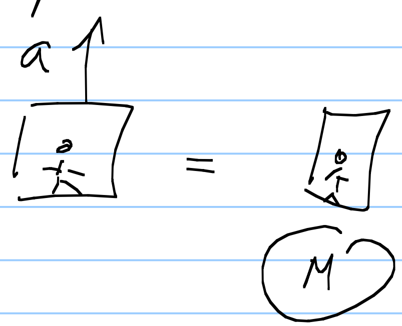
Gravity as a Gauge Theory

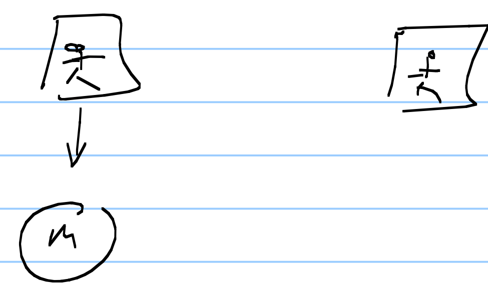
Note Title 12/8/2009

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu$$

$$\hookrightarrow \Lambda^\mu{}_\nu(x) \implies \text{gravity}$$

<u>Equiv princ.</u>





Lorentz Trans on Fermions

$$x^{\mu\prime} = \Lambda^\mu{}_\nu x^\nu$$

$$\psi'(x') = S(\Lambda)\,\psi$$

$\uparrow$ 4×4 matrix

$$\mathcal{L} \to \mathcal{L}'(x') = \bar{\psi}'(x')\left(i\gamma^\mu \partial'_\mu - m\right)\psi'(x')$$

$$= \bar{\psi}\,\gamma_0 S^\dagger \gamma_0 \left(i\gamma^\mu \partial'_\mu - m\right) S\psi$$

$$\gamma^0 S^\dagger \gamma^0 = S^{-1}$$

$$S^{-1}\gamma^\mu \partial'_\mu S = \gamma^\mu \partial_\mu$$

or $\quad S^{-1}\gamma^\mu \Lambda_\mu{}^\nu S = \gamma^\nu \quad$ **

$$\Lambda^{\mu}{}_{\nu} = \underset{\uparrow\, \delta^{\mu}_{\nu}}{g^{\mu}{}_{\nu}} + \omega^{\mu}{}_{\nu}$$ ← infinitessimal ← small trans

$$\Lambda^{\mu}{}_{\nu} \Lambda^{\nu}{}_{\sigma} = \delta^{\mu}_{\sigma} \implies \omega^{\mu\nu} = -\omega^{\nu\mu}$$

$$S = 1 - \frac{i}{4} \sigma^{\mu\nu} \omega_{\mu\nu} \qquad \sigma^{\mu\nu} = \frac{i}{2} [\gamma_\mu, \gamma_\nu]$$

Finite trans

$$S \Rightarrow \lim_{N \to \infty} \left(1 - \frac{i}{4} \frac{\sigma^{\mu\nu} \omega_{\mu\nu}}{N}\right)^N = e^{-\frac{i}{4} \sigma^{\mu\nu} \omega_{\mu\nu}}$$

Local Lorentz Transformations

$$x'^\mu = \underbrace{\Lambda^\mu{}_\nu(x)}\, x^\nu$$

$$\phi(x) \to \phi(x')$$

$$S = \int d^4x \, \frac{1}{2} \left\{ g^{\mu\nu} \partial_\mu \phi \, \partial_\nu \phi - m^2 \phi^2 \right]$$

$$\to \int d^4x \left| \frac{\partial x}{\partial x'} \right| \left\{ g^{\mu\nu} \frac{\partial x'^\alpha}{\partial x^\mu} \frac{\partial x'^\beta}{\partial x^\nu} \partial'_\alpha \phi \, \partial'_\beta \phi - m^2 \phi^2 \right\}$$

↑ jacobean

$\Rightarrow g_{\mu\nu} \to g_{\mu\nu}(x)$ with $g^{\mu\nu}(x) \to \frac{\partial x_\alpha}{\partial x^\mu} \frac{\partial x'_\beta}{\partial x^\nu} g^{\alpha\beta}(x')$

$$g = -\det g_{\mu\nu} \Rightarrow g \to \left|\frac{\partial x'}{\partial x}\right|^2 g(x')$$

Volume $\int d^4x \sqrt{g} \to \int d^4x' \sqrt{g(x')}$

Invariant action

$$S = \int d^4x \sqrt{g(x)} \left\{ g^{\mu\nu}(x)\, \partial_\mu \phi\, \partial_\nu \phi - m^2 \phi^2 \right\}$$

Vierbein $V_\mu^{\ a}$

$\rightarrow$ Princ of Equiv $\quad g_{\mu\nu}(x) = \dfrac{\partial \xi^a}{\partial x^\mu} \dfrac{\partial \xi^b}{\partial x^\nu} \eta_{ab}$ Flat

$$\Rightarrow \quad \underline{\underline{V_\mu^{\ a}(x) = \frac{\partial \xi^a}{\partial x^\mu}}}$$

$a, b \ldots$ flat coordinates
$\mu\nu$

with $\quad V_\mu^{\ a}(x) \rightarrow V'^{\ a}_\mu(x') = \dfrac{\partial x'^\nu}{\partial x^\mu} V_\nu^{\ a}(x)$

$$\sqrt{g} = \det[V_\mu^{\ a}]$$

Covariant deriv for scalars $\quad D_a = V_a^{\ \mu} \partial_\mu$

$$\Rightarrow \quad \mathcal{L} = \frac{1}{2}\left[\eta^{ab} D_a \phi\, D_b \phi - m^2 \phi^2\right]$$

Fermions

Inf. $x'^{\mu} = x^{\mu} + a^{\mu} = \Lambda^{\mu}{}_{\nu} x_{\nu} = (\delta^{\mu}{}_{\nu} + \omega^{\mu}{}_{\nu}) x^{\nu}$

$a^{\mu} = \omega^{\mu}{}_{\nu} x^{\nu}$

$S^{ab} = \frac{-i}{2} \sigma^{ab}$

$\psi'(x') = S(\Lambda)\psi = \left(1 + \frac{1}{2}\omega^{ab}(x) S_{ab}\right)\psi$

$\mathcal{L} \to \mathcal{L}' = \bar{\psi}'(i\gamma^{a} \underbrace{V'^{\mu}_{a}}_{} \partial'_{\mu} - m)\psi'$

invariant with $V'^{\mu}_{a} = \frac{\partial x^{\mu}}{\partial x'^{\nu}} V^{\nu}_{a}$

Spin Transformation

$$\Psi' = \left(1 + \frac{1}{2} \omega^{ab}_{(x)} S_{ab}\right) \Psi = S \Psi$$

$$\mathcal{L}' = \bar{\Psi} S^{-1} \left[i \gamma^a V_a^{\mu} \partial_\mu - m \right] S \Psi$$

$$\partial_\mu \omega_\mu^{ab} S_{ab}$$

Add gauge field A_μ^{ab}

matrix

$$D_a = V_a^{\mu} \left(\partial_\mu + \frac{1}{2} S_{cd} A_\mu^{cd} \right) = V_a^{\mu} \left(\partial_\mu + A_\mu \right)$$

$$\mathcal{L} = \bar{\Psi}(\gamma^a D_a - m)\Psi$$

Transformations

$$V_a^{\mu\,\prime} = V_a^\mu + \omega_a{}^b V_b^\mu$$

$$A'_\mu = A_\mu + \frac{1}{2}\omega^{ab}[S_{ab}, A_\mu] - \frac{1}{2} S_{ab}\,\partial_\mu \omega^{ab}$$

$$\mathcal{L}' = \mathcal{L}$$

Action for A_μ^{ab}

Yang Mills $\quad D_\mu = \partial_\mu - igA_\mu \qquad A_\mu = \frac{\tau^i A^i}{2}$

$$[D_\mu, D_\nu] = -igF^i_{\mu\nu}\,\tau^i$$

$$\Rightarrow F^i_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + g\,\underbrace{\epsilon^{ijk} A^j_\mu A^k_\nu}_{[A,A]}$$

Repeat for A_μ^{ab}

$$D_\mu = \partial_\mu + A_\mu \qquad A_\mu = \frac{1}{2} S_{ab} A_\mu^{ab}$$

$$[D_\mu, D_\nu] = R_{\mu\nu}^{\ ab} \frac{S_{ab}}{2}$$

$$R^{ab}_{\ \ \mu\nu} = \partial_\mu A_\nu^{ab} - \partial_\nu A_\mu^{ab} + A^a_{\ c\mu} A_\nu^{cb} - A^a_{\ c\nu} A_\mu^{cb}$$

Then

$$YM = \mathcal{L} = -\frac{1}{4} F^i_{\mu\nu} F^{i\mu\nu}$$

Gravity $\quad R = V_a^{\ \mu} V_b^{\ \nu} R^{ab}_{\ \ \mu\nu}$ $\quad$ invariant

$$\mathcal{L}_G = \text{const}\, R = \frac{1}{8\pi G} R \qquad + \underbrace{R^{ab}_{\ \ \mu\nu} R_{ab}^{\ \ \mu\nu} + R^2}_{\text{also}}$$

Einstein Eq

2 fields $V_a^{\mu}(x)$; $A_\mu^{ab}(x)$ ← spin connection

$$S = \int d^4x \left[-\frac{1}{8\pi G} R + \frac{1}{2}\left[\eta^{ab} D_a \phi D_b \phi - m^2 \phi^2\right]\right]$$

R ← V, A, A^2

D_a ← V

Vary w.r.t A_μ^{ab}

$$A_\mu^{ab} = g^{\sigma\sigma} V_\varphi^a \left[\partial_\mu V_\sigma^b \right]$$ ← ties fields together

g ← VV

Modify it

$\rightarrow \Gamma^{\mu\nu}{}_\lambda = V_a^\mu V_b^\nu A_\lambda^{ab}$ ←

↖ affine connection

Vary V_a^{μ} , work $ab \to \mu, \nu$

$2 \frac{\partial \mathcal{L}_s}{\delta g_{\mu\nu}}$

$$\Rightarrow \quad R_{\mu\nu} - g_{\mu\nu} R = 8\pi G \, T_{\mu\nu}$$ ← Einstein

$$R_{\mu\nu} = g_{\mu\sigma} V_a^{\lambda} V_b^{\sigma} R^{ab}{}_{\lambda\nu}$$