AMERICAN JOURNAL of PHYSICS

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Citation: American Journal of Physics **54**, 827 (1986); doi: 10.1119/1.14423
View online: http://dx.doi.org/10.1119/1.14423
View Table of Contents: http://scitation.aip.org/content/aapt/journal/ajp/54/9?ver=pdfcov
Published by the American Association of Physics Teachers

cross-sectional area and the amount of material lumped on the end depends only on the maximum value of ω allowed. As ω_{max} is allowed to increase material is taken from the lump and placed in the supporting rod, and the maximum energy that can be stored also increases asymptotically approaching the limit $E_{max} = \frac{1}{2}S_m V_o$ as the lump is depleted.

This problem has several relevant aspects, summarized by the following:

(1) By using simple calculus a student can deduce the four physical constraints on A.

(2) After obtaining the four mathematical relations that describe what is demanded of A, simple reasoning will lead a student in the proper direction to obtain a solution.

(3) The introduction of the Dirac delta function in mechanics to describe the simple physical fact of wanting to put a finite amount of material at one location will make it easier to introduce and use the delta function in other areas of physics.

(4) After simple reasoning suggests that the stress should be constant in order to produce the maximum energy it can be proved to be correct with simple calculus that a junior physics major has had.

(5) Investigating the effect that can be produced by using a discontinuous function to describe something physical in mechanics will make the idea easier when it is encountered in other areas of physics.

(6) The problem is very timely/current in that energy storage devices are being researched extensively.

[1]R. Post and S. Post, Sci. Am. **229** (6), 17–23 (1973).

Quantum mechanics in curved space

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(Received 28 March 1985; accepted for publication 24 September 1985)

The quantum mechanical wave equation for a particle in a Schwarzschild metric is derived using general relativity in the linearized approximation. The significance of the various terms in the effective Hamiltonian is discussed and shown to be associated with the post-Newtonian terms for the corresponding classical motion.

I. INTRODUCTION

Although general relativity is nearly a decade older than quantum mechanics, only the latter is a staple of the undergraduate curriculum. One of the reasons for this is the mathematical sophistication required to handle the full general relativistic formalism. However, in most applications the full machinery is unnecessary. Since general relativistic corrections depend upon[1]

$$-\frac{\phi_g(r)}{c^2} = \frac{GM}{rc^2} \approx \begin{cases} 10^{-9}, & \text{surface of the Earth,} \\ 10^{-6}, & \text{surface of the Sun,} \\ 10^{-4}, & \text{surface of a white dwarf,} \end{cases} \quad (1)$$

where $\phi_g(r)$ is the usual gravitational potential, corrections to Newtonian mechanics are in most situations very small and can be treated perturbatively via the simple linearized version of the general relativistic equations.[2]

Many articles in this Journal have dealt with classical aspects of general relativity,[3] but only a few have been concerned with the result of merging general relativity and quantum mechanics.[4] Nevertheless, some fascinating physics arises when the two subjects are married, and this could provide stimulating supplementary material in a quantum mechanics course.

In this paper we discuss the quantum mechanics of a particle moving in a weak gravitational field—$\phi_g/c^2 \ll 1$—including kinematics which go beyond the simple nonrelativistic limit. The relativistic effects reveal interesting aspects of general relativity and allow a derivation of the bending of starlight by the sun from a wave mechanical viewpoint.

For simplicity and relevance to a realistic situation, we consider the case of the Schwarzschild metric, which describes space-time in the vicinity of a spherically symmetric mass distribution, say the Earth or the Sun. In "isotropic" coordinates, the proper time interval can be written as[5]

$$d\tau^2 = \left(\frac{1 - GM/2rc^2}{1 + GM/2rc^2}\right)^2 dt^2 - \left(1 + \frac{GM}{2rc^2}\right)^4 \frac{1}{c^2} d\mathbf{r} \cdot d\mathbf{r}$$
$$\equiv g_{\mu\nu} dx^\mu dx^\nu . \quad (2)$$

In the linearized approximation referred to above, wherein only terms up to first order in ϕ_g/c^2 are retained we have then

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} , \quad (3)$$

where

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad (4)$$

is the usual Minkowski metric and

$$h_{\mu\nu} = (2\phi_g/c^2)\delta_{\mu\nu}$$

$$= \begin{pmatrix} 2\phi_g/c^2 & 0 & 0 & 0 \\ 0 & 2\phi_g/c^2 & 0 & 0 \\ 0 & 0 & 2\phi_g/c^2 & 0 \\ 0 & 0 & 0 & 2\phi_g/c^2 \end{pmatrix} \quad (5)$$

is the linearized correction term.

Our plan for the paper is as follows. In Sec. II we review the classical motion arising in the linearized Schwarzschild approximation, while in Sec. III we present the parallel quantum mechanical discussion. Our results are summarized in Sec. IV.

II. CLASSICAL MECHANICS IN THE SCHWARZSCHILD FIELD

Before undertaking our discussion of quantum mechanics in curved space-time, we first review the more familiar classical situation. Recall that in flat space the relativistic action may be taken to be[6]

$$S(\tau) = \int_0^\tau \frac{1}{2} m \frac{dx^\mu}{ds}\frac{dx^\nu}{ds}\eta_{\mu\nu} ds\,, \quad (6)$$

where τ is the proper time. Making a variation

$$x^\mu(s) \to x^\mu(s) + \delta x^\mu(s) \quad (7)$$

and the demanding that the action be an extremum,

$$\delta S = 0\,, \quad (8)$$

subject to the boundary conditions $\delta x^\mu(0) = \delta x^\mu(\tau) = 0$, we find the Euler–Lagrange equation corresponding to the motion of a freely moving particle

$$\left(\frac{d}{ds}\frac{\partial}{\partial(dx^\lambda/ds)} - \frac{\partial}{\partial x^\lambda}\right)\frac{1}{2} m\eta_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = m\frac{d}{ds}\frac{dx_\lambda}{ds} = 0\,. \quad (9)$$

Here s is a parameter which for massive particles can be identified with the proper time τ. However we will use the symbol s to simplify later discussion of massless particles.

In curved space the only change is that the flat space metric tensor $\eta_{\mu\nu}$ is replaced by the curved space metric[7]

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}\,. \quad (10)$$

Working to first order in $h_{\mu\nu}$ we find

$$\begin{aligned} 0 &= \left(\frac{d}{ds}\frac{\partial}{\partial(dx^\mu/ds)} - \frac{\partial}{\partial x^\mu}\right)\frac{m}{2} g_{\lambda\nu}\frac{dx^\lambda}{ds}\frac{dx^\nu}{ds} \\ &= \frac{d}{ds}\left(m\, g_{\mu\nu}\frac{dx^\nu}{ds}\right) - \frac{m}{2}\frac{\partial}{\partial x^\mu} g_{\lambda\nu}\frac{dx^\lambda}{ds}\frac{dx^\nu}{ds} \\ &= m\left(g_{\mu\nu}\frac{d^2x^\nu}{ds^2} + \frac{\partial}{\partial x^\lambda} g_{\mu\nu}\frac{dx^\lambda}{ds}\frac{dx^\nu}{ds} - \frac{1}{2}\frac{\partial}{\partial x^\mu} g_{\lambda\nu}\frac{dx^\lambda}{ds}\frac{dx^\nu}{ds}\right). \end{aligned} \quad (11)$$

Then

$$g_{\mu\lambda}\frac{d^2x^\lambda}{ds^2} = \frac{1}{2}\partial_\mu h_{\lambda\nu}\frac{dx^\lambda}{ds}\frac{dx^\nu}{ds} - \partial_\lambda h_{\mu\nu}\frac{dx^\lambda}{ds}\frac{dx^\nu}{ds}\,, \quad (12)$$

which is the equation of motion for a classical particle. If we now specialize to the Schwarzschild metric in isotropic coordinates

$$h_{\mu\nu} = \delta_{\mu\nu}(2\phi_g/c^2)\,, \quad (13)$$

we find

$$\frac{d^2x^\mu}{ds^2} = \partial^\mu\phi_g\frac{dx^\lambda}{ds}\frac{dx^\nu}{ds}\delta_{\lambda\nu} - 2\partial_\lambda\phi_g\frac{dx^\lambda}{ds}\frac{dx^\sigma}{ds}\eta^{\sigma\mu}\,. \quad (14)$$

Finally, one can calculate the ordinary acceleration, d^2x^μ/dt^2, yielding

$$\begin{aligned} \frac{d^2}{ds^2}x^\mu &= \left(\frac{dt}{ds}\frac{d}{dt}\right)\frac{dt}{ds}\frac{dx^\mu}{dt} \\ &= \partial^\mu\phi_g\left(\frac{dt}{ds}\right)^2\frac{dx^\lambda}{dt}\frac{dx^\nu}{dt}\delta_{\lambda\nu} - 2\partial_\lambda\phi_g\left(\frac{dt}{ds}\right)^2\frac{dx^\lambda}{dt}\frac{dx^\sigma}{dt}\eta^{\sigma\mu} \\ &= \left(\frac{ds}{dt}\right)^{-2}\frac{d^2x^\mu}{dt^2} - \left(\frac{ds}{dt}\right)^{-3}\frac{d^2s}{dt^2}\frac{dx^\mu}{dt}\,. \end{aligned} \quad (15)$$

Then

$$\frac{dv_i}{dt} \cong -\nabla_i\phi_g(1 + v^2/c^2) + (4/c^2)\mathbf{v}\cdot\nabla\phi_g v_i\,, \quad (16)$$

which is the equation of motion of a particle moving in curved space described by the Schwarzschild metric.[8]

To lowest order in v/c, we have

$$\frac{d\mathbf{v}}{dt} = -\nabla\phi_g\,, \quad (17)$$

which is Newton's law for a particle moving in a gravitational field described by potential ϕ_g. At large velocities one must use the full post-Newtonian form given in Eq. (16). Thus for a photon, one has

$$\frac{du_i}{dt} = -2\nabla_i\phi_g + 4\hat{\mathbf{u}}\cdot\nabla\phi_g\hat{u}_i\,. \quad (18)$$

This factor of 2 multiplying the gradient of the potential is well known. For a photon passing near the rim of the Sun (see Fig. 1) we calculate the deflection angle to be

$$\frac{\Delta u_y}{u_x} \cong \Delta\theta = \frac{2}{c}\int_{-\infty}^{\infty}\frac{GM}{r^3(t)}b\,dt = \frac{2GM}{c}b\int_{-\infty}^{\infty}\frac{dt}{(b^2 + c^2t^2)^{3/2}} = \frac{4GM}{c^2b}\,, \quad (19)$$

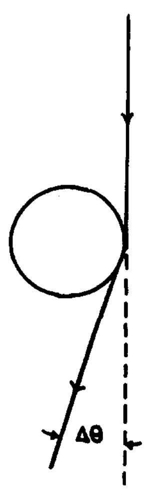

Fig. 1. Deflection of starlight as it passes the rim of the Sun.

which differs by this factor of 2 from the simple nonrelativistic result obtained by use of Eq. (17).

In order to see how corresponding results arise quantum mechanically, one must study how space-time curvature modifies usual wave equations. This is done in the succeeding section.

III. QUANTUM MECHANICS IN THE SCHWARZSCHILD FIELD

The wave equation for a quantum mechanical particle in curved space-time has been the subject of previous articles in this Journal.[4] However, these works have by and large been devoted to formal discussions of relativistic aspects and have not concentrated on the physics contained therein, as shall be our purpose here. We shall consider the more familiar (spin $\frac{1}{2}$) case first. As shown by Leiter the Dirac equation becomes[4]

$$[i\gamma^a e_a^\mu(\partial_\mu - \tfrac{1}{4}\sigma^{ab}e_a^\nu e_{b\nu;\mu}) - m]\psi = 0\,, \qquad (20)$$

where γ^a and $\sigma^{ab} = (i/2)[\gamma^a,\gamma^b]$ are the conventional Dirac matrices.[9] Here $e_a^\mu(x)$, which satisfies

$$g_{\mu\nu}(x) = \eta_{ab}e_\mu^a(x)e_\nu^b(x)\,, \qquad (21)$$

is called the vierbein, and

$$e_{b\nu;\mu} = \partial_\mu e_{b\nu} - \Gamma^\lambda_{\nu\mu}e_{b\lambda}\,, \qquad (22)$$

where

$$\Gamma^\lambda_{\nu\mu} = \tfrac{1}{2}g^{\lambda\eta}(g_{\lambda\nu,\mu} + g_{\lambda\mu,\nu} - g_{\mu\nu,\lambda}) \qquad (23)$$

is the usual affine connection. This is a rather complex equation in general. However, if we simplify to the linearized Schwarzschild approximation, we find

$$e_a^\mu(x) = \delta_a^\mu + \tfrac{1}{2}h_a^{\ \mu}(x) \qquad (24)$$

and the wave equation becomes (note from hereon we shall take $c = 1$ in order to simplify the equations)

$$\begin{aligned}0 &= \Big[i\gamma^\mu\partial_\mu - m + \frac{i}{2}\gamma^a h^{\ \mu}_a\partial_\mu \\ &\quad - \tfrac{1}{8}\gamma^\mu\sigma^{ab}(\partial_b h_{a\mu} - \partial_a h_{b\mu})\Big]\psi \\ &= [i\gamma^0(1+\phi_g)\partial_0 + i\boldsymbol{\gamma}\cdot(1-\phi_g)\nabla - m]\psi\,. \end{aligned} \qquad (25)$$

Rewriting this result in Schrödinger formalism, we have

$$i\frac{\partial}{\partial t}\psi = H\psi\,, \qquad (26)$$

where

$$\begin{aligned}H = &-i\gamma_0\boldsymbol{\gamma}(1-\phi_g)\cdot\nabla + m\gamma_0 - i\phi_g\frac{\partial}{\partial t} + i\frac{1}{2}\gamma_0\boldsymbol{\gamma}\cdot\nabla\phi_g \\ &- i\gamma_0\boldsymbol{\gamma}(1-2\phi_g)\cdot\nabla + \gamma_0 m(1-\phi_g) + \frac{i}{2}\gamma_0\boldsymbol{\gamma}\cdot\nabla\phi_g\,.\end{aligned} \qquad (27)$$

This Hamiltonian is Hermitian when the requisite spatial integrations are carried out using the correct measure[10]

$$\langle H\rangle = \int d^3r\, J\psi^\dagger H\psi\,, \qquad (28)$$

where the Jacobian J is given by

$$J = \sqrt{\det g_{ij}} = (1 - 3\phi_g)\,. \qquad (29)$$

However, it is more convenient to write the wavefunction so that H is Hermitian with respect to the usual measure $\int d^3r$. We do this by means of a unitary transformation

$$\begin{aligned}&\psi\to\tilde\psi = (1-\tfrac{3}{2}\phi_g)\psi \equiv U\psi\,, \\ &H\to\tilde H = UHU^{-1} = -i\gamma_0\boldsymbol{\gamma}(1-2\phi_g)\cdot\nabla + i\gamma_0\boldsymbol{\gamma} \\ &\qquad\cdot\nabla\phi_g + \gamma_0 m(1-\phi_g)\,.\end{aligned} \qquad (30)$$

In order to understand the significance of this Hamiltonian we write

$$\tilde H = \gamma_0 m + \gamma_0\epsilon + O\,, \qquad (31)$$

where

$$\begin{aligned}&\epsilon = \phi_g\,, \\ &O = -i\gamma_0\boldsymbol{\gamma}\cdot[(1-2\phi_g)\nabla - \tfrac{3}{2}\nabla\phi_g]\,,\end{aligned} \qquad (32)$$

and apply a Foldy–Wouthuysen transformation as outlined by Bjorken and Drell,[9] yielding the effective Schrödinger Hamiltonian

$$\begin{aligned}H_s &= m + \gamma_0\epsilon + \frac{1}{2m}O^2 - \frac{\gamma_0}{8m^2}[O,[O,\epsilon]] \\ &= m + m\phi_g + \frac{p^2}{2m} - \frac{p^4}{8m^3} + \frac{3}{2} \\ &\quad\times\left(-\frac{\phi_g}{m}p^2 + \frac{i\hbar}{m}\nabla\phi\cdot\mathbf{p} - \frac{h}{2m}\boldsymbol{\sigma}\cdot\nabla\phi_g\ x\mathbf{p}\right) + \cdots\end{aligned} \qquad (33)$$

Let us now see if we can understand the origin of these terms. The piece of the Hamiltonian independent of the potential is simply the relativistic energy

$$\sqrt{m^2+p^2} = m + p^2/2m - p^4/8m^3 + \cdots\,. \qquad (34)$$

In the case of ϕ_g-dependent terms, the component

$$V_0 = m\phi_g \qquad (35)$$

is obviously the usual gravitational potential energy. However, the other contributions are less familiar. The term

$$V_1 = -(3/2m)\phi_g p^2 \qquad (36)$$

is simply the post-Newtonian correction to the potential required by general relativity. We can see this by calculating the quantum mechanical acceleration via

$$i\frac{d}{dt} = [H\,, \qquad (37)$$

which yields the usual Newtonian result

$$\mathbf{a} = -[H_0,[H_0,\mathbf{r}]] = -\nabla\phi_g\,, \qquad (38)$$

if we employ the simple form

$$H_0 = m + p^2/2m + m\phi_g\,. \qquad (39)$$

However, if we now append the next-order terms

$$H = m + \frac{p^2}{2m} - \frac{p^4}{8m^3} + m\phi_g - \frac{3}{2m}\phi_g p^2\,, \qquad (40)$$

we determine

$$\begin{aligned}\mathbf{a} &= -[H,[H,\mathbf{r}]] \\ &= -\nabla\phi_g - (1/m^2)p^2\nabla\phi_g + (4/m^2)\nabla\phi_g\cdot\mathbf{pp}\,,\end{aligned} \qquad (41)$$

in agreement with the classical expression, Eq. (16).

The origin of the remaining contributions

$$V_2 = (3/2m)(i\hbar\nabla\phi_g\cdot\mathbf{p} - \tfrac{1}{2}\hbar\boldsymbol{\sigma}\cdot\nabla\phi_g\times\mathbf{p}) \qquad (42)$$

is more subtle. That these are quantum mechanical in origin can be inferred from the factors of $\hbar$. They arise from the problem of writing the quantum mechanical Hamiltonian when both momentum and position dependence is involved. Thus, the correct quantum mechanical form of the classical term

$$V_1 = (3/2m)\phi_g \boldsymbol{\sigma}\cdot\mathbf{p}\boldsymbol{\sigma}\cdot\mathbf{p}\,, \tag{42}$$

required by general relativity must be obtained using so-called Weyl ordering[11]

$$\begin{aligned} V_1^{\mathrm{Weyl}} &= -(3/2m)(\tfrac{1}{4}\boldsymbol{\sigma}\cdot\mathbf{p}\boldsymbol{\sigma}\cdot\mathbf{p}\phi_g + \tfrac{1}{2}\boldsymbol{\sigma}\cdot\mathbf{p}\phi_g\boldsymbol{\sigma}\cdot\mathbf{p} \\ &\quad + \tfrac{1}{4}\phi_g\boldsymbol{\sigma}\cdot\mathbf{p}\boldsymbol{\sigma}\cdot\mathbf{p}) \\ &= -(3/2m)(\phi_g p^2 - i\hbar\nabla\phi_g\cdot\mathbf{p} \\ &\quad + (\hbar/2)\boldsymbol{\sigma}\cdot\nabla\phi_g\times\mathbf{p}) \end{aligned} \tag{43}$$

in agreement with the Foldy–Wouthuysen result, Eq. (33). The "physical" origin of these additional terms can be understood from the feature that, quantum mechanically, the position and momentum of a particle are uncertain by amounts $\delta\mathbf{x}$ and $\delta\mathbf{p}$, respectively. Of course, the Heisenberg uncertainty relation requires that

$$\delta x_i\,\delta p_j \sim \hbar\delta_{ij}\,. \tag{44}$$

Thus, if we write the post-Newtonian term [Eq. (42)] and take account of this "Zitterbewegung" motion we find[12]

$$\begin{aligned} V &\sim (3/2m)(\phi_g(\mathbf{r}) + \nabla\phi_g\cdot\delta\mathbf{r})(p_i+\delta p_i)(p_i+\delta p_i) \\ &\sim (3/2m)(\phi_g(\mathbf{r})p^2 + 2\hbar\nabla\phi_g\cdot\mathbf{p})\,. \end{aligned} \tag{45}$$

The post-Newtonian potential V_1, of course, can be "observed" via the classical motion of the particle. It is interesting to ask whether there is any way to observe the corresponding "Zitterbewegung" terms. The answer is yes, in principle. The point is that such terms are parity violating. Thus in a hydrogen atom they will produce an interaction potential of the form

$$V \sim (3/2m_e)\hbar\nabla\phi_g\cdot(\nabla - (i/2)\boldsymbol{\sigma}_e\times\nabla)\,. \tag{46}$$

This term, which is P and CP violating will lead to a mixing between say S and P states and can be detected, for example, by measuring a circular polarization in the radiative decay of a hydrogen atom. This effect would be negligible ($P_\gamma \sim 10^{-9}$) on the surface of the earth but could approach unity near a neutron star. Other possible methods of detection have been catalogued by Fischbach, Freedman, and Cheng.[13]

The universality of these effects can be seen by considering the case of a spin zero particle. In flat space the particle will obey the Klein–Gordon equation[14]

$$(\eta^{\mu\nu}\partial_\mu\partial_\nu + m^2)\phi = 0\,. \tag{47}$$

However, in curved space the derivatives must be replaced by co-variant derivatives and the equation becomes

$$[g^{\mu\nu}(\partial_\mu\partial_\nu - \Gamma^\lambda_{\mu\nu}\partial_\lambda) + m^2]\phi = 0\,. \tag{48}$$

Specializing now to the linearized Schwarzschild case we find

$$[(1+2\phi_g)\partial_0^2 - (1-2\phi_g)\nabla^2 + m^2]\phi = 0\,. \tag{49}$$

In order to understand the physics of this equation we again follow Bjorken and Drell in writing the Klein–Gordon equation in first-order form[15]

$$i\frac{\partial}{\partial t}\Phi = H\Phi\,, \tag{50}$$

where

$$\begin{aligned} \Phi &= \begin{pmatrix}\psi\\ \chi\end{pmatrix}, \\ \psi &= \frac{1}{2}\left(\phi + \frac{i}{m(1+\phi_g)}\dot{\phi}\right), \\ \chi &= \frac{1}{2}\left(\phi - \frac{i}{m(1+\phi_g)}\dot{\phi}\right), \\ H &= -(1-3\phi_g)\begin{pmatrix}1 & 1\\ -1 & -1\end{pmatrix}\frac{\nabla^2}{2m} \\ &\quad + m(1+\phi_g)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}. \end{aligned} \tag{51}$$

Defining the two by two matrices

$$\eta = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \quad\text{and}\quad \rho = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}, \tag{52}$$

we can write

$$H = m(1 + \phi_g\eta + \epsilon + O)\,, \tag{53}$$

where

$$\begin{aligned} \epsilon &= -(1-3\phi_g)(\nabla^2/2m)\eta\,, \\ O &= -\rho(1-3\phi_g)(\nabla^2/2m)\,, \end{aligned} \tag{54}$$

and may now use the Foldy–Wouthuysen transformation as before to yield the effective Schrödinger equation

$$\begin{aligned} H_S &= m(1+\phi_g) + \eta\epsilon + O^2/2m + \cdots \\ &= m + m\phi_g + \frac{p^2}{2m} - \frac{p^4}{8m^3} - \frac{3}{2m}\phi_g p^2 + \cdots. \end{aligned} \tag{55}$$

Finally, we transform the Hamiltonian in terms of the measure d^3r as before yielding

$$\begin{aligned} H_S' &= (1+\tfrac{3}{2}\phi_g)H_S(1-\tfrac{3}{2}\phi_g) \\ &= m + m\phi_g + \frac{p^2}{2m} - \frac{p^4}{8m^3} - \frac{3}{2m} \\ &\quad \times(\phi_g p^2 - i\hbar\nabla\phi_g\cdot\mathbf{p})\,, \end{aligned} \tag{56}$$

which is identical to that in the spin $\frac{1}{2}$ case except for the absence of spin dependence. Obviously the physics is the same.

An interesting feature arises if we consider the massless case. Instead of seeking an effective Hamiltonian (which is only valid nonrelativistically) we instead attempt to solve the equation

$$[(1+2\phi_g)\partial_0^2 - (1-2\phi_g)\nabla^2]\phi = 0 \tag{57}$$

directly. This can also be written as

$$[(1+4\phi_g)\partial_0^2 - \nabla^2]\phi = 0\,. \tag{58}$$

Now assume a plane waveform

$$\phi(\mathbf{r},t) = f(\mathbf{r},t)e^{i\mathbf{k}\cdot\mathbf{r}}\,. \tag{59}$$

Then an approximate solution is

$$f(\mathbf{r},t) = e^{-i|k|[1-2\phi_g(\mathbf{r})]t}\,. \tag{60}$$

That is, it is as if there exists a position-dependent index of refraction[16]

$$\omega = |\mathbf{k}|[1/n(\mathbf{r})]\,, \tag{61}$$

where

$$n(\mathbf{r}) \cong 1 + 2\phi_g(\mathbf{r})\,. \tag{62}$$

In order to understand the significance of this suppose we consider the region about a particular point **R** in space. The wavefunction can then be written as

$$\phi(\mathbf{r},t) = C \exp\{i\mathbf{k}\cdot\mathbf{r} - i|k|t[1 - 2\phi_g(R) - 2(\mathbf{r}-\mathbf{R})\cdot\nabla\phi_g(\mathbf{R})]\}. \quad (63)$$

Choosing **R** as our origin of coordinates and

$$\mathbf{x} = \mathbf{r} - \mathbf{R}, \quad (64)$$

we have then

$$\phi(\mathbf{x},t) = C' \exp i\{[\mathbf{k} + 2|k|t\nabla\phi_g(R)]\cdot\mathbf{x} - i|k|[1 - 2\phi_g(R)]t\}, \quad (65)$$

so that the effective wavenumber is space and time-dependent. As the wave propagates its effective wavenumber varies with

$$d\mathbf{k} = 2|k|dt\,\nabla\phi_g(R). \quad (66)$$

If we consider a wave moving in the x direction at time $t = -\infty$ along a trajectory with impact parameter b with respect to the gravitational source, then we find

$$\Delta k_y = 2|k| \int_{-\infty}^{\infty} dt \frac{GbM}{(b^2+t^2)^{3/2}} = 4GM|k|/b. \quad (67)$$

Thus the angle of deflection is

$$\Delta\theta = \frac{\Delta k_y}{|k|} = \frac{4GM}{b}, \quad (68)$$

as before, but now derived from a wavefunction point of view.[17]

IV. SUMMARY

We have discussed the quantum mechanical wave equation relevant to the situation that a particle finds itself in the vicinity of a spherically symmetric mass distribution. To lowest order, of course, one finds the simple gravitational potential

$$V_0 = m\phi_g. \quad (69)$$

However, additional terms are also present, of two types. One term

$$V_1 = -(3/2m)\phi_g p^2, \quad (70)$$

represents the post-Newtonian correction to the classical motion. A second type

$$V_2 = (3/2m)i\hbar\nabla\phi_g\cdot\mathbf{p} \quad (71)$$

is quantum mechanical in origin and results from the fact that the position and momentum of a quantum mechanical particle are indefinite. Although effects of this term are small, it may be detectable through characteristic parity violating effects which can become sizeable near large gravitational sources.

In addition, we examined the wavefunction of a massless quantum mechanical particle and demonstrated how the Einstein deflection of starlight can be derived via a wave picture.

The presentation of these features by means of a general relativistic framework is unusual and yet elementary enough to be presented as supplementary material in an advanced undergraduate or a graduate quantum mechanics course.

ACKNOWLEDGMENT

Research supported in part by the National Science Foundation.

[1]See, e.g., H. C. Ohanian, *Gravitation and Spacetime* (Norton, New York, 1976); S. Weinberg, *Gravitation and Cosmology* (Wiley, New York, 1972); R. H. Price, Am. J. Phys. **50**, 300 (1982), and references therein. There exist many other standard references, of course, Misner, Thorne, and Wheeler; Adler, Bazin, and Schiffer; etc. However, for simplicity we shall reference items only to the three sources cited above.

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[10]The point here is that the invariant volume element is $\sqrt{\det g}\, d^4x$. (S. Weinberg, Ref. 1, pp. 98–100.)

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[12]J. Bjorken and S. Drell, Ref. 9 pp. 58–60; T. A. Welton, Phys. Rev. **74**, 1157 (1948).

[13]E. Fischbach, B. S. Freedman, and W. K. Cheng, Phys. Rev. D **23**, 2157 (1981); E. Fischbach in *Cosmology and Gravitation*, edited by P. G. Bergmann and V. De Sabbata (Plenum, New York, 1980), pp. 359–73.

[14]H. Kragh, Am. J. Phys. **52**, 1024 (1984); J. Bjorken and S. Drell, Ref. 10, pp. 198–203.

[15]J. Bjorken and S. Drell, Ref. 14.

[16]H. C. Ohanian, Ref. 1, p. 139.

[17]Of course, a photon is a spin 1 particle. However, general relativity assures us that all massless particles travel the same geodesic. Thus, we have treated the mathematically simpler spin 0 case.

PROBLEM: OSCILLATING BUOYANT SPHERE

In a familiar problem in elementary physics textbooks one shows that a floating vertical cylinder executes simple harmonic motion when it is displaced from its equilibrium position. This result is exact even if the oscillations have a large amplitude, as long as part of the cylinder remains submerged. If the cylinder is replaced by a sphere the problem is more complicated; the motion of the sphere is not simple harmonic.

Consider a sphere with radius R and specific gravity $\rho < 1$ which is held under water so that its entire volume is just submerged. An alternate choice of the initial state of the sphere will yield a somewhat different solution. If friction and the motion of the water are neglected, what is the motion of the sphere after it is released? (Solution is on p. 848.)