Path Integrals 3

Note Title 4/27/2010

1) QM

$$D(x_f, x_i, t) = \sum_n \psi_n(x_f)\,\psi_n^*(x_i)\, e^{-iE_n t}$$

$$= \int \mathcal{D}(x(t))\, e^{\frac{i}{\hbar}\underbrace{\int dt\, L(x, \dot{x}(t))}_{S = \text{action}}}$$

Classical limit $S/\hbar$ large

- $x(t) = x_{cl}(t) + \delta x(t)$

$S = \underbrace{S_{cl}}_{\text{large}} + \delta S$ $\Rightarrow e^{i\frac{S}{\hbar}}$ oscillates a lot as change δx

$\delta S = 0$ oscill quadratic $\Rightarrow$ dominates P.I.

2) Project out ground state

$t \to \infty \qquad e^{-i E_n t}$ fluctuates wildly $\Rightarrow E_0$ survives

Math 1) $i\varepsilon \qquad H \to (1 - i\varepsilon) H$

2) Wick rotate $\quad t \to -i\tau \quad , \quad \tau \to \infty \qquad e^{-H\tau} \quad , \quad e^{-E_n \tau}$

E_0 survives

3) Functional method $\qquad$ "source" J

$$Z[J] = \int \mathcal{D}(x)\, e^{i \int dt [L + \vec{J} \cdot \vec{x}]}$$

$$\frac{\delta}{\delta J_i} \quad - \quad -$$

QFT

$$y_i(t) \longrightarrow \phi(x,t)$$

$$L(y_i, \dot{y}_i) \longrightarrow \mathcal{L}(\phi, \partial_\mu \phi)$$

$$\int \mathcal{D} y_i(t)) \longrightarrow \int [d\phi(x,t)]$$

What is $[d\phi(x,t)]$?

- little block $\delta x\, \delta y\, \delta z\, \delta t$
- in each block $\phi(x_i, y_i, z_i\, t_i)$

$$[d\phi(x,t)] = \prod_i d\phi(x_i\, y_i\, z_i\, t_i)$$

$$PI = N \int [d\phi]\, e^{iS[\phi, \partial_\mu \phi]}$$

$\backslash \int d^3x\, dt\, \mathcal{L}$

Project out ground state $\quad t_f - t_i \to \infty$

Generating functional

$$Z[J] = N \int [d\phi]\, e^{i \int d^4x \left[\mathcal{L}(\phi, \partial_\mu \phi) + J(x,t)\, \phi(x,t) \right]} \qquad **$$

Functional Diff.

$$\frac{\delta}{\delta J(x)} J(x') \equiv \delta^4(x - x')$$

$$\frac{\delta}{\delta J(x)} \int d^4x'\, f(x')\, J(x') = \int d^4x'\, f(x')\, \delta^4(x - x') = f(x)$$

Then:

$$\frac{\delta}{\delta J(x)} Z[J] = N \int [d\phi] \; i\phi(x) \; e^{i\int d^4x' [\mathcal{L} + J\phi]}$$

Greens functions:

$$G^{(n)}(x_1, \dots x_n) = \frac{(-i)^n}{Z[0]} \frac{\delta^n}{\delta J(x_1)\,\delta J(x_2) \cdots \delta J(x_n)} Z[J] \Big|_{J=0}$$

$$= \frac{N \int [d\phi] \; \phi(x_1) \cdots \phi(x_n) \, e^{i\int d^4x' \mathcal{L}}}{N \int [d\phi] \; e^{i\int d^4x' \mathcal{L}}}$$

To make $Z[J]$ defined:

A) $i\varepsilon$ trick $m^2 \to (m^2 - i\varepsilon)$

B) Wick $t \to -i\tau$ $\quad \tau \to \infty$

Interpretation of $J(x)$ = "Source"

A) QM $\qquad \mathcal{L} = \frac{1}{2} m\dot{x}^2 - \frac{1}{2} m\omega^2 x^2 + Jx$

Eq of motion $\quad m\ddot{x} = m\omega^2 x - J(t)$ ← arbitrary force

Probing HO with arbitrary force

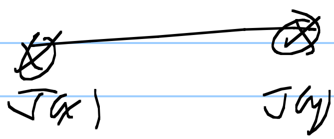
B) QFT $\qquad \mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{m^2}{2}\phi^2 + J\phi$

Eq of motion $\quad (\Box + m^2)\phi = J(x,t)$ ← source of field

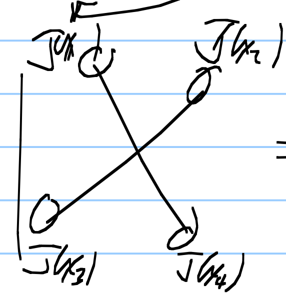
$$\phi(x) = \int d^4x'\, G(x-x')\, J(x')$$

J ~ creates or removes fields

Probing groundstate



$= i D_F(x-y)$



$= D_F(\ ) D_F D_F D_F (-6i\lambda)$

Free Fields

$$Z_0[J] \equiv N \int [d\phi]\, e^{i\int d^4x \left[\frac{1}{2}(\partial_\mu \phi)^2 - \frac{m^2}{2}\phi^2 + J\phi\right]}$$

$\uparrow$ free field

$$= N \int [d\phi]\, e^{i\int d^4x \left[-\frac{1}{2}\phi\underbrace{(\Box + m^2)}_{A = (\Box + m^2 - i\varepsilon)}\phi + J\phi\right]}$$

Previously $x' = x - A^{-1} J$ to complete square

For field theory $(\Box + m)^{-1} = ?$

$(\Box + m^2) D_F(x-y) = -\delta^4(x-y) \implies D_F = -(\Box + m^2)^{-1}$

Complete the square:

$$\phi'(x) = \phi(x) + \int d^4y\, D_F(x-y) J(y)$$

Then:

$$\int d^4x \; -\frac{1}{2}\phi'(x)(\Box+m^2)\phi'(x) = \int d^4x \; -\frac{1}{2}\left[\phi(\Box+m^2)\phi - 2J(x)\phi(x)\right]$$

$$+\frac{1}{2}\int d^4x\, d^4y \; J(x)\, D_F(x-y)\, J(y)$$

$$Z[J] = N \underbrace{\int [d\phi]}_{=[d\phi']} \; e^{i\int d^4x \, \phi'(\Box+m^2)\phi'} \; e^{-\frac{i}{2}\int d^4x\, d^4y \, J(x)\, D_F(x-y)\, J(y)}$$

$$= Z[0] \; e^{-\frac{i}{2}\int d^4x\, d^4y\, J(x)\, D_F(x-y)\, J(y)}$$

☆☆

Our First Calculation

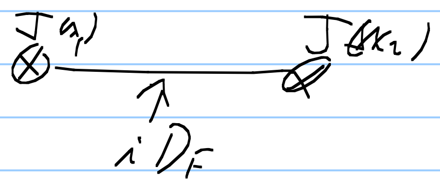
Canonical quant

$\downarrow \quad \phi = (a + a^\dagger)$

$$G^{(2)}(x_1,x_2) = \frac{(-i)^2}{Z[0]} \frac{\delta^2 Z[J]}{\delta J(x_1)\,\delta J(x_2)}\Bigg|_{J=0} = \langle 0| T(\phi(x_1)\,\phi(x_2))|0\rangle$$

$$= \frac{(-1)}{\cancel{Z[0]}} \frac{\delta}{\delta J(x_2)} \cancel{Z[0]}\left[-\frac{i}{2}\times 2 \int d^4y\; D_F(x_1-y)\,J(y)\; e^{-\frac{i}{2}\int d^4x\, d^4z\, J D_F J}\right]\Bigg|_{J=0}$$

$$= +i\, D_F(x_1-x_2) \qquad \checkmark$$



— no operator — just function

— $i\varepsilon$ trick $\rightarrow$ causality

$\rightarrow$ projecting out ground state

Wick's Theorem

no vac. value

Recall $T(\phi_1 \phi_2 \phi_3 \phi_4) = :\phi_1 \phi_2 \phi_3 \phi_4:$

$+ \text{perm} \langle 0 | T(\phi_1 \phi_2) | 0 \rangle :\phi_3 \phi_4:$

$+ \text{Perm} \langle 0 | T(\phi_1 \phi_2) | 0 \rangle \langle 0 | T(\phi_3 \phi_4) | 0 \rangle$

Here

$$G^4(x_1, x_2, x_3, x_4) = \langle 0 | T(\phi(x_1) \phi(x_2) \phi(x_3) \phi(x_4)) | 0 \rangle$$

$$= iD_F(x_1 - x_2)\, iD_F(x_3 - x_4) + iD_F(x_1 - x_3)\, iD_F(x_2 - x_4) + iD_F(x_1 - x_4)\, iD_F(x_2 - x_3)$$

$$= \frac{(-i)^4}{Z[0]} \frac{\delta^4}{\delta J(x_1)\, \delta J(x_2)\, \delta J(x_3)\, \delta J(x_4)} Z[J] \Big|_{J=0} \quad \checkmark$$