{\mu\nu} + \overline{\psi}(i\not{D}' - m)\psi$$

← only coupling

$$g = g(\lambda)$$

$$\frac{\delta S}{\delta \lambda} = \int d^4x \frac{\partial}{\partial\lambda}\left[-\frac{1}{4g^2(\lambda)}\right] F'^2 = \frac{1}{2g^3}\underbrace{\frac{\partial}{\partial\lambda} g}_{\beta(g)} F'^2$$

Now back to original

$$= \int d^4x \, \frac{\beta(g)}{2g} F^2$$

$$T^\mu_\mu = \frac{\beta}{2g} F^2$$

$$\mu \frac{\partial}{\partial \mu} \to \frac{\delta}{\delta \lambda}$$

$$\delta \mu = \delta \lambda \, \mu$$

# Axial Anomaly

## 2 examples in SM

- "U(1) problem"
- $\pi \to \gamma\gamma$

U(1) $\quad (m \to 0\,,\ m_u, m_d, m_s)$ $\qquad$ U(3)

N.T. $\quad \partial^\mu J^i_{5\mu} = 0 \qquad i = 1 \dots 9$

$$J^i_\mu = \bar{\psi} \lambda^i \gamma_\mu \gamma_5 \psi \qquad i = 1 \dots 8$$

$$J^0_\mu = \bar{\psi} \gamma_\mu \gamma_5 \psi$$

$$\psi = \begin{pmatrix} u \\ d \\ s \end{pmatrix}$$

$\Rightarrow$ 9 Goldstone bosons $\underline{3\pi,\ 4K,\ \eta(549)}$, ? $\qquad$ ( $\overset{?}{\eta'(960)}$ )

But $\partial_\mu J_5^{0\mu} = \frac{3\alpha_s}{8\pi} F^a_{\mu\nu}\tilde{F}^{a\mu\nu}$ $\qquad \tilde{F}_{\mu\nu} = \varepsilon_{\mu\nu\alpha\beta}F^{\alpha\beta}$

<u>PI version</u>

$\psi \to e^{-i\beta\gamma_5}\psi \qquad \mathcal{L}_{QCD} \to \mathcal{L}_{QCD}$ if $m=0$

But $\int d\psi\, d\bar{\psi} = \int d\bar{\psi}'\, d\psi'\, J$

$J = e^{-2i\,\mathrm{Tr}\,\beta\gamma_5} = \left[\det\left(e^{i\beta\gamma_5}\right)\right]^{-2}$

$\mathrm{Tr}\,\beta\gamma_5 = \int d^4x\, \langle x|\,\beta\gamma_5\, e^{-\cancel{D}\cancel{D}/M^2}\,|x\rangle$

use heat kernal

$$\mathcal{J} = e^{-i \int d^4x \, \beta(x) \frac{3\alpha_5}{4\pi} F\tilde{F}}$$

$$\Longrightarrow \quad \partial^\mu J^0_{5\mu} = \frac{3\alpha}{8\pi} F\tilde{F}$$

Feynman Diagrams

$$\langle G_1 \, G_2 \,|\, J_{5\mu} \,|\, 0 \rangle = \varepsilon_1^{\alpha} \, \varepsilon_2^{\beta} \, T_{\mu\alpha\beta}$$

$$\partial^\mu \langle \qquad \rangle \overset{?}{=} 0 \Rightarrow k^\mu T_{\mu\alpha\beta} = 0$$

$$U(3)_V \times \underbrace{U_A(3)}_{\uparrow\, U_A(1)}$$

overall quark #

$$\downarrow$$

$$U_V(1)$$

$$U(3) = SU(3) + U(1)$$

$$e^{i(\alpha_0 + \alpha^a \lambda^a)}$$