QFT II Sept 17

Note Title 9/17/2009

Chiral Fermions

$m = 0$ $\qquad i\not\partial\psi = 0$

another solution $\qquad i\not\partial\gamma_5\psi = 0$ $\qquad\qquad \gamma_5\gamma_\mu = -\gamma_\mu\gamma_5$

Linear comb

$$\psi_L = \frac{1}{2}(1+\gamma_5)\psi \quad ; \quad \psi_R = \frac{1}{2}(1-\gamma_5)\psi$$

$$u(p) = N\begin{pmatrix} \chi \\ \frac{\vec\sigma\cdot\vec p}{E+m}\chi \end{pmatrix} \xrightarrow[m=0]{} N\begin{pmatrix} \chi \\ \frac{\vec\sigma\cdot\hat p}{p}\chi \end{pmatrix}$$

$$u_L = \frac{1}{2}(1+\gamma_5)u = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} \chi \\ \sigma\cdot\hat p\,\chi \end{pmatrix} = \frac{N}{2}\begin{pmatrix} (1-\sigma\cdot\hat p)\chi \\ -(1-\sigma\cdot\hat p)\chi \end{pmatrix}$$

Helicity basis $\qquad \vec\sigma\cdot\hat p\,\chi = \underset{\pm 1}{\lambda}\,\chi$

ψ_L is purely $\lambda = -1$ $\qquad$ LH

ψ_R " $\qquad \lambda = +1$ $\qquad$ RH

$$\Pi_L = \frac{1}{2}(1+\gamma_5) \quad , \quad \Pi_R = \frac{1}{2}(1-\gamma_5)$$

$$\Pi_L + \Pi_R = 1$$

$$\Pi_L^2 = \Pi_L \quad , \quad \Pi_R^2 = \Pi_R \quad , \quad \Pi_L \Pi_R = 0$$

All fields

$$\psi = \psi_L + \psi_R \qquad \psi_{\substack{L\\R}} = \Pi_{\substack{L\\R}}\,\psi$$

$$\bar{\psi}_L = \psi^\dagger \underbrace{\Pi_L^\dagger}_{\Pi_L} \gamma^0 = \bar{\psi}\,\Pi_R \;\checkmark \qquad \text{since } (1+\gamma_5)\gamma_0 = \gamma_0(1-\gamma_5)$$

Dirac Lagrangian

$$\mathcal{L} = \bar{\Psi}(i\not{D} - m)\Psi$$

$$\Psi = \Psi_L + \Psi_R$$

Mass $\quad \bar{\Psi}_L \Psi_L = \bar{\Psi} \Gamma_R \Gamma_L \Psi = 0$

$$\bar{\Psi}_L \Psi_R = \bar{\Psi} \Gamma_R \Gamma_R \Psi \neq 0$$

Kinetic Term $\quad \bar{\Psi}_L i\not{D} \Psi_L = \bar{\Psi} \Gamma_R i\not{D} \Gamma_L \Psi = \bar{\Psi} i\not{D} \Gamma_L \Gamma_L \Psi \cancel{\neq} 0$

$$\bar{\Psi}_L i\not{D} \Psi_R = \bar{\Psi} \Gamma_R i\not{D} \Gamma_R \Psi = 0$$

$$\mathcal{L} = \bar{\Psi}_L i\not{D} \Psi_L + \bar{\Psi}_R i\not{D} \Psi_R - m(\bar{\Psi}_L \Psi_R + \bar{\Psi}_R \Psi_L)$$

In massless limit — 2 invariance

$$\Psi_L \to e^{-i\alpha_L} \Psi_L \quad ; \quad \Psi_R \to \Psi_R$$

$\psi_R \to e^{-i\alpha_R}\psi_R \quad ; \quad \psi_L \to \psi_L$

$L^\mu = \bar{\psi}_L \gamma^\mu \psi_L = \frac{1}{2}\bar{\psi}\gamma^\mu(1+\gamma_5)\psi$ $\qquad \partial_\mu L^\mu = 0$ if $m=0$

$R^\mu = \bar{\psi}_R \gamma^\mu \psi_R = \frac{1}{2}\bar{\psi}\gamma^\mu(1-\gamma_5)\psi$

$V^\mu = L^\mu + R^\mu = \bar{\psi}\gamma^\mu\psi$

$A^\mu = L^\mu - R^\mu = \psi\gamma^\mu\gamma^5\psi$

with mass $\quad \partial_\mu V^\mu = 0$ $\qquad$ still $\psi_{L,R} = e^{-i\alpha}\psi_{L,R}$

$\partial_\mu A^\mu = 2im\bar{\psi}\gamma_5\psi$

Majorana Mass

Lorentz inv $\quad \bar{\Psi}\psi \qquad \bar{\psi}\, i\not{\partial}\psi$

$$\psi(x) \to \psi'(x') = S(\Lambda)\,\psi(x) \qquad x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu}$$

$\uparrow$ 4x4 matrix on Dirac indices

$$\bar{\Psi}\, i\not{\partial}\Psi \to \bar{\Psi}'\, i\not{\partial}'\psi'$$

$$\Rightarrow \quad S\gamma^{\mu}S^{-1} = \Lambda^{-1\mu}{}_{\nu}\gamma^{\nu}$$

$$\Lambda^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + \omega^{\mu}{}_{\nu} \quad \Rightarrow \quad S = e^{-\frac{i}{4}\omega^{\mu\nu}\sigma_{\mu\nu}}$$

Another invariant?

$$g^{\mu\nu} g_{\mu\gamma} = \delta^{\nu}_{\gamma}$$

$$I = M_{\alpha\beta}\psi_{\alpha}\psi_{\beta} \longrightarrow M_{\alpha\beta} S_{\alpha\gamma} S_{\beta\delta} \psi'_{\gamma}\psi'_{\delta} = M_{\gamma\delta}\psi'_{\gamma}\psi'_{\delta}$$

$$S^{T} M S = M$$

Solution $\quad M = i\gamma_0\gamma_2 = C$

$$\mathcal{L}_M = -\frac{m}{2}\left(\psi^T \underset{\nwarrow C}{M} \psi + \bar{\psi}^T \underset{C}{M} \bar{\psi}\right)$$ Majorana mass term

$$\psi_L^T C \psi_L = \psi^T \Gamma_L\, i\gamma_0\gamma_2\, \Gamma_L \psi \neq 0$$

$$\psi_R^T C \psi_L = 0$$

Independent

$$\mathcal{L} = \left[\bar{\psi}_L i\not{D}\psi_L + \frac{m_L}{2}\left(\psi_L^T C \psi_L + h.c.\right)\right] \quad \leftarrow 2 \text{ D.o.F.}$$

$$+ \left[\bar{\psi}_R i\not{D}\psi_R + \frac{m_R}{2}\left(\psi_R^T C \psi_R + h.c.\right)\right] \quad \leftarrow 2 \text{ D.o.F.}$$

No longer particle # conservation $\quad \psi \to e^{i\alpha}\psi$ not symmetry

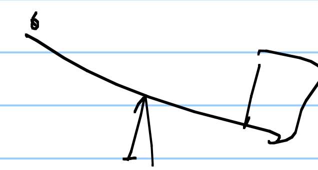
$\psi + \psi \to 0$ Majorana particle is own antiparticle

0 ψ_L, ψ_R

$$\begin{pmatrix} m_{M_L} & m_D \\ m_D & m_{M_R} \end{pmatrix} \to \begin{pmatrix} 0 & m_D \\ m_D & m_{RM} \end{pmatrix}$$

M_M large M_M

$\frac{m_D^2}{M_M}$ $\leftarrow$ seesaw



Spin Sums

$\swarrow$ 2x2 matrix

Non Rel. $\mathcal{M} = \chi_f^\dagger M \chi_i$ $\qquad \chi = \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}$

average over initial, sum over final

$$|\mathcal{M}|^2 = \chi_f^\dagger M \chi_i \chi_i^\dagger M^\dagger \chi_f$$

$$\sum_{s_i} \chi_{ia} \chi_{ib}^\dagger = \begin{pmatrix}1\\0\end{pmatrix}_a (1, 0)_b + \begin{pmatrix}0\\1\end{pmatrix}_a (0\ 1)_b = \delta_{ab}$$

$$\sum_{s_i} \chi_i \chi_i^\dagger = \underline{1} = \begin{pmatrix}1\\0\end{pmatrix} \otimes (1\ 0) + \begin{pmatrix}0\\1\end{pmatrix} \otimes (0, 1) = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$$

$$\sum_{s_i} |\mathcal{M}|^2 = \chi_f^\dagger M M^\dagger \chi = (M M^\dagger)_{dc} \chi_c \chi_d^\dagger$$

$$\sum_{s_i, f} |\mathcal{M}|^2 = \mathrm{Tr}(M M^\dagger)$$

Dirac Case

$$\mathcal{M} = \bar{u}(p')\,\Gamma\, u(p)$$

$$|\mathcal{M}|^2 = \bar{u}(p')\,\Gamma\, u(p)\ \underbrace{u^\dagger(p)\gamma^0\gamma^0\Gamma^\dagger\gamma^0}\, u(p')$$

$$\bar{u}(p)\,\bar{\Gamma}\, u(p') \quad \text{with} \quad \bar{\Gamma} = \gamma_0 \Gamma^\dagger \gamma_0$$

$$\sum_s u_\alpha(p)\,\bar{u}_\beta(p) = \left(\frac{E+m}{2E}\right)\begin{pmatrix} \chi \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\chi \end{pmatrix}_\alpha \left(\chi^\dagger\ ,\ -\chi^\dagger \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\right)_\beta$$

$$= \frac{E+m}{2E}\begin{pmatrix} 1 & -\frac{\vec{\sigma}\cdot\vec{p}}{E+m} \\ +\frac{\vec{\sigma}\cdot\vec{p}}{E+m} & -\frac{(\vec{\sigma}\cdot\vec{p})^2}{(E+m)^2} \end{pmatrix}_{\alpha\beta} = \frac{1}{2E}\left(\not{p}+m\right)_{\alpha\beta}$$

$$\frac{p^2}{(E+m)^2} = \frac{E^2-m^2}{(E+m)^2} = \frac{E-m}{E+m}$$

Consistency

$$(\not{p}-m)\, u(p)\,\bar{u}(p) = 0$$

$$\sum_s v(p)\,\bar{v}(p) = \frac{\not{p}-m}{2E}$$

$$\sum_{s,s'} |\mathcal{M}|^2 = \mathrm{Tr}\left(\Gamma \frac{(\not{p}+m)}{2E} \bar{\Gamma} \frac{(\not{p}'+m)}{2E'} \right)$$

$$\bar{\Gamma} = \gamma_0 \Gamma^\dagger \gamma_0$$

$$\Gamma = 1 \quad , \quad \bar{\Gamma} = 1$$

$$\Gamma = \gamma_5 \quad \bar{\Gamma} = -\gamma_5$$

$$\Gamma = \gamma_\mu \quad \bar{\Gamma} = \gamma_0 \gamma_\mu^\dagger \gamma^0 = \gamma_\mu$$

$$\Gamma = \gamma_\mu \gamma_5 \quad \bar{\Gamma} = \gamma_\mu \gamma_5$$

$$\Gamma = \not{a}\,\not{b} \ldots \not{n} \quad , \quad \bar{\Gamma} = \not{n} \ldots \not{b}\,\not{a}$$

Photon spin sums

$\mathcal{M} = \varepsilon_\mu M^\mu$ gauge inv $k_\mu M^\mu = 0$

Pick z along $\vec{k}$

$k^\mu = k(1, 0, 0, 1)$

$\varepsilon_\mu = (0, 1, 0, 0)$ or $(0, 0, 1, 0)$

$k_\mu M^\mu = 0 = k M_0 - k M_3 \Longrightarrow M_0 = M_3$

Spin sum

$\sum_\lambda |\varepsilon_\mu \mathcal{M}^\mu|^2 = M_1^2 + M_2^2 + (M_3^2 - M_0^2) = -g_{\mu\nu} M^\mu M^\nu$

Rule $\sum_\lambda \varepsilon_\mu(\lambda) \varepsilon_\nu^*(\lambda) \doteq -g_{\mu\nu}$

Traces of γ Matrices

$$\mathrm{Tr}(\gamma_\mu) = 0$$

$$\mathrm{Tr}(\gamma_\mu \gamma_5) = 0$$

$$\mathrm{Tr}(\not{a}\not{b}) = \mathrm{Tr}(\not{b}\not{a}) = \frac{1}{2}\mathrm{Tr}(\not{a}\not{b} + \not{b}\not{a}) = \frac{1}{2}\mathrm{Tr}(\underbrace{\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu}_{2g^{\mu\nu}\mathbb{1}})\, a^\mu b^\nu = 4\, a\cdot b$$

$$\mathrm{Tr}(\not{a}\not{b}\not{c}) = 0 = \mathrm{Tr}(\text{odd } \#)$$

$$\mathrm{Tr}(\not{a}\not{b}\not{c}\not{d}) = 4\left[a\cdot b\; c\cdot d + a\cdot d\; b\cdot c - (a\cdot c)\; b\cdot d\right]$$

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta\gamma^5) = -4i\,\varepsilon^{\mu\nu\alpha\beta}$$

$$\gamma^\mu \gamma_\mu = 4 \quad (= d\ )$$

$$\gamma_\mu \not{a} \gamma^\mu = -2 \not{a}$$

$$\gamma_\mu \not{a} \not{b} \gamma^\mu = 4\, a \cdot b$$

$$\gamma_\mu \gamma_\sigma \gamma_\nu = g_{\mu\sigma} \gamma_\nu + g_{\sigma\nu} \gamma_\mu - g_{\mu\nu} \gamma_\sigma - i\, \epsilon_{\mu\sigma\nu\beta} \gamma^\beta \gamma^5$$

Partial sample calc. — Compton scattering

$$-i\mathcal{M} = e^2\, \epsilon_\mu(k)\, \epsilon^*_\nu(k')\, \bar{u}(p') \Big[\underbrace{\gamma_\mu \frac{1}{\not p + \not k - m} \gamma_\nu}_{\Gamma_{\mu\nu}} - \gamma_\nu \frac{1}{\not p - \not k' - m} \gamma_\mu \Big] u(p)$$

photon pol

e spin

$$\frac{1}{2} \times \frac{1}{2} \sum |\mathcal{M}|^2 = \frac{1}{4} \times -g_{\mu\alpha} \; -g_{\nu\beta} \; \mathrm{Tr}\Big(\Gamma_{\mu\nu} \frac{\not p + m}{2E} \; \bar{\Gamma}_{\alpha\beta} \frac{\not p' + m}{2E'} \Big)$$

Note
$$\Gamma_{\mu\nu} = \gamma_\mu \frac{\not p + \not k + m}{(p+k)^2 - m^2} \gamma_\nu = \frac{\gamma_\mu \not k \gamma_\nu + \gamma_\mu \Big[\overbrace{\{\not p, \gamma_\nu\}}^{2p_\nu} + \gamma_\nu \underbrace{(-\not p + m)}_{\to 0} \Big]}{(\qquad\qquad)} \qquad (\not p - m) u(p)$$