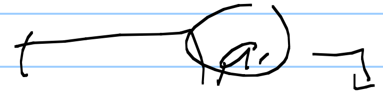
Path Integrals 4

Note Title 4/29/2010

Interaction



canonical $|p_1, p_2\rangle$, $U_I(\infty, -\infty)$, $\langle p_3 p_4 | \lambda \phi^4 | p_1 p_2\rangle$

P.I $G(x_1 \cdots x_n)$

LSZ Reductions - connect ...

No longer gaussian integral

$$Z[J] = \int [d\phi]\, e^{i\int d^4x\left[\underbrace{\frac{1}{2}(\partial_\mu\phi)^2 - \frac{m^2\phi^2}{2}}_{\text{did}} \underset{\uparrow\ ?}{-\frac{\lambda}{4}\phi^4} + J\phi\right]}$$

Path in Pert. Theory

$$Z[J] = \int [d\phi]\; \underbrace{e^{-i\int d^4y\, \phi^4 \frac{\lambda}{4}}}_{\left[1 - i\frac{\lambda}{4}\int d^4y\, \underset{\overset{\uparrow}{(-i)^4\frac{\delta^4}{\delta J(y)^4}}}{\phi^4} + \frac{(-i)^2}{2}\int d^4y\, d^4z\, \frac{\lambda}{4}\phi^4(y)\, \frac{\lambda}{4}\phi^4(z) \cdots \right]}\; e^{i\int d^4x \left[\frac{1}{2}(\partial_\mu\phi)^2 - \frac{m^2}{2}\phi^2 + J\phi\right]}$$

~~∝~~ ∗

$$= \left[1 - \frac{i\lambda}{4}\int d^4y\, (-i)^4\frac{\delta^4}{\delta J(y)^4} + \cdots\right] \underbrace{\int [d\phi]\; e^{i\int d^4x\left[\;''\;\; + J\phi\right]}}_{Z_0[J]}$$

$\uparrow$ free P.I.

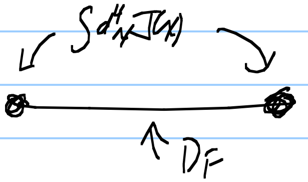
$$= e^{\left[-i\int d^4y \frac{\lambda}{4}\frac{\delta^4}{\delta J(y)^4}\right]} Z_0[J]$$

$$= e^{i\int d^4y\, \mathcal{L}_{Int}\left(\frac{-i\delta}{\delta J(y)}\right)} Z_0[J]$$

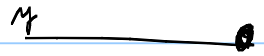
Work to first order in λ

$$Z[J] = Z_0[J] + \underset{\uparrow \lambda^1}{Z_1[J]} + \underset{\uparrow \lambda^2}{Z_2[J]}$$

Shorthand $\int d^4x\, d^4y\, J(x)\, D_F(x-y)\, J(y)$



$\int d^4x\, J(x) D_F(x-y)$ ↓

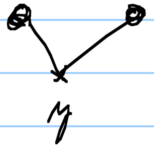


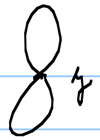
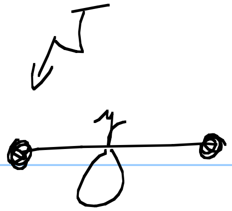
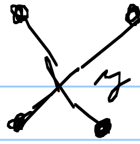
↓

$Z_0[J] = Z_0[0]\, e^{-\frac{i}{2}}$ 

$$\frac{\delta}{\delta J(y)} Z_0 = \left[\frac{-i}{2} \times 2 \int d^4x\, J(x)\, D_F(x-y) \right] Z_0[J]$$

$$\frac{\delta}{\delta J(y)} \frac{\delta}{\delta J(y)} Z_0 = \left[-i\, D_F(y-y) + \left[-i \int d^4x\, J(x)\, D_F(x-y)\right]\left[-i\int d^4z\, J(z)\, D_F(z-y)\right]\right] Z_0[J]$$

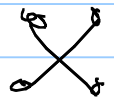
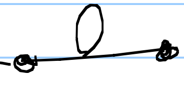
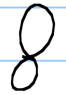


$Z_1[J] = -i\frac{\lambda}{4}\int d^4y \Big[$  $+$  $+$  $\Big] Z_0[J]$

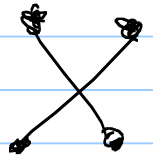
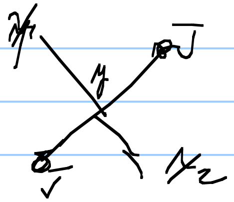
$Z_1[0] = -i\frac{\lambda}{4} d^4y \Big[$  $\Big] Z_0[0]$

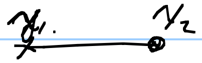
Two point functions $G^{(2)}(x_1, x_2)$

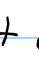
$$G^{2}(x_1, x_2) = \frac{(-i)^2}{Z[0]} \frac{\delta^2}{\delta J(x_1)\,\delta J(x_2)} \left[Z_0[J] + Z_1[J] + \cdots \right] \Big|_{J=0}$$

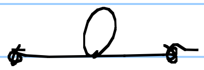
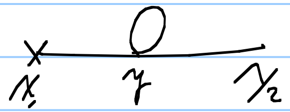
$= \dfrac{(-i)^2 \dfrac{\delta^2}{\delta J \delta J} \left[1 - i \dfrac{\lambda}{4} \int d^4y \,\Big(\right.}{}$  $+$  $+$  $\left.\Big)\right] Z_0[J] \Big|_{J=0}$

$\overline{\left[1 - i \dfrac{\lambda}{4} d^4x \right.}$  $\left.\right] Z_0[0]$

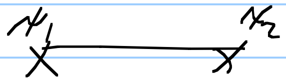
i) $\dfrac{\delta^2}{\delta J \delta J}$  $\Big|_{J=0} = 0 =$ 

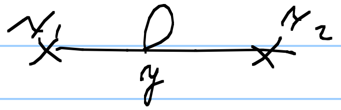
ii) if Both act on $Z_0[J]$ $\dfrac{[1+}{}$  $]$ 

$\overline{[1+}$  $]$ $= i D_F(x_1 - x_2)$

iii) Both on  $=$ 

$$G^{(2)}(x_1,x_2) = i D_F(x_1-x_2) + \int d^4y\,(-6i\lambda)\,iD_F(x_1-y)\,iD_F(y-y)\,iD_F(y-x_2)$$

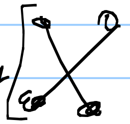




If Fourier transform $e^{ip\cdot x_1}$

$$\frac{i}{p^2-m_1^2} + \frac{i}{p^2-m^2} \underbrace{\int \frac{d^4k}{(2\pi)^4}(-6i\lambda)\frac{i}{k^2-m^2}}_{-i\Sigma} \frac{i}{p^2-m^2}$$ ✓

4 point functions at $\mathcal{O}(\lambda)$

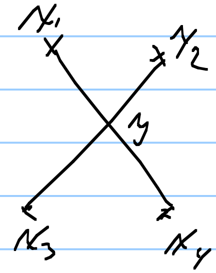
$$G^4(x_1, \dots x_4) = \frac{(-i)^4}{Z[0]} \frac{\delta^4}{\delta J(x_1) \cdots \delta J(x_4)} \left[1 - \frac{i\lambda}{4} \int d^4y \left[\right. \right.$$

$+$

$+$

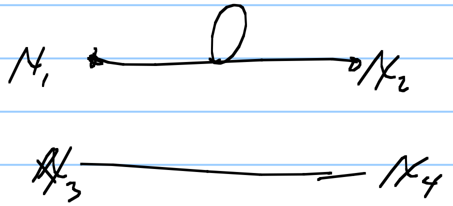
$$\left. \left. \right] \right] Z[J] \Big|_{J=0}$$

Connected piece

$$= \int iD_F(x_1 - y)\, iD_F(x_2 - y)\, iD_F(x_3 - y)\, iD_F(x_4 - y) \; - \cancel{6} i\lambda \; d^4y =$$

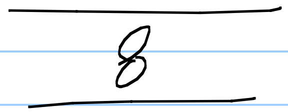


disconnected diagrams



$+ \dots$

Drop out $\rightarrow$



from $\frac{1}{Z[0]}$

Fourier transform each external x $\quad e^{\pm i p_i \cdot x_i}$

A) Disconnected drop out $p_2 \neq p_1$

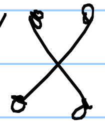
B) Connected gives

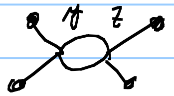
$$\frac{i}{p_1^2-m^2}\;\frac{i}{p_2^2-m^2}\;\frac{i}{p_3^2-m^2}\;\frac{i}{p_4^2-m^2}\;\underbrace{-6i\lambda}_{i\mathcal{M}}\;\underbrace{(2\pi)^4\,\delta^4(p_1+p_2-p_3-p_4)}_{\int d^4y\; e^{i(p_1+p_2-p_3-p_4)y}}$$

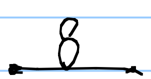
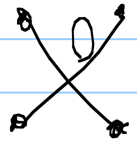
Rule

$$\left.\begin{array}{l}\text{F.T. w.r.t external variables}\\ \text{Drop propagators}\end{array}\right\}\quad \begin{array}{l}\text{scattering amp}\\ (2\pi)^4\,\delta^4(\quad)\;\underline{\underline{i\mathcal{M}}}\end{array}$$

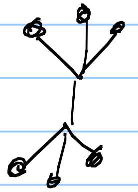
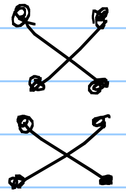
The Rest of Pert Theory $(O(\lambda^2))$

$$\frac{Z[J]}{Z[0]} = \Big[1 - i\frac{\lambda}{4}\int d^4y \Big($$  $+$  $\Big)$

$$-\frac{1}{2}\left(\frac{-i\lambda}{4}\right)^2 \int d^4y\, d^4z \Big($$ 

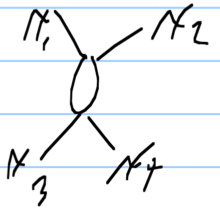
$+$  $+$  $+$  $+ \ldots$

↗

 $+$  $\Big) Z_0[J]$

$\underbrace{\qquad}_{\phi+\phi \to 4\phi}$

Give usual diagrams



$$= \frac{(-6i\lambda)^2}{2}\int d^4y\, d^4z\; iD_F(x_1-y)\, iD_F(x_2-y) \ldots$$

i F.T. + Drop externa $\frac{1}{p^2-m^2}$ ⇒ usual amplitude

⇒ all Feynman rules

$Z\{J\}$ contains everything

$$Z\{J\} = Z[0] \sum_n \frac{1}{n!} \int d^4x_1\, J(x_1) \int d^4x_2\, J(x_2) \cdots \int d^4x_N\, J(x_N)\; G^n(x_1, x_2 \cdots x_n)$$

$$G^n(\quad) = \frac{(-i)^n}{Z[0]} \frac{\delta^n}{\delta J(x_1) \ldots \delta J_n} \qquad Z_\phi[J] \qquad ⤢$$

$Z[J]$ vacuum amplitude, contains scattering

J = source

LSZ reduction - afrivated

- connection canonical quant + P. I $\underbrace{}_{G^{n}(x_1 - - x_n)}$

Recall

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k} \left[a(k) e^{-ik\cdot x} + a^{+}(k) e^{+ik\cdot x} \right]$$

Calculate $a^{+}(p)$ $\qquad (a^{+} \sim x - ip)$ in H-W.

$$a^{+}(p) = \int d^3x \, e^{-ip\cdot x} \omega_p \left(1 - \frac{i}{\omega_p} \frac{\partial}{\partial t} \right) \phi(x)$$

$$= \int d^3x \, e^{-ip\cdot x} \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k} \omega_p \left[a(k) \underbrace{\left(1 - \frac{\omega_k}{\omega_p}\right)}_{0} e^{-ik\cdot x} + a^{+}(k) \underbrace{\left(1 + \frac{\omega_k}{\omega_p}\right)}_{2} e^{+ik\cdot x} \right]$$

$$= a^{+}(p) \quad \checkmark$$

Goal

$$\langle \beta\ \text{out} | p, \overset{\swarrow\ \text{rest}}{\alpha}\ \text{in}\rangle = \overbrace{\langle \beta - p\ \text{out} | \alpha_{\text{in}}\rangle}^{0}$$

$$+ i \underbrace{\int d^4x\, e^{-ip\cdot x}}_{\text{Go to momentum space}} \underbrace{(\Box + m^2)}_{\text{remove propagator}} \underbrace{\langle \beta | \phi(x) | \alpha \rangle}_{\text{Keep reducing until}}$$

$$\underbrace{\langle 0 | T(\phi\phi\ \phi\phi\ \ )\ | 0\rangle}_{G^{n}(x_1 -- x_n)}$$

Derivation

$$\langle \beta \text{ out} | a^\dagger(p) | \alpha \text{ in} \rangle = \lim_{t \to -\infty} \int d^3x \, e^{-ip\cdot x} \, -i \overleftrightarrow{\partial_0} \, \langle \beta | \phi(x) | \alpha \rangle$$

$$\uparrow \; \omega - i\frac{\partial}{\partial t} = \omega\left(1 - \frac{i}{\omega}\frac{\partial}{\partial t}\right)$$

$$- \lim_{t \to +\infty} \int d^3x \, e^{-ip\cdot x} \, -i \overleftrightarrow{\partial_0} \, \langle \beta | \phi | \alpha \rangle$$

$$+ \langle (\beta - p) \text{ out} | \alpha \text{ in} \rangle \quad \Leftarrow$$

$$= \int d^4x \, \partial_0 \left[e^{-ip\cdot x} \, -i \overleftrightarrow{\partial_0} \, \langle \beta \text{ out} | \phi(x) | \alpha \text{ in} \rangle \right]$$

$$+ \langle \beta - p | \alpha \rangle$$

$$\frac{\partial^2}{\partial x_0^{\,2}} e^{-ip\cdot x} = \underset{\uparrow \text{ int by parts}}{(\nabla^2 - m^2)}\, e^{i\overrightarrow{p\cdot x}}$$

Generates $\underbrace{\left[\frac{\partial^2}{\partial t^2} + (\nabla^2 - m^2)\right)}_{\Box + m^2} \phi(x)$

All particles reduced ← done with a & a^+

$$\langle p_3\, p_4 \text{ out} \,|\, p_1\, p_2 \text{ in}\rangle$$

$$= \prod \int d^4x_i\, e^{\mp i p_i x_i} (\Box_i + m^2) \quad \underbrace{\langle 0 | T(\phi(x_1) \cdots \phi(x_4)) | 0\rangle}_{G^4(\qquad)}$$

↑ done with P. I.