


# Path Integrals 4

Note Title

4/29/2010

## Interaction

canonical  $|p_1, p_2\rangle$ ,  $U_I(\infty, -\infty)$ ,

$$\langle p_1, p_2 | \phi^4 | p_1, p_2 \rangle$$


P.I  $G(N_1, \dots, N_n)$

LSZ Reduction - connect ...

No longer gaussian integral

$$Z[J] = \int [d\phi] e^{i \int d^4x \left[ \frac{1}{2} (\partial_\mu \phi)^2 - \frac{m^2}{2} \phi^2 - \frac{\lambda}{4} \phi^4 + J\phi \right]}$$

did ↑ ?

Path in Pert. Theory

$$Z[J] = \int [d\phi] e^{-i \int d^4y \left[ \frac{1}{4} \phi^4 + \dots \right]} e^{i \int d^4x \left[ \frac{1}{2} (\partial_\mu \phi)^2 - \frac{m^2}{2} \phi^2 + J\phi \right]}$$

$$\left[ 1 - i \frac{\lambda}{4} \int d^4y \phi^4 + \frac{(-i)^2}{2} \int d^4y d^4y' \frac{\lambda}{4} \phi(y) \phi(y') \dots \right]$$

$$\frac{(-i)^4}{4!} \int d^4y \phi^4$$

↑  
\*\*\*

$$= \left[ 1 - \frac{i\lambda}{4} \int d^4y \frac{(-i)^4 \delta^4}{\delta J(y)^4} + \dots \right] \int [d\phi] e^{i \int d^4x \left[ \dots + J\phi \right]}$$

$Z_0[J]$

↑ free l. I.

$$= e^{\left[ -i \int d^4y \frac{\lambda}{4} \frac{\delta^4}{\delta J(y)^4} \right]} Z_0[J]$$

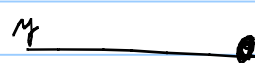
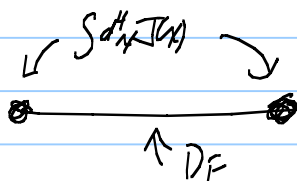
$$= e^{i \int d^4y \mathcal{L}_{int} \left( \frac{-i\delta}{\delta J(y)} \right)} Z_0[J]$$

Work to first order in  $\lambda$

$$Z[J] = Z_0[J] + Z_1[J] + Z_2[J]$$

Shorthand  $\int d^4x d^4y J(x) D_F(x-y) J(y)$

$$\int d^4x J(x) D_F(x-y)$$



$\lambda^1$   $\lambda^2$

$$Z_0[J] = Z_0[0] e^{-\frac{i}{2} \dots}$$

$$\frac{\delta}{\delta J(y)} Z_0 = \left[ -\frac{i}{2} \times 2 \int d^4x J(x) D_F(x-y) \right] Z_0[J]$$

$$\frac{\delta}{\delta J(y)} \frac{\delta}{\delta J(y)} Z_0 = \left[ -i D_F(y-y) + \left[ -i \int d^4x J(x) D_F(x-y) \right] \left[ -i \int d^4z J(z) D_F(z-y) \right] \right] Z_0[J]$$



$$Z_1[\mathcal{J}] = -i \frac{\lambda}{4} \int d^4 y \left[ \delta_4 + \text{diagram 1} + \text{diagram 2} \right] Z_0[\mathcal{J}]$$

The first diagram is a circle with a horizontal line through its center. The left and right ends of the horizontal line are marked with small circles. An arrow labeled  $\mathcal{J}$  points downwards from the top of the circle.

The second diagram is a four-point vertex where two lines cross each other at a central point. Each of the four ends of the lines is marked with a small circle.

$$Z_1[0] = -i \frac{\lambda}{4} \int d^4 y \left[ \delta \right] Z_0[0]$$

Two point functions  $G^{(2)}(x_1, x_2)$

$$G^2(x_1, x_2) = \frac{(i)^2}{Z[0]} \frac{\delta^2}{\delta J(x_1) \delta J(x_2)} \left[ Z_0[J] + Z_1[J] + \dots \right] \Big|_{J=0}$$

$$= \frac{(i)^2 \delta^2}{\delta J \delta J} \left[ 1 - i \frac{\lambda}{4} \int d^4 y \left( \text{diagram 1} + \text{diagram 2} + \text{diagram 3} \right) \right] Z_0[J] \Big|_{J=0}$$

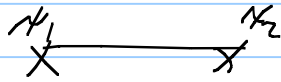
$$\frac{\overline{\left[ 1 - i \frac{\lambda}{4} \int d^4 y \delta \right]}}{Z_0[0]}$$

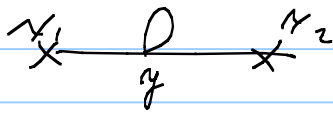
i)  $\frac{\delta^2}{\delta J \delta J} \text{diagram 1} \Big|_{J=0} = 0 = \text{diagram 2}$

ii) if both act on  $Z_0[J]$   $\frac{[1 + \delta] \text{diagram 3}}{[1 + \delta]} = i D_F(x_1, -x_2)$

iii) Both on  $\text{diagram 2} = \text{diagram 3}$

$$G^{(2)}(x_1, x_2) = i D_F(x_1, -x_2) + \int d^4y (-6i\lambda) i D_F(x_1, -y) i D_F(y, -y) i D_F(y, -x_2)$$





If Fourier transform  $e^{iP \cdot X_1}$

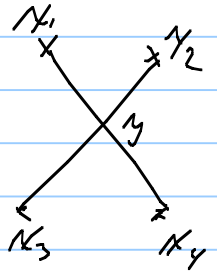
$$\frac{i}{p^2 - m^2} + \frac{i}{p^2 - m^2} \underbrace{\int \frac{d^4k}{(2\pi)^4} (-6i\lambda) \frac{i}{k^2 - m^2}}_{-i\lambda} \frac{i}{p^2 - m^2} \quad \checkmark$$

4 point functions at  $\mathcal{O}(\lambda)$

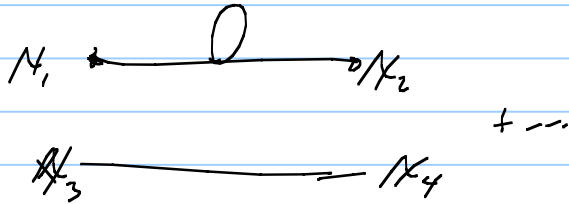
$$G^4(k_1, \dots, k_4) = \frac{(-i)^4}{Z[0]} \delta^4 \left[ 1 - \frac{i\lambda}{4} \int d^4 y \left[ \text{diagram 1} + \text{diagram 2} + \text{diagram 3} \right] Z[J] \right]_{J=0}$$

connected piece

$$= \int iD_F(k_1 - y) iD_F(k_2 - y) iD_F(k_3 - y) iD_F(k_4 - y) - \text{kit} \int d^4 y =$$



disconnected diagrams



Drop out  $\rightarrow$   $\frac{8}{Z[0]}$  from  $\frac{1}{Z[0]}$

Fourier transform each external  $X$   $e^{\pm i p_i \cdot X_i}$

A) Disconnected drop out  $p_2 \neq p_1$

B) Connected gives

$$\frac{i}{p_1^2 - m^2} \frac{i}{p_2^2 - m^2} \frac{i}{p_3^2 - m^2} \frac{i}{p_4^2 - m^2} \underbrace{-6i\lambda}_{iM} \underbrace{(2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4)}_{\int d^4y e^{i(p_1 + p_2 - p_3 - p_4) \cdot y}}$$

Rule

F.T. w.r.t external variables

Drop propagators

} scattering amp  
 $(2\pi)^4 \delta^4(\dots) iM$



# The Rest of Pert Theory ( $O(\lambda^2)$ )

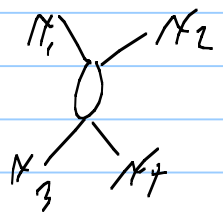
$$\frac{Z[J]}{Z[0]} = \left[ 1 - i \frac{\lambda}{4} \int d^4y \left( \text{diagram 1} + \text{diagram 2} \right) \right]$$

$$- \frac{1}{2} \left( \frac{-i\lambda}{4} \right)^2 \int d^4y d^4z \left( \text{diagram 3} + \text{diagram 4} + \text{diagram 5} + \text{diagram 6} + \dots \right)$$

$$\uparrow \left( \text{diagram 7} + \text{diagram 8} \right) Z_0[J]$$

$\phi + \phi \rightarrow 4\phi$

Give usual diagrams



$$= \left( \frac{-6i\lambda}{2} \right)^1 \int d^4y d^4z i D_F(k_1 - y) i D_F(k_2 - y) \dots$$

i.F.T. + Drop external  $\frac{1}{p^2 - m^2} \Rightarrow$  usual amplitude

$\Rightarrow$  all Feynman rules

$Z[J]$  contains everything

$$Z[J] = Z[0] \sum_n \frac{1}{n!} \int d^4x_1 J(x_1) \int d^4x_2 J(x_2) \dots \int d^4x_n J(x_n) G^n(x_1, x_2, \dots, x_n)$$

$$G^n(x_1, \dots, x_n) = \frac{(-i)^n \delta^n}{Z[0] \delta^n J(x_i)} \cdot \text{FTn} \quad Z[J] \quad \swarrow$$

$Z[J]$  vacuum amplitude, contains scattering

$J = \text{source}$

## LSZ reduction - abbreviated

- connection canonical quant + P.I  
 $\underbrace{\quad}_{G^m(x_1, \dots, x_n)}$

### Recall

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k} \left[ a(k) e^{-ik \cdot x} + a^\dagger(k) e^{+ik \cdot x} \right]$$

Calculate  $a^\dagger(p)$   $\left( a^\dagger \sim \nabla x - ip \right)$  in H.W.

$$a^\dagger(p) = \int d^3x e^{-ip \cdot x} \omega_p \left( 1 - \frac{i}{\omega_p} \frac{\partial}{\partial t} \right) \phi(x)$$

$$= \int d^3x e^{-ip \cdot x} \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k} \omega_p \left[ a(k) \underbrace{\left( 1 - \frac{\omega_k}{\omega_p} \right)}_0 e^{-ik \cdot x} + a^\dagger(k) \underbrace{\left( 1 + \frac{\omega_k}{\omega_p} \right)}_2 e^{+ik \cdot x} \right]$$

$$= a^\dagger(p) \quad \checkmark$$

Goal

rest



$$\langle \beta \text{ out} | p, \alpha \text{ in} \rangle = \langle \beta - p \text{ out} | \alpha \text{ in} \rangle$$

$$+ i \int d^4x e^{-i p \cdot x} (\square + m^2) \langle \beta | \phi(x) | \alpha \rangle$$

go to momentum space

remove propagator

Keep reducing until  
 $\langle 0 | \Pi(\phi \phi \phi \phi) | 0 \rangle$

$$G^4(x_1, \dots, x_n)$$

## Derivation

$$\langle \beta \text{ out} | a^\dagger(p) | \alpha \text{ in} \rangle = \lim_{t \rightarrow -\infty} \int d^3x e^{-i p \cdot x} \overleftrightarrow{\partial}_0 \langle \beta | \phi(x) | \alpha \rangle$$

$\uparrow \omega - i\epsilon \frac{\partial}{\partial t} = \omega \left(1 - \frac{1}{\omega} \frac{\partial}{\partial t}\right)$

$$- \lim_{t \rightarrow +\infty} \int d^3x e^{-i p \cdot x} \overleftrightarrow{\partial}_0 \langle \beta | \phi | \alpha \rangle$$

$$+ \langle \beta - \beta | \text{out} | \alpha \text{ in} \rangle \leftarrow$$

$$= \int d^4x \partial_0 \left[ e^{-i p \cdot x} \overleftrightarrow{\partial}_0 \langle \beta \text{ out} | \phi(x) | \alpha \text{ in} \rangle \right]$$

$$+ \langle \beta - p | \alpha \rangle$$

$$\frac{\partial^2}{\partial X_0^2} e^{-i p \cdot X} = \underbrace{(\nabla^2 - m^2)}_{\substack{\uparrow \\ \text{int by parts}}} e^{i \vec{p} \cdot \vec{A}}$$

Generates  $\underbrace{\left[ \frac{\partial^2}{\partial t^2} - (\nabla^2 - m^2) \right]}_{D + m^2} \phi(x)$

All particles reduced  $\leftarrow$  done with  $a$  &  $a^\dagger$   
 $\langle P_3, P_4 \text{ out} | P_1, P_2 \text{ in} \rangle$

$$= \prod \int d^4 X_i e^{\pm i p_i \cdot X_i} (D_i + m^2) \underbrace{\langle 0 | T(\phi(x_1) \dots \phi(x_n)) | 0 \rangle}_{\mathcal{G}^n(\quad)}$$

$\mathcal{G}^n(\quad)$

$\uparrow$  done with P.T.