Gauge Theory 2 9/29 QFT II

Path Integral Quantization

$$Z = \int dA \int d\theta \; \Delta(A^\theta) \, \delta(f(A^\theta)) \qquad e^{iS}$$

$$\Delta(A) = \det\left(\frac{\partial f}{\partial \theta}\right) \leftarrow \text{jacobean}$$

__Why the det?__

2 variables & 2 constraints

$$1 = \int du_1 \, du_2 \, \Delta \; \delta(f_1(u_1, u_2)) \, \delta f_2(u_1, u_2) \qquad \text{What is } \Delta?$$

$$\int dx_1 \, dx_2 \, \delta(x_1) \, \delta(x_2) = \int du_1 \, du_2 \, \det\left|\frac{\partial f_i}{\partial u_j}\right| \delta(f_1(u_1, u_2)) \, \delta(f_2(u_1, u_2))$$

Change to $x_1 = f_1(u_1, u_2)$
$x_2 = f_2(u_1, u_2)$ $\Bigg\}$ $\int dx_1 dx_2 \, F(x_1, x_2) = \int du_1 du_2 \, F(f_1, f_2) \, J$

$$J = \det\left(\frac{\partial f_i}{\partial u_j}\right)$$

Then Δ independent of θ

$\Rightarrow$ factor out $\int [d\theta] = \infty$

Result

$$Z = \int [dA] \, \underline{\det\left(\frac{\partial f}{\partial \theta}\right)} \, \underline{\delta(f(A))} \, e^{i\int d^4x\left[\mathcal{L} + J_\mu A^\mu\right]}$$

QED

$$f(A) = \partial_\mu A^\mu - F \qquad\qquad A_\mu \to A_\mu + \partial_\mu \theta$$

$$f(A^\theta) = \partial_\mu A^\mu - \Box\theta - F$$

$$\frac{\partial f}{\partial \theta} = \Box$$

$$\Delta = \det \frac{\partial f}{\partial \theta} = \det \Box = \text{constant} \quad \text{no } A \text{ dependency}$$

$$Z[J] = \int dA \, \delta(\partial^\mu A_\mu - F) \, e^{i\int d^4x \left[\frac{1}{2} A_\mu (\Box g^{\mu\nu} - \partial^\mu \partial^\nu) A_\nu + J_\mu A^\mu \right]}$$

Then usual trick $A'_\mu(x) = A_\mu(x) + \underline{\int d^4y \, D_F^{\mu\nu}(x-y) J_\nu(y)}$

$$Z = Z[0] \, e^{-\frac{i}{2}\int d^4x \, d^4y \, J D J}$$

Condition $\partial^\mu A'_\mu - F = 0$ ✓

Check $D_F^{\mu\nu} = \int \frac{d^4k}{(2\pi)^4} \frac{-i g^{\mu\nu}}{k^2 + i\epsilon} e^{ik\cdot(x-y)}$ ← int by parts

$$\int d^4y\, \partial_x^\mu D_F^{\mu\nu}(x-y) J^\nu(y) = \int d^4y \left[\partial_y^\mu D_F^{\mu\nu}(x-y)\right] J^\nu(y)$$

$$= \int d^4y\, D_F(x-y)\, \underbrace{\partial_\mu J^\nu(y)}_{0} = 0 \checkmark$$

$$\partial_\mu A'^\mu = \partial_\mu A^\mu \quad \checkmark$$

 $= \dfrac{-i g^{\mu\nu}}{k^2 + i\epsilon}$

Covariant gauges

inserted $\delta(\partial_\mu A^\mu - F)$ $\Longrightarrow$ $\delta(\partial_\mu A^\mu(x) - F(x))$

$A^\mu(x)$

Covariant trick

$$\int [dF(x)]\ \delta(\partial^\mu A_\mu(x) - F(x))\ e^{-\frac{i}{2\xi}\int d^4x\, F^2(x)} = e^{-\frac{i}{2\xi}\int d^4x\, (\partial_\mu A^\mu)^2}$$

$$Z[J_\mu] = \int [dA_\mu] = e^{i\int d^4x \left[\frac{1}{2} A_\mu(\Box g^{\mu\nu} - \partial^\mu\partial^\nu)A_\nu - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 + J_\mu A^\mu\right]}$$

$-A_\mu \partial^\mu \partial^\nu A_\nu$

$$D_{F\mu\nu} = \frac{i}{k^2+i\varepsilon}\left[-g^{\mu\nu} + (1-\xi)\frac{k^\mu k^\nu}{k^2+i\varepsilon}\right]$$

$\xi = 1$ Feynman gauge

$\xi \to \infty$ blows up

ξ drops out at end

R_ξ gauges

Fadeev-Popov trick

$$\det M = e^{\mathrm{Tr} \ln M} = e^{\int d^4x \langle x | \ln M | x \rangle}$$

Recall

$$\int d\Psi d\bar{\Psi}\, e^{i\int d^4x\, \bar{\Psi} \mathcal{O} \Psi} = N \det \mathcal{O}$$

Reverse this

$$\det M = \int dc\, d\bar{c}\, e^{i\int d^4x\, \bar{c} M c}$$

$c, \bar{c}$ "ghosts"
- not in external states
- propogating in loops
- fermions

$$Z[J] = \int [dA\, dc\, d\bar{c}]\, \delta(f(A))\, e^{i\int d^4x [\mathcal{L} + \bar{c}' \frac{\partial f}{\partial \theta} c}$$

in QED $\frac{\partial f}{\partial G} = \Box \quad \Rightarrow \quad \mathcal{L}_{ghost} = \bar{c} \Box c = -\partial_\mu \bar{c}\, \partial^\mu c$

Yang Mills

Small gauge trans $A^{a\prime}_\mu = A^a_\mu + \frac{1}{g}\partial_\mu \alpha^a - f^{abc} A^b_\mu \alpha^c$

Notation $\alpha^a \to \theta^a$

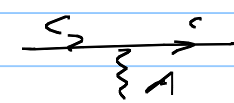
$$f(A^a_\mu) = \partial^\mu A_\mu - F$$

$$f(A^\theta_\mu) = \partial^\mu A^a_\mu + \frac{1}{g}\partial_\mu(\partial^\mu \theta^a - f^{abc} A^b \theta^c) - F$$

$$\frac{\partial f}{\delta \theta^a} \quad \frac{1}{g}\left(\delta^{ab}\Box - g f^{a\cancel{cb}}\, \partial^\mu A^b_\mu\right)$$

rescale $\frac{1}{g}\bar c c \to \bar c c$

$$\mathcal{L}_{ghost} = \bar c^a\left[\delta^{ab}\Box + g f^{abc}\partial^\mu A^c_\mu\right] c^b$$



Physics

Proposed by Feynman quantum gravity 1961

- $h_{\mu\nu}$ 2 D.O.F.

  10 components

  - too many DOF in loops
  - extra minus sign ghosts cancel unphysical DOF

    ↑ fermion loop (-1)

$$\frac{1}{[\det \Theta]^{\#}} \quad [\det \Theta]$$

- de Witt more formal
- Fadeev Popov gauge theories

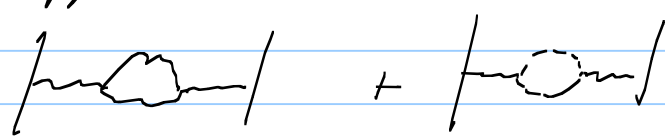
Ghosts

c^a $\quad a = 1 - \cdots (N^2-1)$ $\quad$ adjoint

Free $\quad \bar{c}^a \Box c^a \quad = \frac{i}{p^2+i\varepsilon}\,\delta^{ij}$

not $i\not{\partial}$

Application



Final YM Lagrangian

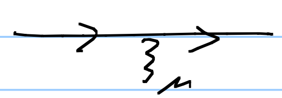
$$\mathcal{L} = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^{a\mu})^2 + \bar{\psi}(i\not{D}-m)\psi - \partial_\mu \bar{c}\,\partial^\mu c - g f^{abc}(\partial^\mu \bar{c}^a) A^b_\mu c^c$$

acts on outgoing field

Feynman rules for YM

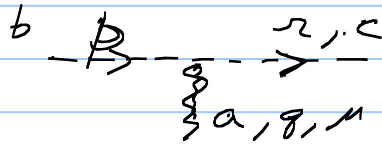
$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g f^{abc} A^b_\mu A^c_\nu$$

$$F^2 = F_0^2 - 2g f^{abc} \underbrace{F_0}_{\partial_\mu A_\nu} AA + g^2 f^{abc} f^{acd} AAAA$$



$$= -ig\,(\gamma_\mu)_{\alpha\beta}\left(\frac{\lambda^a}{2}\right)_{ij}$$

$$= -g f^{abc}\left[g_{\mu\nu}(p+q)_\lambda - g_{\nu\lambda}(r+q)_\mu + g_{\lambda\mu}(r-p)_\nu\right]$$



$$= g f^{abc} r_\mu$$

$$= -ig^2 f_{abd} f_{cde}\left(g_{\mu\lambda} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\lambda}\right) + \text{perm.}$$