QED 4

Note Title 4/20/2010

Group theory

$SU(2)$ $\quad \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \longrightarrow \psi' = U\psi$

$\uparrow$ 2×2 unitary

$U = e^{i\vec{\alpha}\cdot\vec{\tau}}$ $\leftarrow$ pauli

$\bar{\psi}\psi$ invariant

$\bar{\psi}(\vec{\tau}\cdot\vec{\pi})\psi$ "

if $\vec{\tau}\cdot\vec{\pi}' = U\,\vec{\tau}\cdot\pi\,U^\dagger$

$Tr\left[\tau^i(\tau\cdot\pi')\right] = -$

$\Rightarrow \pi'^i = R^{ij}\pi^j$

$R^{ij} = \frac{1}{2}Tr(\tau^i U \tau^j U^\dagger)$

SU(N)

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_N \end{pmatrix} \qquad \psi \to \psi' = U\psi$$

U: $N \times N$ unitary

$$U = e^{iH} = e^{i(\alpha_0 + \vec{\alpha}\cdot\vec{\lambda})}$$

$\vec{\lambda}$: $N \times N$ Hermetian

$\lambda^i = N \times N$ Hermetian, traceless

$\Rightarrow N^2 - 1$ matrices $\qquad i = 1 \cdots (N^2 - 1)$

$$\mathrm{Tr}(\lambda^i \lambda^j) = 2\delta^{ij} \Leftarrow \text{convention}$$

$$[\lambda^i, \lambda^j] = 2i f^{ijk} \lambda^k$$

f^{ijk} antisymmetric

$$([\lambda^i, \lambda^j])^\dagger = \lambda^{j\dagger}\lambda^{i\dagger} - \lambda^{i\dagger}\lambda^{j\dagger} = -[\lambda^i, \lambda^j]$$

Invariant

$\psi^{+}\psi$ invariant

ψ — "fundamental" N

$\lambda^i V^i \leftarrow i = 1, \ldots (N^2-1)$

V^i — "adjoint" (N^2-1)

$$\bar\psi\, \vec\lambda\cdot\vec V\, \psi \rightarrow \bar\psi\, U^{+}(U\lambda\cdot V U^{+})U\psi$$

$$\Rightarrow \vec\lambda\cdot\vec V' = U\vec\lambda\cdot\vec V\, U^{+}$$

$$Tr\, \lambda^i(\lambda\cdot V') = 2V_i' = Tr(\lambda^i U \lambda^j U^{+})\, V^j$$

$$V_i' = R_{ij} V^j$$

$$R_{ij} \leftarrow \tfrac{1}{2} Tr(\lambda^i U \lambda^j U^{+})$$

$$Tr\,(\vec\lambda\cdot\vec V)(\vec\lambda\cdot\vec V) = 2V^iV^i \rightarrow Tr(U\lambda\cdot V \underbrace{U^{+} U}\lambda\cdot V U^{+}) = Tr\,((\vec\lambda\cdot\vec V)(\vec\lambda\cdot\vec V))$$

SU(N) Gauge Theory

U(1) $\qquad \psi \to \psi' = e^{i\alpha_0}\psi \qquad \alpha_0 = \alpha(x)$

$$A_\mu \Rightarrow A'_\mu = \left(A_\mu - \frac{1}{e}\partial_\mu \alpha_0\right)$$

SU(N) $\qquad \psi \to \psi' = U\psi \qquad U = e^{-i\frac{\vec{\alpha}(x)\cdot\vec{\lambda}}{2}}$

SU(N) gauge trans

$\bar{\psi}\psi$ invariant but $\bar{\psi}\not{\partial}\psi$ is not

D_μ = gauge covariant deriv.

$$\boxed{D_\mu \to D'_\mu = U D_\mu U^+}$$

if that $\quad \bar{\psi}\, i\not{D}\psi \to \bar{\psi}U^+(U\not{D}U^+)U\psi = \bar{\psi}\, i\not{D}\psi$

Constructions

$$D_\mu = \partial_\mu + i\frac{g}{2}\vec{\lambda}\cdot\vec{A}_\mu$$

gauge field A^i_μ — $i = 1 \ldots (N^2-1)$

$$\vec{\lambda}\cdot\vec{A}_\mu = \lambda^i A^i_\mu$$

$$D'_\mu = \partial_\mu + i\frac{g}{2}\vec{\lambda}\cdot\vec{A}'_\mu = U\left[\partial_\mu + ig\frac{\vec{\lambda}\cdot\vec{A}_\mu}{2}\right]U^\dagger$$

$$= \partial_\mu + U(\partial_\mu U^\dagger) + i\frac{g}{2}U(\vec{\lambda}\cdot\vec{A}_\mu)U^\dagger$$

$$\frac{\vec{\lambda}\cdot\vec{A}'_\mu}{2} = U\frac{\vec{\lambda}\cdot\vec{A}}{2}U^\dagger - \frac{i}{g}U(\partial_\mu U^\dagger) \qquad \star\star\star$$

$$A'^i_\mu = \mathrm{Tr}\left(\lambda^i \frac{\lambda\cdot A'_\mu}{2}\right) = R_{ij}A^j_\mu - \frac{i}{g}\mathrm{Tr}\left(\lambda^i U(\partial_\mu U^\dagger)\right)$$

$\uparrow$ 'rotation' $\qquad\qquad \uparrow$ gauge

small $\vec{\alpha}$ $\qquad A'^i_\mu = A^i - \frac{1}{g}\partial_\mu\alpha^i + f^{ijk}A^k_\mu\alpha^j$

Invariants $\bar{\Psi} i \not{D} \Psi$ ✓

$\underbrace{-\frac{1}{4} F_{\mu\nu} F^{\mu\nu}}_{U(1)} \longrightarrow ?$ $\qquad \mathcal{L} = -\frac{1}{4} \underbrace{F^i_{\mu\nu}}_{} F^{i\mu\nu}$

$F^i_{\mu\nu} = ?$

Construction

$$[D_\mu , D_\nu] \rightarrow [U D_\mu U^+ , U D_\nu U^+] = U [D_\mu , D_\nu] U^+$$

$$[D_\mu , D_\nu] = \frac{ig}{2} \lambda^i F^i_{\mu\nu} \Rightarrow \lambda^i F'^i_{\mu\nu} = U(\lambda^i F^i_{\mu\nu}) U^+$$

$$= \left[\left(\partial_\mu + i\frac{g}{2}\lambda^i A^i_\mu\right), \left(\partial_\nu + i\frac{g}{2}\lambda^j A^j_\nu\right)\right] = i\frac{g}{2} \lambda^i \left(\partial_\mu A^i_\nu - \partial_\nu A^i_\mu\right) + \left(\frac{ig}{2}\right)^2 \underbrace{[\lambda^i A^i_\mu , \lambda^j A^j_\nu]}_{2 i f^{ijk} \lambda^k A^i_\mu A^j_\nu}$$

$$F^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu - g f^{ijk} A^j_\mu A^k_\nu \qquad \cancel{\star}\cancel{\star}\star$$

Find invariant

$$Tr(\lambda^i F^i_{\mu\nu} \; \lambda^j F^{j\,\mu\nu}) \longrightarrow Tr(\underbrace{U \lambda^i F^i_{\mu\nu} U^+}\; \underbrace{U \lambda^j F^{j\,\mu\nu} U^+}) = \text{inv.}$$

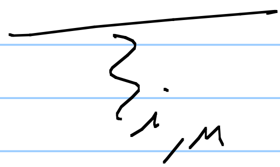
$$\hookrightarrow 2 F^i_{\mu\nu} F^{i\,\mu\nu}$$

Overall

$$\mathcal{L} = -\frac{1}{4} \underbrace{F^i_{\mu\nu} F^{i\,\mu\nu}} + \bar{\psi}(i\not{D} - m)\psi \qquad \leftarrow \star\star$$

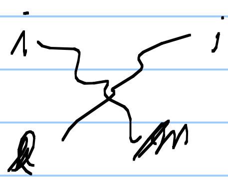
Content $\quad \psi_F = \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_N \end{pmatrix} \qquad N$ fields

$A^i_\mu \qquad (N^2-1)$ fields



$$= -i\frac{g}{2}\lambda^i \gamma_\mu$$

$$= -ig f^{ijk}(p_1 - p_2)^\mu + \text{permutations}$$



$$= -g^2 f^{ijk} f^{lmk} g_{\mu\nu} g_{\lambda\sigma} + \dots \text{perms}$$

Applications

"colors"

1) QCD $\qquad N=3 \qquad \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}$

$A^i \qquad i = 1 \ldots 8$ gluons

2) Weak $\qquad N=2 \qquad \psi = \begin{pmatrix} u \\ d \end{pmatrix} \begin{pmatrix} \nu \\ e^- \end{pmatrix} \begin{pmatrix} c \\ s \end{pmatrix} \ldots$

$A^i \to i = 1, 2, 3$

$A^{1 \pm i 2} \to W^{\pm}$

$A^3 \text{ "}\to\text{" } Z^0 \qquad$ (really $SU(2) \times U(1)$)

B_μ

$A^3_\mu \, B_\mu \to Z^0, \gamma$