Extension 5 9/22/09 QFT II

Note Title 9/22/2009

Practical Group Theory

SU(N) groups – $N \times N$ unitary $\det U = 1$

$U = e^{iH}$ H hermitian

$\det U = e^{\mathrm{Tr} \ln U} = e^{i \mathrm{Tr} H} \Rightarrow H$ is traceless

Parameters $N \times N$ Hermetian N^2

Traceless $N^2 - 1$

Diagonal $N - 1$

Invariances of 2 objects

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \longrightarrow \psi \to \psi' = U\psi \qquad U^\dagger U = 1$$

$$\psi^\dagger \psi \to \psi'^\dagger \psi' = \psi^\dagger \psi$$

Unitary 2x2 $\qquad U = e^{-i(\alpha \mathbb{1} + \vec{\alpha}\cdot\vec{\tau})}$

$\alpha \mathbb{1}$ ↑ $\underline{U(1)}$ $\qquad \vec{\alpha}\cdot\vec{\tau}$ ↑ $\underline{SU(2)}$

τ^i = Pauli matrices (σ^i)

$$[\tau^i, \tau^j] = 2i\,\varepsilon^{ijk}\tau^k$$

$$\mathrm{Tr}(\tau^i \tau^j) = 2\delta^{ij}$$

$$\tau^i \tau^j = \mathbb{1}\,\delta_{ij} + i\,\varepsilon^{ijk}\tau^k$$

$$(\tau^i)_{ab}(\tau^i)_{cd} = 2(\delta_{ad}\delta_{bc} - \frac{1}{2}\delta_{ab}\delta_{cd})$$

Checks

$$\delta^{ab}(\tau^i)_{ab}(\tau^i)_{cd} = 2(\delta_{cd} - \frac{1}{2}\times 2\,\delta_{cd}) = 0$$

$$(\tau^i)_{ab}(\tau^i)_{ba} = Tr(\tau^i\tau^i) = 6 = 2(2\times 2 - \frac{1}{2}\times 2) = 6 \checkmark$$

$$(\tau^j)_{ba}\tau^i_{ab}\,\tau^i_{cd} = 2\delta_{ij}\,\tau^i_{cd} = 2\,\tau^j_{cd} = 2\,\tau^j_{cd}$$

Invariants $\bar{\Psi} \mathcal{O} \Psi \to \bar{\Psi} U^{+} \mathcal{O} U \Psi$ $\quad$ if $\mathcal{O} \propto \mathbb{1}$

$\bar{\Psi} \mathcal{O} \Psi$ $\quad\quad$ τ 2×2

What about

$\bar{\Psi} \tau^i \Psi \longrightarrow \bar{\Psi} \underbrace{U^{+} \tau^i U}_{} \Psi$

2x2, Hermetian, Traceless $\Rightarrow R_{ij} \tau^j$ $\quad \leftarrow U = e^{-i\vec{\alpha}\cdot\vec{\tau}}$

Find

$R^{ij} = \frac{1}{2} Tr(R_{ik} \tau^k \tau^j) = R^{ij} = \frac{1}{2} Tr(U^{+} \tau^i U \tau^j)$

Claim R^{ij} = Orthogonal 3x3 matrix $\quad O(3)$

$R^{ij} R^{kj} = \delta^{ik} = \frac{1}{2} Tr(\underbrace{U^{+} \tau^i U}_{(U^{+}\tau^i U)_{ba}} \tau^j_{ab}) \; \frac{1}{2} Tr(\underbrace{U^{+} \tau^k U}_{(U^{+}\tau^k U)_{dc}} \tau^j_{cd})$

$= \frac{1}{4} * 2 \left[(U^{+} \tau^i U)_{ba} (U^{+} \tau^k U)_{ab} - \frac{1}{2} \cancel{(U^{+} \tau^i U)_{ab} \delta^{ab}} (U^{+} \tau^k U)_{dc} \delta^{cd} \right]$

$$= \frac{1}{2} \times Tr(u^\dagger \tau^i u u^\dagger \tau^k u) = \delta^{ik} \quad \checkmark$$

$\Rightarrow O(3)$ vector $\quad V_i \to R_{ij} V_j$

$U = e^{-i\alpha\tau_z}$

$$\Rightarrow R_{ij} = \begin{pmatrix} \cos 2\alpha & \sin 2\alpha & 0 \\ -\sin 2\alpha & \cos 2\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \theta = 2\alpha$$

two fold mapping $SU(2) \to O(3)$ $\qquad \alpha = \pi \qquad R_{ij} = 1$

$\qquad \alpha = 2\pi \qquad R_{ij} = 1$

SU(2) Representation

doublet $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ $\psi \to U\psi$ "fundamental"

vector $\pi^i = (\pi^1, \pi^2, \pi^3)$ $\to R^{ij}\pi^j$ "adjoint"

$\bar\psi\psi$ singlet

$\bar\psi\tau^i\psi$ vector

$2 \otimes 2 = 1 \oplus 3$

$\frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1$ ← "spin"

Invariants

$$\mathcal{L} = \bar{\Psi} \tau^i \Psi \pi^i \longrightarrow \underbrace{R^{ij} \bar{\psi} \tau^j \psi R^{ik}}_{\delta_{jk}} \pi^k = \bar{\Psi} \tau^j \Psi \pi^j$$

Matrix notation:

$$\Pi = \tau^i \pi^i \qquad 2\times 2 \text{ matrix}$$

Transform

$$\Pi \to \tau^i R^{ij} \pi^j = \frac{1}{2} \underset{\tau_{ab}}{\tau^i} \, Tr\big(\underbrace{U^+ \tau^i U \tau^j}_{(\tau^i)_{cd} (U \tau^j U^+)_{dc}}\big) \pi^j$$

$$\tau^i_{ab} R^{ij} \pi^j = \frac{1}{2} * 2 \left(\delta_{ad} \delta_{bc} - \frac{1}{2} \delta_{ab} \delta_{cd}\right) (U \tau^j U^+)_{dc} \, \pi^j$$

$$= (U \tau^j \pi^j U^+)_{ab}$$

$$\left[\; \begin{aligned} &Tr(AB) = Tr(BA) \\ &Tr(U^+ \tau^j U) = Tr(\tau^j U U^+) \end{aligned} \;\right]$$

$$\Pi = \tau \cdot \vec{\pi} \rightarrow U \pi U^{+} = U(\vec{\tau} \cdot \vec{\pi}) U^{+}$$

Invariants

$$\bar{\Psi} \vec{\tau} \cdot \pi \Psi \rightarrow (\bar{\Psi} U^{+})(U \tau \cdot \pi U^{+})(U \Psi) = \bar{\Psi} \vec{\tau} \cdot \vec{\pi} \Psi$$

adjoint $(\vec{\tau} \cdot \vec{V}) \rightarrow U(\vec{\tau} \cdot \vec{V}) U^{+}$

Another invariant $\frac{1}{2} \mathrm{Tr}(\vec{\tau} \cdot \vec{\pi}\, \vec{\tau} \cdot \vec{\pi}) = \frac{1}{2} \mathrm{Tr}(\tau^{i} \tau^{j}) \pi^{i} \pi^{j} = \pi^{i} \pi^{i}$

$$\rightarrow \frac{1}{2} \mathrm{Tr}(U \vec{\tau} \cdot \vec{\pi}\, U^{+} U \vec{\tau} \cdot \vec{\pi}\, U^{+}) = \frac{1}{2} \mathrm{Tr}(\vec{\tau} \cdot \vec{\pi}\, \vec{\tau} \cdot \vec{\pi})$$

Calculations

$$\psi = \begin{pmatrix} \psi_p \\ \psi_N \end{pmatrix}$$

$$\vec{\tau} \cdot \vec{\pi} = \tau_3 \pi^3 + \frac{1}{2}\left[(\tau_1 + i\tau_2)(\pi_1 - i\pi_2) + (\tau_1 - i\tau_2)(\pi_1 + i\pi_2)\right]$$

$$\tau_{\pm} = \frac{1}{2}(\tau_1 \pm i\tau_2) \qquad \pi^+ = \frac{1}{\sqrt{2}}(\pi_1 - i\pi_2)$$

$$\pi^- = \frac{1}{\sqrt{2}}(\pi_1 + i\pi)$$

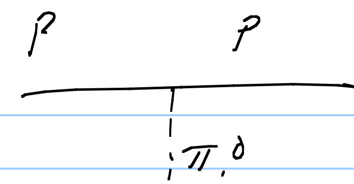
$$\Rightarrow \vec{\tau} \cdot \vec{\pi} = \begin{pmatrix} \pi^3 & \sqrt{2}\pi^+ \\ \sqrt{2}\pi^- & -\pi^3 \end{pmatrix}$$

$$\pi \cdot \vec{\pi} = \pi^3 \pi^3 + 2\pi^+ \pi^-$$

$$\mathcal{L} = ig\,\bar{\psi}\,\vec{\tau}\cdot\vec{\pi}\,\gamma_5\,\psi \xrightarrow{\text{drop } \gamma_5} ig\,(\bar{P}, \bar{N}) \begin{pmatrix} \pi^0 & \sqrt{2}\pi^+ \\ \sqrt{2}\pi^- & -\pi^0 \end{pmatrix} \begin{pmatrix} P \\ N \end{pmatrix}$$

$$= ig(\bar{P}P\pi^0 - \bar{N}N\pi^0) + \sqrt{2}g\bar{P}N\pi^+ + \sqrt{2}g\bar{N}P\pi^-$$

Coupling



$= i g_{pp\pi^0} \gamma_5$

$g_{pp\pi^0} = g$

$g_{nn\pi^0} = -g$

$g_{np\pi^+} = \sqrt{2}\, g$

$g_{pn\pi^-} = \sqrt{2}\, g$

$\downarrow p \qquad \downarrow \pi \qquad \downarrow I = \tfrac{1}{2},\ I_z = +\tfrac{1}{2} \Rightarrow p$

Group Theory

Clebsch Gordon

$\langle \tfrac{1}{2}\,\tfrac{1}{2};\ 1, 0 \ |\ \tfrac{1}{2}\,\tfrac{1}{2} \rangle = -\dfrac{1}{\sqrt{3}}$

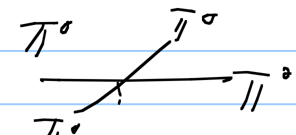
$\langle \tfrac{1}{2}\,-\tfrac{1}{2}\ \ 1\ +1 \ |\ \tfrac{1}{2}\,\tfrac{1}{2} \rangle = -\sqrt{\dfrac{2}{3}}$

$\uparrow N \qquad \uparrow \pi^+$

$\mathcal{M} = \langle\ |\ |\ \rangle\ C^{I\,I'\,I''}_{I_3\,I_3'\,I_3''}$

Also $\mathcal{L} = -\frac{\lambda}{4}(\pi^i\pi^i)^2 = -\frac{\lambda}{4}\left(\pi^{0\,2} + 2\pi^+\pi^-\right)^2 = -\frac{\lambda}{4}\left(\pi^{0\,4} + 4\pi^{0\,2}\pi^+\pi^- + 4\pi^{+2}\pi^{-2}\right)$

$\langle \pi^0\pi^0|\mathcal{L}|\pi^0\pi^0\rangle = -6\lambda$



$= 6i\lambda$

$\langle \pi^0\pi^-|\mathcal{L}|\pi^0\pi^-\rangle = -\frac{\lambda}{4}\times 4\times 2 = -2\lambda$

$\langle \pi^+\pi^-|\mathcal{L}|\pi^+\pi^-\rangle = -4\lambda$

$O(N)$ scalar theory

N real fields ϕ^i $\quad i = 1 \ldots N$

$\phi^i \phi^i$ $\quad O(N)$ invariant $\qquad \phi^i \to R^{ij} \phi^j$

$R^{ij} R^{ik} = \delta^{jk}$

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi^i \partial^\mu \phi^i - V(\phi^i \phi^i)$$

$$V = \pm \frac{\mu^2}{2} \phi^i \phi^i - \frac{\lambda}{4} (\phi^i \phi^i)^4$$

Linear σ Model

Gell Mann + Levy 1960

$$\psi = \begin{pmatrix} \psi_p \\ \psi_N \end{pmatrix}$$

$$(\sigma, \pi^i) \sim (\pi^i, \pi^4) \to (\sigma^2 + \pi^i \pi^i)$$ $O(4)$ invariant

$$V(\sigma^2+\pi^2) = \pm \frac{\mu^2}{2}(\sigma^2+\vec{\pi}^2) - \frac{\lambda}{4}(\sigma^2+\vec{\pi}^2)^2$$

$$\mathcal{L} = \bar{\psi} i \not\partial \psi - g \bar{\psi}(\sigma - i \vec{\tau}\cdot\vec{\pi}\gamma_5)\psi - V(\sigma^2+\vec{\pi}^2)$$

no mass term

Symmetry $SU(2)_L \otimes SU(2)_R$

$\psi_L \to \psi_L' = L\,\psi_L$ $\qquad L = e^{-i\vec{\alpha}_L\cdot\vec{\tau}}$

$\psi_R \to \psi_R' = R\,\psi_R$ $\qquad R = e^{-i\vec{\alpha}_R\cdot\vec{\tau}}$

$$(\sigma - i\,\vec{\tau}\cdot\vec{\pi}) \to \sigma' - i\,\vec{\tau}\cdot\vec{\pi}' = L\,(\sigma - i\,\vec{\tau}\cdot\vec{\pi})\,R^{+}$$

SU(N)

$$\psi = \begin{pmatrix} \psi_1 \\ \vdots \\ \vdots \\ \psi_N \end{pmatrix} \qquad \text{with } \psi \to \psi' = U\psi \qquad \begin{matrix} U^\dagger U = 1 \\ \det U = 1 \end{matrix}$$

$$U = e^{-i\vec{\alpha}\cdot\vec{\lambda}} = e^{-i\alpha^i\lambda^i} \qquad i = 1 \ldots (N^2-1)$$

$$= e^{-i\alpha^A t^A}$$

$\lambda^i = 3\times 3$ Traceless Hermitian

$$\mathrm{Tr}(\lambda^i \lambda^j) = 2\delta^{ij}$$

$$\lambda^i = \begin{pmatrix} \tau^i & 0 \\ 0 & 0 \end{pmatrix} \qquad i = 1, 2, 3$$

$$(\lambda^i)_{ab}(\lambda^i)_{cd} = 2\left(\delta_{ad}\delta_{cb} - \frac{1}{N}\delta_{ab}\delta_{cd}\right)$$

Representations

$\Psi \Rightarrow U\Psi$ fundamental N

$\vec{\lambda}\cdot\vec{V} \to U\,\vec{\lambda}\cdot\vec{V}\,U^{+}$ adjoint N^2-1

$V^{i} \to R^{ij}V^{j}$ with $R^{ij} = \frac{1}{2}\,\mathrm{Tr}\left(U\tau^{i}U^{+}\tau^{j}\right)$

Invariants

$\overline{\Psi}\,\vec{\lambda}\cdot\vec{V}\,\Psi$

$V^{i}V^{i} = \frac{1}{2}\,\mathrm{Tr}(\lambda\cdot V\,\lambda\cdot V)$

$\mathcal{I}_3 = \frac{1}{2}\,\mathrm{Tr}\left((\lambda\cdot V_1)(\lambda\cdot V_2)(\lambda\cdot V_3)\right)$