Path Integral 1

4/20/10

Basic Integrals – "Gaussian"

$$\underline{I}(a) = \int_{-\infty}^{\infty} dx \, e^{-\frac{1}{2} a x^2} = \left(\frac{2\pi}{a}\right)^{1/2}$$

Variations

A) $$I_J(a) = \int_{-\infty}^{\infty} dx \, e^{-\frac{1}{2} a x^2 + Jx} = \overbrace{\int_{-\infty}^{\infty} dx}^{\int_{-\infty}^{\infty} dx'} \, e^{-\frac{1}{2} a (\overbrace{x - J/a}^{x'})^2} \, e^{+J^2/2a}$$

$$= \left(\frac{2\pi}{a}\right)^{1/2} e^{J^2/2a}$$

Note $$\int_{-\infty}^{\infty} dx \, x^n \, e^{-\frac{1}{2} a x^2} = \frac{d^n}{dJ^n} I_J(a) \Big|_{J=0}$$

B) $J \to iJ$

$$\int_{-\infty}^{\infty} dx\, e^{-\frac{1}{2}ax^2 + iJx} = \left(\frac{2\pi}{a}\right)^{1/2} e^{-J^2/2a}$$

c) $a \to -ia$

$$\int_{-\infty}^{\infty} dx\, e^{i\left[\frac{1}{2}ax^2 + Jx\right]} = \left(\frac{2\pi i}{a}\right)^{1/2} e^{-iJ^2/2a}$$

D) <u>Many variables</u>

$x \to x_i$ $\qquad i = 1, 2 \ldots N$

$a \to A_{ij}$ $\qquad i, j$ " $\qquad$ Symmetric

$J \to J_i$

$$\int_{-\infty}^{\infty} dx_1 \ldots dx_N \, e^{-\frac{1}{2} x_i A_{ij} x_j + J_i x_i}$$

Diagonalize A $\quad y_i = O_{ij} x_j \quad$ or $\quad x = O^{-1} y$

$$x^T A x = \underbrace{x O^{-1}}_{y^T} \underbrace{(O A O^{-1})}_{A_D} \underbrace{O x}_{y}$$

$\uparrow$ diagonal

Use $\int dx_1 \ldots dx_N = \int dy_1 \ldots dy_N$

$$\Rightarrow \int dy_1 \ldots dy_N \, e^{-\frac{1}{2} y^T A_D y - \vec{J}' \vec{y}} \qquad J x = J O^{-1} y = J' y$$

$$= \prod_{i=1}^{N} \left(\frac{2\pi}{a_{D i}} \right)^{1/2} e^{-\frac{1}{2} J'_i A_D^{-1} J'_j}$$

$$A_D^{-1} = \begin{pmatrix} \frac{1}{a_1} & & 0 \\ & \frac{1}{a_2} & \\ 0 & & \ddots \ \frac{1}{a_N} \end{pmatrix}$$

$$= \frac{(2\pi)^{N/2}}{\left[\det A_D\right]^{1/2}} \, e^{-\frac{1}{2} J \left(O^{-1} A_D^{-1} O\right) J}$$

$$\det A_D = \det A = \det\left[\underbrace{O A O^{-1}}\right]$$

$$O^{-1} A_D^{-1} O = A^{-1} \quad \text{since} \quad A^{-1} A = \underbrace{O^{-1} \overbrace{A_D^{-1} \underbrace{O \, O^{-1}} A_D}^{1} O}_{1} = 1$$

$$\Rightarrow \int_{-\infty}^{\infty} dx_1 \cdots dx_N \, e^{-\frac{1}{2} x \cdot A \cdot x + J \cdot x} = \frac{(2\pi)^{N/2}}{\left[\det A\right]^{1/2}} \, e^{\frac{1}{2} J \cdot A^{-1} J}$$

$$Z[J] = \int dx_1 \cdots dx_N \, e^{i\left[\frac{1}{2} x \cdot A \cdot x + J \cdot x\right]} = \frac{(2\pi i)^{\frac{N}{2}}}{[\det A]^{\frac{1}{2}}} \, e^{-\frac{i}{2} J \cdot A^{-1} \cdot J}$$