Introducing the Fields 4

Note Title 2/9/2010

Dirac Field

$\psi_\alpha(x)$

α ← 4 components

$$(i\,\gamma_\mu \partial^\mu - m)\psi = 0$$

↑ 4×4 Hermitian matrices

$$\{\gamma_\mu, \gamma_\nu\} = 2 g_{\mu\nu}$$

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

2×2 identity

$$(\gamma^0)^2 = 1$$

$$\gamma_\mu \partial^\mu = \not\partial$$

$$\not A = \gamma_\mu A^\mu$$

$$(i\not\partial - m)\psi$$

$$\bar\psi = \psi^\dagger \gamma_0 = (\psi_1^*, \psi_2^*, \psi_3^*, \psi_4^*)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\mathcal{L} = \bar{\psi}\left[i\not{\partial} - m\right]\psi$$

or $\frac{i}{2}(\overrightarrow{\not{\partial}} - \overleftarrow{\not{\partial}}) = \frac{i}{2}\overleftrightarrow{\not{\partial}}$

$$\partial^\mu\left[\frac{\partial\mathcal{L}}{\partial\partial_\mu\bar{\psi}}\right] - \frac{\partial\mathcal{L}}{\partial\bar{\psi}} = 0 \Rightarrow (i\not{\partial} - m)\psi = 0$$

$$\partial^\mu\left[\frac{\partial\mathcal{L}}{\partial\partial_\mu\psi}\right] - \frac{\partial\mathcal{L}}{\partial\psi} = 0 \qquad \bar{\psi}(-i\overleftarrow{\not{\partial}} - m) = 0$$

Hamiltonian

$$\bar{\pi} = \frac{\partial\mathcal{L}}{\partial\partial_0\bar{\psi}} = -\frac{i}{2}\gamma_0\psi \quad , \quad \pi = \frac{\partial\mathcal{L}}{\partial\partial_0\psi} = +\frac{i}{2}\bar{\psi}\gamma_0$$

$\vec{\alpha} = \gamma_0\vec{\gamma}$ $\quad$ $\gamma_0 = \beta$

$$\mathcal{H} = \pi\partial_0\psi + \partial_0\bar{\psi}\bar{\pi} - \mathcal{L}$$

$$= \bar{\psi}(-i\vec{\gamma}\cdot\vec{\nabla} + m)\psi = \psi^\dagger(-\vec{\alpha}\cdot\vec{\nabla} + \beta m)\psi$$

Solution

$$\psi(x) = e^{-ip\cdot x}\, u(p)$$

$$(i\not{\partial} - m)\psi = 0 \implies (\not{p} - m)\, u(p) = 0$$

$$u(p) = \sqrt{\frac{E+m}{2E}} \begin{pmatrix} \chi \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\chi \end{pmatrix} \qquad \chi = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$u^\dagger(p)\, u(p) = 1$$

$$\bar{u}(p)\, u(p) = \frac{m}{E}$$

$$p^2 = m^2 \implies E^2 - \vec{p}^{\,2} = m^2$$

$$\psi(x) = e^{+ip\cdot x}\, v(p) \implies (\not{p} + m)\, v(p) = 0$$

$$v(p) = \sqrt{\frac{E+m}{2E}} \begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\chi \\ \chi \end{pmatrix}$$

$$v^\dagger v = 1 \quad , \quad \bar{v} v = \frac{m}{E}$$

Useful relation

χ_m $\left[\text{use } \sum_m \chi_{m\,a} \chi^\dagger_{m\,b} = \begin{pmatrix}1\\0\end{pmatrix}_a (1,0)_b + \begin{pmatrix}0\\1\end{pmatrix}_a (0,1)_b \right.$

$\left. = \delta_{ab} = (\mathbb{1})_{ab} \right]$

$$\sum_m u_\alpha(p,m)\,\bar{u}_\beta(p,m)$$

$$= \sum_m \left(\frac{E+m}{2E}\right) \begin{pmatrix} \chi_m \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\chi_m \end{pmatrix}_\alpha \left(\chi_m^\dagger , -\chi_m^\dagger \frac{\vec{\sigma}\cdot\vec{p}}{E+m} \right)_\beta$$

$$= \frac{E+m}{2E} \begin{pmatrix} \mathbb{1} & -\frac{\vec{\sigma}\cdot\vec{p}}{E+m} \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m} & -\frac{(\vec{\sigma}\cdot\vec{p})^2}{(E+m)^2} \end{pmatrix}_{\alpha\beta} = \frac{E+m}{2E}\,\frac{1}{(E+m)} \begin{bmatrix} (E+m) & -\vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & -\frac{\vec{p}^{\,2}}{E+m} \end{bmatrix}_{\alpha\beta}$$

$$-\frac{(E^2-m^2)}{E+m} = -(E-m)$$

$$= \frac{1}{2E}\left(\not{p} + m\right)_{\alpha\beta}$$

$$\sum_m v_\alpha(p)\,\bar{v}_\beta(p) = \frac{1}{2E}\left(\not{p} - m\right)$$

Quantization

$$\psi(x) = \sum_m \int \frac{d^3p}{(2\pi)^3} N_p \left\{ e^{-ip\cdot x} u(p,m) b(p,m) + e^{+ip\cdot x} v(p,m) d^\dagger(p,m) \right\}$$

Logic 1) Try $\boxed{\underbrace{\{\psi_\alpha(x), \psi^\dagger_\beta(x')\}}_{\equiv 1} = \delta^3(x-x')} \Rightarrow H = \ldots$

2) Invert $b(p,m) = \int d^3x \, e^{ip\cdot x} u^\dagger(p,m) \psi(x)$

$$\boxed{\{b(p,m), b^\dagger(p',m')\} = \delta_{mm'} (2\pi)^3 \delta^3(\vec{p}-\vec{p}')}$$

$$\Rightarrow \{\psi_\alpha, \psi^\dagger_\beta\} = \delta^3(x-x')$$

$$\boxed{\{d(p,m), d^\dagger(p',m') = \delta_{mm'} (2\pi)^3 \delta^3(p-p')}$$

$\Rightarrow N_p = 1$

Hamiltonian

$$H = \int d^3x \; \bar{\psi} \left(-i \vec{\gamma} \cdot \vec{\nabla} + m \right) \psi$$

$\leftarrow b + d^{+}$

? algebra

$$= \sum_{m} \int \frac{d^3p}{(2\pi)^3} \; \omega_p \left[b^{+}(p,m)\, b(p,m) - \underbrace{d(p,m)\, d^{+}(p,m)}_{} \right]$$

$$\{ d(p,m), d^{+}(p',m') \} = \delta^3(p-p')\, \delta_{mm'} (2\pi)^3$$

$$d\, d^{+} = -d^{+} d \;\; + \underbrace{(2\pi)^3 \delta^3(0)}_{V}$$

$$H = \sum_{m} \frac{d^3p}{(2\pi)^3} \; \omega_p \left[b^{+}(p,m)\, b(p,m) + d^{+}(p,m)\, d(p,m) \right] + E_{0F}$$

$$E_{0F} = -\overset{\text{2 spins}}{2}\, V \int \frac{d^3p}{(2\pi)^3} \; \omega_p \quad < 0$$

Conserved Current (EM current)

$$J_\mu = \bar{\psi}\gamma_\mu\psi \quad = (u^+u,\ \bar{u}^+\vec{\gamma}u) = (1, \tfrac{\vec{p}}{E})$$

$\vec{v}$

Conserved

$$i\partial_\mu J^\mu = i\partial_\mu(\bar{\psi}\gamma^\mu\psi) = i(\bar{\psi}\overrightarrow{\not{\partial}}\psi + \bar{\psi}\overleftarrow{\not{\partial}}\psi) = (\bar{\psi}m\psi - \bar{\psi}m\psi) = 0$$

$$= 0$$

Charge

$$Q = \int d^3x\, \bar{\psi}\gamma_0\psi = \int d^3x\, \psi^+_{(x)}\psi_{(x)}$$

$$\frac{dQ}{dt} = \int d^3x\, \partial_0(\bar{\psi}\gamma_0\psi) = \underbrace{\int d^3x\, -\vec{\nabla}\cdot(\bar{\psi}\vec{\gamma}\psi)}_{\text{total derivative}} = 0$$

$$Q = \sum_m \int \frac{d^3p}{(2\pi)^3}\left[b^+(p,m)\,b(p,m) + d(p,m)\,d^+(p,m)\right]$$

$$= \sum_m \int \frac{d^3p}{(2\pi)^3} \left[b^+ b + d^+(p,m)\, d(p,m) \right] \underbrace{+ Q_0}_{\substack{\text{drop} \\ \text{equal } \# (\pm)}}$$

Interpretation

$$|f(p,m)\rangle = b^+(p,m)\,|0\rangle$$

$$|\bar{f}(p,m)\rangle = d^+(p,m)\,|0\rangle$$

$$H\,|f(p,m)\rangle = \omega_p\,|f(p,m)\rangle$$

$$Q\,|f(p,m)\rangle = +1\,|f(p,m)\rangle$$

$$H\,|\bar{f}(p,m)\rangle = \omega_p\,|\bar{f}(p,m)\rangle$$

$$Q\,|\bar{f}(p,m)\rangle = -1\,|\bar{f}(p,m)\rangle$$

All particles energies > 0

Q = "particle #"

$Q = +1$ = particle

$Q = -1$ = antiparticle

Feynman propagator

$$i S_F(x-x')_{\alpha\beta} \equiv \langle 0 | T\, \psi_\alpha(x,t)\ \bar\psi_\beta(x',t')\ | 0 \rangle \qquad (4\times 4 \text{ matrix})$$

$$T\, \psi_\alpha(x)\bar\psi_\beta(x') = \Theta(x_0-x_0')\,\psi_\alpha(x)\,\bar\psi_\beta(x') - \Theta(x_0'-x_0)\,\bar\psi_\beta(x')\,\psi_\alpha(x)$$

$S_F(x-x')$ is Green Function

$$(i\not\partial_x - m)_{\alpha'\alpha}\, i S_F(x-x')_{\alpha\beta} = \langle 0 | T\,(i\not\partial - m)_{\alpha'\alpha}\,\psi_\alpha\ \bar\psi_\beta\ | 0 \rangle$$

$$+\, i(\gamma_0)_{\alpha'\alpha}\langle 0 | \delta(x_0-x_0')\,\psi_\alpha\bar\psi_\beta + \delta(x_0'-x_0)\,\bar\psi_\beta\,\psi_\alpha\ | 0 \rangle$$

$$= i\,\delta^4(x-x')\,\delta_{\alpha\beta} \qquad \gamma_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$(i\not\partial - m) S_F = i\,\delta^4(x-x')\,\mathbb{1}$$

Solution

$$S_F(x) = \int \frac{d^4p}{(2\pi)^4} \frac{\not{p}+m}{p^2-m^2+i\varepsilon} e^{-ip\cdot x} \equiv \int \frac{d^4p}{(2\pi)^4} \left(\frac{1}{\not{p}-m+i\varepsilon}\right) e^{-i\vec{p}\,\vec{x}}$$

$$(\not{p}-m)(\not{p}+m) = \not{p}\not{p} - m^2 = p^2 - m^2$$

$$\not{A}\not{A} = \frac{1}{2} A_\mu A_\nu \{\gamma^\mu, \gamma^\nu\}$$

$2g^{\mu\nu}$

$$= \frac{1}{2}(\not{A}\not{A} + \not{A}\not{A})$$

$$= A_\mu A^\mu = A^2$$

$$(i\not{\partial}-m) S_F(x) = \int \frac{d^4p}{(2\pi)^4} \underbrace{\left((\not{p}-m)\frac{(\not{p}+m)}{p^2-m^2+i\varepsilon}\right)}_{1} e^{-ip\cdot x}$$

$$= \delta^4(x). \checkmark$$