Gauge Theory 1 9/24

Note Title 9/24/2009

Goals

1) Extend to YM
2) Teach right
3) Survey of major lessons

Nonabelian – Yang Mills

QED $\quad U(1)$
YM $\Rightarrow SU(N)$

Local

$\psi \to U(x)\psi \quad \longrightarrow \quad \psi = \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_N \end{pmatrix}$

$\in N \times N$

$U = e^{-i\alpha(x)\cdot\frac{\lambda}{2}} \leftarrow *$

To form invariant

$\mathcal{L} = \bar{\psi}\, i\not{D}\psi \longrightarrow \bar{\psi}'\, i\not{D}'\psi' = \bar{\psi}\, U^\dagger \not{D}' U \psi$

$\Rightarrow \not{D} \to \not{D}' = U \not{D} U^\dagger \qquad$ or $\quad \not{D}\psi \to U(\not{D}\psi)$

SU(N) gauge Fields

$$\partial_\mu U \sim -i\, \underline{\partial_\mu \vec{\alpha}(x)} \cdot \frac{\vec{\lambda}}{2}$$

$$D_\mu = \partial_\mu + i g A_\mu^i \frac{\lambda^i}{2}$$

Make it work:

$$D_\mu' = U\left[\partial_\mu + i g \frac{\vec{A}\cdot\vec{\lambda}}{2}\right]U^\dagger = \partial_\mu + i g \frac{\vec{A}'\cdot\vec{\lambda}}{2}$$

$$= \partial_\mu + U \partial_\mu U^\dagger + i g\, U \frac{\vec{A}\cdot\vec{\lambda}}{2} U^\dagger$$

$\uparrow$ gauge trans $\quad\quad \leftarrow$ rotations

$\swarrow$ Hermetian & Traceless

$$\frac{\vec{\lambda}\cdot \vec{A}^{\mu\,\prime}}{2} = U \frac{\lambda\cdot A}{2} U^\dagger - \frac{i}{g} U \partial_\mu U^\dagger$$

Small transformations

$$U = \left(1 - i\frac{\vec{\alpha}\cdot\vec{\lambda}}{2}\right)$$

$$\frac{\vec{\lambda}}{2}\cdot\vec{A}_\mu' = \left(1 - i\frac{\alpha\cdot\lambda}{2}\right)\frac{\lambda\cdot A_\mu}{2}\left(1 + i\frac{\alpha\cdot\lambda}{2}\right) - \frac{i}{g}\partial_\mu\left(i\frac{\vec{\alpha}\cdot\lambda}{2}\right)$$

$$= \frac{\lambda\cdot A_\mu}{2} + \frac{i}{4}\underbrace{\left[\lambda\cdot A_\mu, \lambda\cdot\alpha\right]} + \frac{1}{g}\partial_\mu\frac{\vec{\alpha}\cdot\lambda}{2}$$

Use $[\lambda^i, \lambda^j] = 2i f^{ijk}\lambda^k$ ← antisymmetric
— $SU(N)$
$SU(2) = \varepsilon^{ijk}$

$$A'^{i}_\mu = A^i_\mu + \underbrace{\frac{1}{g}\partial_\mu\alpha^i}_{\text{gauge}} + \underbrace{f^{ijk}A^k\alpha^j}_{\text{rotation}}$$

$$A'^{i}_\mu = R^{ij}A^j_\mu - \frac{i}{g}\mathrm{Tr}\left(\lambda^i u\,\partial_\mu u^\dagger\right) \qquad R^{ij} = \frac{1}{2}\mathrm{Tr}\left(u\lambda^i u^\dagger\lambda^j\right)$$

To show Hermetian & Traceless

1) Hermitian $\quad (i U \partial_\mu U^+)^\dagger = -i(\partial_\mu U) U^+ = +i U \partial_\mu U^+$

$U^+ U = 1 \Rightarrow \partial_\mu (U^+ U) = 0 \Rightarrow (\partial_\mu U^+) U = -U^+ \partial_\mu U$

$U U^+ = 1 \qquad\qquad\qquad\qquad \Rightarrow (\partial_\mu U) U^+ = -\cancel{U} \partial_\mu U^+$

2) Traceless - no elegant proof

$$U \partial_\mu U^+ = \left(1 - i\frac{\alpha \cdot \lambda}{2} + \cdots\right) \partial_\mu \left(1 + i\frac{\alpha \cdot \lambda}{2} + \cdots\right)$$

$$= \lambda^i \cdot (i \partial \alpha + \cdots)$$

To get $F^i_{\mu\nu}$

$$[D_\mu, D_\nu] \longrightarrow [U D_\mu U^\dagger, U D_\nu U^\dagger] = U [D_\mu, D_\nu] U^\dagger$$

$$\equiv ig \frac{\vec{\lambda}}{2} \cdot \vec{F}_{\mu\nu}$$

$$[D_\mu, D_\nu] = ig\left[\partial_\mu \frac{\vec{\lambda}\cdot\vec{A}_\nu}{2} - \partial_\nu \frac{\vec{\lambda}\cdot\vec{A}_\mu}{2}\right] + \left(\frac{ig}{2}\right)^2 \underbrace{[\vec{\lambda}\cdot\vec{A}_\mu, \vec{\lambda}\cdot\vec{A}_\nu]}_{2if^{ijk}\lambda^k A^i_\mu A^j_\nu}$$

Then

$$F^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu + g f^{ijk} A^j_\mu A^k_\nu$$

$$\vec{\lambda}\cdot\vec{F}_{\mu\nu} = \partial_\mu \vec{\lambda}\cdot\vec{A}_\nu - \partial_\nu \vec{\lambda}\cdot\vec{A}_\nu + \frac{g}{2} [\vec{\lambda}\cdot\vec{A}, \vec{\lambda}\cdot\vec{A}]$$

$$\vec{\lambda}\cdot\vec{F}_{\mu\nu} \rightarrow U\,\vec{\lambda}\cdot\vec{F}_{\mu\nu}\,U^{\dagger}$$

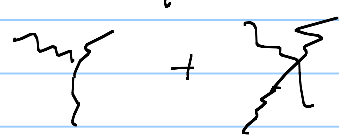
Invariant

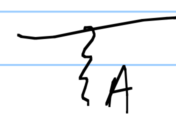
$$\frac{1}{2}\mathrm{Tr}\left(\vec{\lambda}\cdot\vec{F}_{\mu\nu}\;\vec{\lambda}\cdot\vec{F}^{\mu\nu}\right) \longrightarrow \frac{1}{2}\mathrm{Tr}\left(U\lambda\cdot F\,U^{\dagger}\,U\lambda\cdot F\,U^{\dagger}\right)$$

$$= F^{i}_{\mu\nu}\,F^{i\mu\nu}$$

Yang Mills Lagrangian

$$\mathcal{L} = -\frac{1}{4}F^{i}_{\mu\nu}\,F^{i\mu\nu} + \bar{\psi}(\not{D} - m)\psi$$





P.I. - Quantization

1) The propogator does not exist

$$\Box A_\mu - \partial_\mu \partial_\nu A^\nu = J_\mu$$

Like

$$(\Box g^{\mu\nu} - \partial^\mu \partial^\nu) D_{F\mu\lambda}(x-x') = g^{\nu}{}_{\lambda}\, \delta^4(x-x')$$

F.T.

$$(-k^2 g^{\mu\nu} + k^\mu k^\nu)\, G_{\mu\lambda}(k) = g^{\nu}{}_{\lambda}$$

$$\uparrow\; a(k^2)\, g_{\mu\lambda} + b(k^2)\, k_\mu k_\lambda$$

$$\Rightarrow \quad a(k^2)\left(-k^2 g^{\nu}{}_{\lambda} + k^\nu k_\lambda\right) = g^{\nu}{}_{\lambda} \qquad \text{impossible}$$

$$(\Box g^{\mu\nu} - \partial^\mu \partial^\nu) \int \frac{d^4k}{(2\pi)^4} G_{\nu\lambda}(k^2)\, e^{-ik\cdot(x-x')} = g^\nu{}_\lambda \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x-x')}$$

<u>Why important?</u>

$$Z[\,] = \int [d\phi]\, e^{i\int \phi\, \mathcal{O}\, \phi + J\phi\rangle} = \frac{1}{\sqrt{\det \mathcal{O}}}\, e^{-\langle J \mathcal{O}^{-1} J\rangle}$$

with QED $\Rightarrow$ no inverse, blows up

<u>Diagnosis</u> $\int dA_\mu$

- ∞ # of gauge copies in PI
- doesn't factor here

Simple example of method

2 d integral

$$Z = \int dx\, dy\, e^{iS(x,y)}$$

$\swarrow (x,y) \to (r,\phi)$

with symmetry $S(x,y) = S(r)$

ϕ does not appear

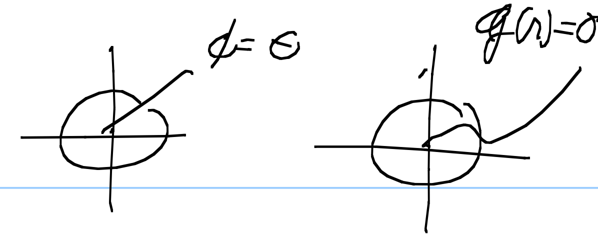
Fancy method

$$1 = \int d\theta\, \delta(\theta - \phi)$$

$$Z = \int r\,dr\, d\phi \int d\theta\, \delta(\theta-\phi)\, e^{iS(r)} = \int d\theta\, \underbrace{\int d^2r\, \delta(\theta-\phi)\, e^{iS}}_{Z_\theta}$$

$$= \int \theta\, Z_\theta = 2\pi Z_\theta$$

Z_θ independent of θ



<u>More realistic</u>

$$g(\vec{r}) = 0$$

if $\quad r_\theta = (r, \phi + \theta)$

$$1 = \int d\theta \left(\frac{\partial g}{\partial \theta}\right)_{g=0} \delta(g(r_\theta)) \qquad \text{or} \qquad \int d\theta\, \delta(g(r_\theta)) = \frac{1}{\left|\frac{\partial g}{\partial \theta}\right|_{g=0}}$$

$$Z = \int d^2 r \left[\int d\theta \left(\frac{\partial g}{\partial \theta}\right) \delta(g(r_\theta))\right] e^{iS(r)}$$

$$= \int d\theta\, Z_\theta = 2\pi Z_\theta$$

with $$Z_\theta = \int d^2 r \left(\frac{\partial g}{\partial \theta}\right) \delta(g(r_\theta))\, e^{iS(r)}$$

Gauge Theory version

$f(A_\mu) = 0$ ← at each point in space time

Identity

Gauge trans $A_\mu \to A_\mu^\theta = A_\mu + \partial_\mu \theta$

$$1 = \int [d\theta(x)] \, \Delta(A) \, \delta(f(A_\mu^\theta))$$

where $\Delta(A) = \det\left(\frac{\partial f}{\partial \theta}\right)\Big|_{f=0}$

example $f(A_\mu) = \partial_\mu A^\mu$

This does factor out the gauge copies

$$Z = \int [dA][d\theta]\ \Delta(A)\ \delta(f(A^{\theta}))\ e^{i\int d^4x\, \mathcal{L}}$$

if $\Delta(A)$ independent of θ (soon) $\Leftarrow$

$\Rightarrow$ factor out $[d\theta]$ (gauge trans)

$\int d\theta(x) = N$ $\Rightarrow$ only constrained A_μ in PI

$$Z_f[J] = N \int [dA]\ \Delta(A)\ \delta(f(A))\ e^{i\int d^4x[\mathcal{L} + J_\mu A^\mu]}$$

$\Rightarrow$ get inverse

Check that $\Delta(A)$ is gauge invariant

$$\int [d\theta] \; \delta(f(A^{\theta})) = \Delta^{-1}(A)$$

$$\Delta^{-1}(A^{\theta_0}) = \int [d\theta] \; \delta(f(A^{\theta\theta_0}))$$

let $\theta\theta_0 = \theta'$

$\int [d\theta] = \int [d\theta']$

$$= \int [d\theta'] \; \delta(f(A^{\theta'}))$$

$$= \Delta^{-1}(A)$$