Twenty years of the Weyl anomaly

REVIEW ARTICLE

Twenty years of the Weyl anomaly†

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Abstract. In 1973 two Salam protégés (Derek Capper and the author) discovered that the conformal invariance under Weyl rescalings of the metric tensor $g_{\mu\nu}(x) \rightarrow \Omega^2(x)g_{\mu\nu}(x)$ displayed by classical massless-field systems in interaction with gravity no longer survives in the quantum theory. Since then these Weyl anomalies have found a variety of applications in black-hole physics, cosmology, string theory and statistical mechanics. We give a nostalgic review.

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When all else fails, you can always tell the truth.

Abdus Salam

1. Trieste and Oxford

Twenty years ago, Derek Capper and I had embarked on our very first postdocs here in Trieste. We were two Salam students fresh from Imperial College filled with ideas about quantizing the gravitational field: a subject which at the time was pursued only by mad dogs and Englishmen. (My thesis title: *Problems in the Classical and Quantum Theories of Gravitation* was greeted with hoots of derision when I announced it at the Cargese Summer School en route to Trieste. The work originated with a bet between Abdus Salam and Hermann Bondi about whether you could generate the Schwarzschild solution using Feynman diagrams. You can—and I did—but I never found out if Bondi ever paid up.)

Inspired by Salam, Capper and I decided to use the recently discovered *dimensional regularization*§ to calculate corrections to the graviton propagator from closed loops of massless particles: vectors [1] and spinors [2], the former in collaboration with Leopold Halpern. This involved the self-energy insertion

$$\Pi_{\mu\nu\rho\sigma}(p) = \int \mathrm{d}^n x \, \mathrm{e}^{\mathrm{i}px} \langle T_{\mu\nu}(x)T_{\rho\sigma}(0)\rangle|_{g_{\mu\nu}=\delta\mu\nu} \tag{1}$$

† Talk given at the *Salamfest*, ICTP, Trieste, March 1993.
‡ Research supported in part by NSF Grant PHY-9106593.
§ Dimensional regularization had just been invented by another Salam student and contemporary of ours, Jonathan Ashmore [3], and independently by Bollini and Giambiagi [4] and by 't Hooft and Veltman [5]. I briefly shared a London house with Jonathan Ashmore and Fritjof Capra. Both were later to leave physics, as indeed was Derek Capper. Ashmore went into biology, Capper into computer science and Capra into eastern mysticism (a decision in which, as far as I am aware, Abdus Salam played no part).

where n is the spacetime dimension and $T_{\mu\nu}(x)$ the energy–momentum tensor of the massless particles. One of our goals was to verify that dimensional regularization correctly preserved the Ward identity

$$p^\mu \Pi_{\mu\nu\rho\sigma}(p) = 0 \tag{2}$$

that follows as a consequence of general covariance. If we denote by ϵ the deviation from the physical spacetime dimension one is interested in, and expand about $\epsilon = 0$, we obtain

$$\Pi_{\mu\nu\rho\sigma} = \frac{1}{\epsilon}\Pi_{\mu\nu\rho\sigma}(\text{pole}) + \Pi_{\mu\nu\rho\sigma}(\text{finite}) \,. \tag{3}$$

Capper and I were able to verify that the pole term correctly obeyed the Ward identity

$$p^\mu \Pi_{\mu\nu\rho\sigma}(\text{pole}) = 0 \tag{4}$$

and that the infinity could then be removed by a generally covariant counterterm. We checked that the finite term also obeyed the identity

$$p^\mu \Pi_{\mu\nu\rho\sigma}(\text{finite}) = 0 \tag{5}$$

and hence that there were no diffeomorphism anomalies†.

We were also aware that since the massless particle systems in question were invariant under the Weyl transformations‡ of the metric

$$g_{\mu\nu}(x) \to \Omega^2(x) g_{\mu\nu}(x) \tag{6}$$

together with appropriate rescalings of the matter fields, this implied that the stress tensors in (1) should be traceless and hence that the self-energy should also obey the trace identity

$$\Pi^\mu{}_{\mu\rho\sigma}(p) = 0 \,. \tag{7}$$

We verified that the pole term was OK:

$$\Pi^\mu{}_{\mu\rho\sigma}(\text{pole}) = 0 \tag{8}$$

consistent with our observation that the counterterms were not only generally covariant but Weyl invariant as well. For some reason, however, I did not get around to checking the finite term until Christmas of '73 by which time I was back in England on my second postdoc, in Oxford, where Dennis Sciama was gathering together a group of quantum-gravity enthusiasts. To my surprise, I found that

$$\Pi^\mu{}_{\mu\rho\sigma}(\text{finite}) \neq 0 \,. \tag{9}$$

I contacted Derek and he confirmed that we hadn't goofed. The Weyl invariance (6) displayed by classical massless-field systems in interaction with gravity, first proposed by Hermann Weyl in 1918 [7–9], no longer survives in the quantum theory! We rushed off a paper [10] to *Nuovo Cimento* (How times have changed!).

I was also able to announce the result at the first Oxford Quantum Gravity Conference, organized by Isham, Penrose and Sciama, and held at the Rutherford Laboratory in February '74 [11]. Unfortunately, the announcement was somewhat overshadowed because Stephen Hawking chose the same conference to reveal to an unsuspecting world his result [12] that black holes evaporate! Ironically, Christensen and Fulling [13] were subsequently to show that in two spacetime dimensions the Hawking effect is due entirely to the trace anomaly. Two-dimensional black holes, and the effects of the Weyl anomaly in particular [14–16], are currently enjoying a revival of interest.

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† Had we looked at closed loops of Weyl fermions or self-dual antisymmetric tensors in 2 mod 4 dimensions, and had we been clever enough, we would have noticed that this Ward identity breaks down. But we did not and we were not, so this had to wait another ten years for the paper by Alvarez-Gaumé and Witten [6].

‡ For fermions this is true $\forall n$; for vectors only for $n = 4$.

2. Anomaly? What anomaly?

Some cynic once said that in order for physicists to accept a new idea, they must first pass through the following three stages:

(1) it's wrong,
(2) it's trivial,
(3) I thought of it first.

In the case of the Weyl anomaly, however, our experience was that (1) and (2) got interchanged. Being in Oxford, one of the first people we tried to impress with our new anomaly was J C Taylor who merely remarked 'Isn't that rather obvious?'. In a sense, of course, he was absolutely right. He presumably had in mind the well known result that theories which are Weyl invariant in n-dimensional curved space are automatically invariant under the conformal group $SO(2, n)$ in the flat-space limit, which implies in particular that the dilatation current $D^\mu \equiv x^\nu T^\mu{}_\nu$ is conserved:

$$\partial_\mu D^\mu = T^\mu{}_\mu + x^\nu \partial_\mu T^\mu{}_\nu = T^\mu{}_\mu = 0. \tag{10}$$

Moreover, one already knew from the work of Coleman and Jackiw [17] that such flat-space symmetries suffered from anomalies. Of course, the Weyl invariance (6), in contrast to the conformal group, is a *local* symmetry for which the dilatation current of (10) is not defined. Nevertheless, perhaps one should not be too surprised to discover that there is a curved-space generalization in the sense of a non-vanishing trace for the stress tensor. Consequently, Capper and I were totally unprepared for the actual response of the rest of the physics community: NO-ONE BELIEVED US! To be fair, we may have put some people off the scent by making the correct, but largely irrelevant, remark that at one loop the anomalies in the *two-point* function could be removed for $n \neq 2$ by adding finite local counterterms [10]. So we wrote another paper [18] stressing that the anomalies were real and could not be ignored, but to no avail. To rub salt in the wounds, among those dismissing our result as spurious were physicists for whom we youngsters had the greatest respect. First the Americans:

... *the finite W_{reg} that is left behind by Schwinger's method, after the infinities have been split off, is both coordinate invariant and conformally invariant*, DeWitt [19]

Something is wrong, Christensen [20]

The form of the conformal anomaly in the trace of the stress tensor proposed by a number of authors violates axiom 5, Wald [21]

Thus we find no evidence of the conformal trace anomalies reported by a number of other authors, Adler *et al* [22],

then the Europeans:

Conformal anomalies in a conformally invariant theory do not arise ..., Englert *et al* [23]

There are important, physically relevant differences: most noticeable, normalized energy–momentum tensors do not possess a 'trace anomaly', Brown and Ottewill [24],

especially the Russians:

The main assumption of our work is that a regularization scheme exists which preserves all the formal symmetry properties (including the Weyl symmetry). ...Therefore we hope dimensional regularization will give no anomalies..., Kallosh [25]

It turns out that conformal anomalies, discovered in gravitating systems, are not true anomalies, since in modified regularizations they do not arise.... The above point of view on conformal anomalies is shared by Englert et al, Fradkin and Vilkovisky [26]

The presence or absence of the conformal anomaly depends on the choice made between

two existing classes of covariant regularizations, Vilkovisky [27]

   and, to be democratic, let us not forget the Greeks:

   *The above method of regularization and renormalization preserves the Ward identities
... and trace anomalies do not arise*, Antoniadis and Tsamis [28]

   *There exists a regularization scheme which preserves both general coordinate invariance
and local conformal invariance (Englert et al) ...*, Antoniadis *et al* [29]

   *In fact one needs a regularization scheme which preserves the general coordinate and
Weyl invariance ... such a scheme exists (Englert† et al)*, Antoniadis *et al* [30].

3. London

As chance would have it, my third postdoc brought me to King's College, London, at the
same time as Steve Christensen, Paul Davies, Stanley Deser, Chris Isham and Steve Fulling.
Bill Unruh was also a visitor. It was destined therefore to become a hot-bed of controversy
and activity in Weyl anomalies. Provoked by Christensen and Fulling, who had not yet
been converted, Deser, Isham and I decided to write down the most general form of the
trace of the energy–momentum tensor in various dimensions [31]. By general covariance
and dimensional analysis, it must take the following form: For $n = 2$,

$$g^{\alpha\beta}\langle T_{\alpha\beta}\rangle = aR \tag{11}$$

where a is a constant. For $n = 4$,

$$g^{\alpha\beta}\langle T_{\alpha\beta}\rangle = \alpha R^2 + \beta R_{\mu\nu}R^{\mu\nu} + \gamma R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + \delta\Box R + c F_{\mu\nu}{}^a F^{\mu\nu a} \tag{12}$$

where $\alpha, \beta, \gamma, \delta$ and c are constants. (In (12) we have allowed for the possibility of an
an external gauge field [32] in addition to the gravitational field.) For $n = 6$, $g^{\alpha\beta}\langle T_{\alpha\beta}\rangle$
would have to be cubic in curvature and so on. (At one-loop, and ignoring boundary terms,
there is no anomaly for n odd). I showed these expressions to Steve Christensen, with
whom I was sharing an office, and asked him if he had seen anything like this before. He
immediately became very excited and told me that these were precisely the Schwinger–
Dewitt b_n coefficients. These are the t-independent terms that appear in the asymptotic
expansion of the heat kernel of the appropriate differential operators Δ:

$$\text{Tr } e^{-\Delta t} \sim \sum_{k=0}^{\infty} B_k t^{(k-n)/2} \qquad t \to 0^+ \tag{13}$$

where

$$B_k = \int \mathrm{d}^n x \, b_k . \tag{14}$$

For example, if Δ is the conformal Laplacian

$$\Delta = -\Box + \frac{R}{6} \tag{15}$$

† If one starts with a classically non-Weyl invariant theory (e.g. pure Einstein gravity) and artificially makes it
Weyl invariant by introducing via a change of variables $g'_{\mu\nu}(x) = e^{2\sigma(x)}g_{\mu\nu}(x)$ an unphysical scalar *spurion*
$\sigma(x)$, then unitarity guarantees that no anomalies can arise because this artificial Weyl invariance of the quantum
theory, $g'_{\mu\nu}(x) \to \Omega^2(x)g'_{\mu\nu}(x)$ with $e^{2\sigma(x)} \to \Omega^2(x) e^{2\sigma(x)}$, is needed to undo the field redefinition and remove
the spurious degree of freedom. Professor Englert informed me in Trieste that this is what the authors of [23]
had in mind when they said that anomalies do not arise. Let us all agree therefore that many of the apparent
contradictions are due to this misunderstanding. To summarize: real Weyl invariance has anomalies; pseudo-Weyl
invariance (i.e. that involving a spurion) does not. This is a regularization-scheme-independent statement.

then b_4 is given by

$$b_4 = \frac{1}{180(4\pi)^2}[-R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + \Box R]. \tag{16}$$

This was the road to Damascus for Steve as far as Weyl anomalies were concerned and, like many a recent convert, he went on to become their most ardent advocate[†]. This was also the beginning of a very fruitful collaboration between the two of us. The significance of my paper with Deser and Isham was that, with the exception of the $\Box R$ term in (12), none of the above anomalies could be removed by the addition of finite local counterterms (hence the title *non-local conformal anomalies*) and thus this laid to rest any lingering doubts about the inevitability of Weyl anomalies (or, at least, it should have done). By this time, or shortly afterwards, the Hawking radiation experts at King's and elsewhere were arriving at the same conclusion [13, 34–37] as in fact was Hawking himself [38]. There followed a series of papers calculating the numerical coefficients in (11) and (12) and confirming that these were indeed just the b_n coefficients [39–43]. These results were further generalized to self-interacting theories [44–48].

One day about this time I answered the phone in my office only to hear those five words most designed to instill fear and trembling into the heart of a young postdoc: 'Hi. This is Steven Weinberg'. Pondering on the non-renormalizability problem, Weinberg had become interested in quantum gravity in $2 + \epsilon$ dimensions [49, 50]. Inspired by workers in statistical mechanics, who frequently work with non-renormalizable field theories but who nevertheless manage to extract sensible predictions, Weinberg wondered whether this might be true for gravity: was the theory 'asymptotically safe'? The answer seemed to rely on the *sign* of the two-dimensional trace anomaly, i.e. on the constant a in (11). Accordingly, Weinberg set $n = 2$ in the n-dimensional calculations of [2] and concluded that fermions had the wrong sign:

$$a = \frac{1}{24\pi}. \tag{17}$$

He repeated our calculation for scalars himself and found the same sign and magnitude (consistent with the observation that in two dimensions there is a Bose–Fermi equivalence, and consistent with the black-hole calculations [13, 34]). In the case of vector bosons, however, he found from [1] that there was a sign flip. His question was simple but crucial: did I agree with him or could there be an overall sign error? Not wanting to be the victim of Weinberg's wrath should I get it wrong, I spent several frantic days and sleepless nights checking and rechecking the calculations. Those who have ever chased a minus sign and those who know Steve Weinberg will appreciate my discomfort! In fact I agreed. Unfortunately, asymptotic safety became asymptotically unpopular but my contact with Weinberg later led to a very fruitful semester in Austin, and to my continuing affection for the state of Texas.

The scalar terms of order $n/2$ in the curvature which appear in the n-dimensional gravitational trace anomaly are reminiscent of the pseudoscalar terms of order $n/2$ in the curvature which appear in the n-dimensional gravitational axial anomaly, as calculated by Delbourgo and Salam [51]. I was musing on this shortly after moving across town to Queen Mary College, when I saw a paper by Eguchi and Freund [52] on the then new and

† A delightful set of reminiscences on the Weyl anomaly by Steve Christensen can also be found in Bryce DeWitt's *festschrift* [33]. During their stay at King's, he and Fulling shared a flat in the London borough of Ealing (home of the famous *Ealing Comedy* movies). According to Christensen, the connection between trace anomalies and the Hawking effect occurred, Archimedes-like, to Fulling while taking a bath. He did not run through the streets of Ealing shouting 'Eureka', but did run upstairs to the payphone to tell Bill Unruh.

exciting topic of gravitational instantons. They considered the two topological invariants, the Pontryagin number, P, and the Euler number, χ, and posed the question: to what anomalies do P and χ contribute? In the case of the Pontryagin number, they were able to answer this question by relating P to the integrated axial anomaly; in the case of the Euler number, however, they found no anomaly. I therefore wrote a short note [53] relating χ to the integrated trace anomaly. As described in section 6, this result was later to prove important in the two-dimensional context of string theory, where

$$\chi = \frac{1}{4\pi} \int d^2x \sqrt{g} R \tag{18}$$

and hence from (11)

$$\frac{1}{4\pi} \int d^2x \sqrt{g} g^{\alpha\beta} \langle T_{\alpha\beta} \rangle = a\chi . \tag{19}$$

Unfortunately, the referee's vision did not extend that far and the paper was rejected. Rather than resubmit it, I decided to incorporate the results into a larger paper [54] which re-examined the Weyl anomaly in the light of its applications to the Hawking effect, to gravitational instantons, to asymptotic freedom and Weinberg's asymptotic safety. In the process, I discovered that the constants α, β, γ and δ are not all independent but obey the constraints

$$4\alpha + \beta = \alpha - \gamma = -\delta . \tag{20}$$

In other words, the gravitational contribution to the anomaly depends on only two constants (call them b and b') so that (12) may be written as

$$g^{\alpha\beta} \langle T_{\alpha\beta} \rangle = b\left(F + \tfrac{2}{3}\Box R\right) + b'G + cH \tag{21}$$

where

$$F = R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 2 R_{\mu\nu} R^{\mu\nu} + \tfrac{1}{3} R^2 \tag{22}$$

$$G = R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 4 R_{\mu\nu} R^{\mu\nu} + R^2 \tag{23}$$

and

$$H = F_{\mu\nu}{}^a F^{\mu\nu a} . \tag{24}$$

In four (but only four) dimensions

$$F = C^{\mu\nu\rho\sigma} C_{\mu\nu\rho\sigma} \tag{25}$$

where $C_{\mu\nu\rho\sigma}$ is the Weyl tensor, and G is proportional to the Euler number density

$$G = {}^* R_{\mu\nu\rho\sigma} {}^* R^{\mu\nu\rho\sigma} \tag{26}$$

where $*$ denotes the dual. Note the absence of an R^2 term in (21). This result was later rederived using the Wess–Zumino consistency conditions [55–59]. Furthermore, the constants a, b, b' and c are those which determine the counterterms

$$\Delta L = \frac{a}{\epsilon} \sqrt{g} R \qquad n = 2 \tag{27}$$

$$\Delta L = \frac{1}{\epsilon} \sqrt{g} (bF + b'G + cH) \qquad n = 4 \tag{28}$$

(and hence the renormalization group β functions) at the one-loop level. The Euler number counterterms are frequently ignored on the grounds that they are total divergences, but will nevertheless contribute in spacetimes of non-trivial topology. We emphasize that the above results are valid only for theories which are classically conformally invariant (e.g.

Maxwell/Yang–Mills for $n = 4$ only, and conformal scalars and massless fermions for both $n = 2$ and $n = 4$). For other theories (e.g. Maxwell/Yang–Mills for $n = 2$, pure quantum gravity for $n = 4$, or any theory with mass terms) the 'anomalies' will still survive, but will be accompanied by contributions to $g^{\alpha\beta}\langle T_{\alpha\beta}\rangle$ expected anyway through the lack of conformal invariance. Since the anomaly arises because the operations of regularizing and taking the trace do not commute, the anomaly in a theory which is not classically Weyl invariant may be defined as

$$\text{Anomaly} = g^{\alpha\beta}\langle T_{\alpha\beta}\rangle_{\text{reg}} - \langle g^{\alpha\beta}T_{\alpha\beta}\rangle_{\text{reg}}. \tag{29}$$

Of course, the second term happens to vanish when the classical invariance is present. This formula, in the context of Pauli–Villars regularization of flat-space dilatation anomalies, may also be found in [60].

Note that in an expansion about flat space with $g_{\mu\nu} = \delta_{\mu\nu} + h_{\mu\nu}$, R is $O(h)$, so it is sufficient to calculate the two-point function as in [10] to fix the a coefficient of R in (11). For $n = 4$, $\Box R$ is $O(h)$ while F and G are $O(h^2)$. Nevertheless, because of the constraint (20), a calculation of the two-point function is again sufficient to fix the b coefficient of $C^{\mu\nu\rho\sigma}C_{\mu\nu\rho\sigma}$ in (21), notwithstanding the ability to remove the $\Box R$ piece by a finite local counterterm, and notwithstanding some contrary claims in the literature. Indeed, this is how the coefficients of the Weyl invariant $C^{\mu\nu\rho\sigma}C_{\mu\nu\rho\sigma}$ counterterms were first calculated [1, 10].

Explicit calculations [1, 10, 2, 61, 13, 39, 37, 42, 40, 54] yield

$$b = \frac{1}{120(4\pi)^2}[N_S + 6N_F + 12N_V] \tag{30}$$

$$b' = -\frac{1}{360(4\pi)^2}[N_S + 11N_F + 62N_V] \tag{31}$$

where N_S, N_F and N_V are respectively the numbers of scalars, spin-$\frac{1}{2}$ Dirac fermions and vectors in the theory. Note that the contributions to the b coefficient are all positive, as they must be by spectral representation positivity arguments [2, 61, 59] on the vacuum polarization proper self-energy part in four dimensions.

I also noted that the non-local effective action responsible for the anomalies would contain a term

$$S_{\text{eff}} \sim \int \mathrm{d}^n x \sqrt{g}\, R \, \Box^{(n-4)/2} R . \tag{32}$$

By setting $n = 2$ one obtains what what later to be known as the *Polyakov action*, discussed in section 6.

4. Cosmology

The role of the Weyl anomaly in cosmology seems to fall into the following categories: inflation in the early universe, the vanishing of the cosmological constant in the present era, particle production and wormholes.

The first is reviewed in papers by Grischuk and Zeldovitch [62] and Olive [63]. Consider the semiclassical Einstein equations

$$R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R = 8\pi G\langle T_{\mu\nu}\rangle \tag{33}$$

where $\langle T_{\mu\nu}\rangle$ is the effective stress-tensor induced by quantum loops. In the inflationary phase, the geometry will be that of de Sitter space. But the trace anomaly for De Sitter

completely determines the energy–momentum because it must be a multiple of the metric by the symmetry†:

$$\langle T_{\mu\nu} \rangle = \tfrac{1}{4} g_{\mu\nu} g^{\alpha\beta} \langle T_{\alpha\beta} \rangle . \tag{34}$$

The idea that the trace anomaly might also have a bearing on the vanishing of the cosmological constant is a recurring theme [64, 28, 65–69] According to Tomboulis [66], models where the cosmological constant relaxes dynamically to zero via some dilaton-like scalar field suffer from an unnatural fine-tuning of the parameters. This problem can be cured, he claims, if the Wess–Zumino functional induced by the conformal anomaly is included. A similar approach has been taken by Antoniadis, Mazur and Mottola [67–69], who argue that four-dimensional gravity is drastically modifed at distances larger than the horizon scale, due to the large infrared quantum fluctuations of the conformal part of the metric, whose dynamics are governed by an effective action induced by the trace anomaly [70], analogous to the *Polyakov action* in two dimensions. Apparently, this leads to a conformally invariant phase in which the effective cosmological term necessarily vanishes. See also [71, 64, 72, 73] for a discussion of the cosmological constant/trace anomaly connection in the context of *spacetime foam* [71].

With regard to particle production, Parker [74] has used the trace anomaly to argue that there is no particle production by a gravitational field if spacetime is conformally flat and quantum fields are conformally coupled, but this has recently been challenged by Massacand and Schmid [75].

Finally, Grinstein and Hill [76] and also Ellis *et al* [77] have claimed that in Coleman's wormhole scenario [78], it is the trace anomaly that controls the behaviour of fundamental coupling constants, particle masses, mixing angles, etc.

5. Supersymmetry

The Weyl anomaly acquires a new significance when placed in the context of supersymmetry. In particular, Ferrara and Zumino [79] showed that the trace of the stress tensor $T^{\mu}{}_{\mu}$, the divergence of the axial current $\partial_{\mu} J^{\mu 5}$, and the gamma trace of the spinor current $\gamma^{\mu} S_{\mu}$ form a scalar supermultiplet. There followed a good deal of activity in calculating the corresponding anomalies in global supersymmetry.

In the period 1977–9, Christensen and I found ourselves in Boston: he at Harvard; I at Brandeis. We decided to look at these anomalies in supergravity. Since the supermultiplets involve fields $e_{\mu}{}^{a}, \psi_{\mu}, A_{\mu}, \chi, \phi$ with spins $2, \tfrac{3}{2}, 1, \tfrac{1}{2}, 0$ we first determined the axial and trace anomalies for fields of arbitary spin [80, 81]. See also [82–86]. One of our main motivations was to calculate the gravitational spin-$\tfrac{3}{2}$ axial anomaly‡ which was at the time unknown. Shortly afterwards (by which time I had come full circle and was back at Imperial College) van Nieuwenhuizen and I noted [87] that the gravitational trace anomaly for a field of given spin could depend on the field representation. Thus a rank two gauge field $\phi_{\mu\nu}$ yielded a different result from a scalar ϕ, even though they are dual to one another. Similarly, a rank three gauge field $\phi_{\mu\nu\rho}$ yielded a non-zero result, even though it is dual to nothing. However, these differences showed up only in the coefficient of the topological Euler number term, essentially because the combination of gauge field and ghost field differential operators Δ contributing to the B_4 coefficients of (14) differed only in their

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† The trace anomaly also determines the energy–momentum completely for a two-dimensional black hole and in the four-dimensional case it determines it up to one function of position [13].

‡ Our result, that the Rarita–Schwinger anomaly was −21 times the Dirac, generated almost as much incredulity as did the Capper–Duff Weyl anomaly, but that's another *festschrift*.

zero modes. The difference between the integrated trace anomalies must therefore be an integer.

At one loop in supergravity after going on-shell, we may write the anomaly as

$$g^{\alpha\beta}\langle T_{\alpha\beta}\rangle = \frac{A}{32\pi^2} {}^*R_{\mu\nu\rho\sigma} {}^*R^{\mu\nu\rho\sigma} \tag{35}$$

so that when (21) applies, $A = 32\pi^2(b+b')$. The contributions to the A coefficient from the various fields are shown in table 1. Note that the fermions are Majorana. The significance of these results lies in their application to the $D = 4, N = 8$ supergravity obtained by dimensional reduction from type II supergravity in $D = 10$ and the $D = 4, N = 4$ supergravity–Yang–Mills supermultiplets which arise from dimensional reduction from $N = 1$ supergravity–Yang–Mills in $D = 10$. As we can see from the field content given in the last three columns of table 1, the combined anomaly exactly cancels†. I have singled out these supermultiplets because these are precisely the field-theory limits of the toroidally compactified type II and heterotic superstrings. Indeed, these results have recently been confirmed in a direct string calculation by Antoniadis *et al* [116].

Table 1. Vanishing anomaly in $N = 8$ and $N = 4$ supermultiplets.

Field	$360A$	$N = 8$ supergravity	$N = 4$ supergravity	$N = 4$ Yang–Mills
e^a_μ	848	1	1	0
ψ_μ	−233	8	4	0
A_μ	−52	28	6	1
χ	7	56	4	4
ϕ	4	63	1	6
$\phi_{\mu\nu}$	364	7	1	0
$\phi_{\mu\nu\rho}$	−720	1	0	0
		$A = 0$	$A = 0$	$A = 0$

Another application of the trace anomaly in the context of supersymmetry concerned the *gauged* N-extended supergravities which exhibit a cosmological constant proportional to the gauge coupling e. By calculating the Weyl anomaly in the presence of a cosmological constant [64, 73], therefore, one can determine the renormalization group beta function $\beta(e)$. One finds, remarkably, that the one-loop β function vanishes for $N > 4$ [72].

See [90] for a review of Weyl anomalies in supergravity, and [91] for those in conformal supergravity.

6. The string era

The history of the Weyl anomaly took a new turn with the advent of string theory. The emphasis shifted away from four-dimensional spacetime to the two dimensions of the string worldsheet. In particular, in two very influential 1981 papers, Polyakov [92, 93] showed that the critical dimensions of the string correspond to the absence of the two-dimensional Weyl anomaly. In the first quantized theory of the bosonic string, one starts with a Euclidean

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† Curiously enough, before the gravitino contribution to the anomaly was calculated explicitly, D'Adda and Di Vecchia [88] attempted to deduce it by assuming that the total anomaly cancels in $N = 4$ supergravity. This was a good idea but they reached the wrong conclusion by working with the dual formulation with two ϕ fields instead of the stringy version with one ϕ and one $\phi_{\mu\nu}$ obtained by dimensional reduction.

functional integral over the worldsheet metric $\gamma_{ij}[\xi]$, $i, j = 1, 2$, and spacetime coordinates $X^\mu[\xi]$, $\mu = 0, 1, \ldots, D - 1$, where ξ^i are the worldsheet coordinates. Thus

$$e^{-\Gamma} = \int \frac{D\gamma\, DX}{\text{Vol(Diff)}} \, e^{-S[\gamma, X]} \tag{36}$$

where

$$S[\gamma, X] = \frac{1}{4\pi\alpha'} \int d^2\xi \sqrt{\gamma} \gamma^{ij} \partial_i X^\mu \partial_j X^\nu \eta_{\mu\nu} . \tag{37}$$

As showed by Polyakov, the Weyl anomaly in the worldsheet stress tensor is given by

$$\gamma^{ij} \langle T_{ij} \rangle = \frac{1}{24\pi} (D - 26) R(\gamma). \tag{38}$$

The contribution of the D scalars follows from (17) while the -26 arises from the diffeomorphism ghosts that must be introduced into the functional integral. In the case of the fermionic string, the result is

$$\gamma^{ij} \langle T_{ij} \rangle = \frac{1}{16\pi} (D - 10) R(\gamma). \tag{39}$$

Thus the critical dimensions $D = 26$ and $D = 10$ correspond to the preservation of the two-dimensional Weyl invariance $\gamma_{ij} \to \Omega^2(\xi)\gamma_{ij}$. One may wonder how Polyakov addressed the controversy of the previous eight years described in section 3. Well he didn't, but merely remarked 'This is the well known trace anomaly'.

Previously, the critical dimensions had been understood from the central charge c of the Virasoro algebra [94]

$$[L_n, L_m] = (n - m)L_{n+m} + \frac{c}{12} n(n^2 - 1)\delta_{n,-m} \tag{40}$$

where the L_n are the coefficents in a Laurent expansion of the stress tensor, namely

$$T(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2} \tag{41}$$

where $T = T_{zz}$ and $z \equiv \exp(\xi^0 + i\xi^1)$. Thus this established a connection between the two-dimensional Weyl anomaly and the central charge of the Virasoro algebra (to be precise, $c = 24\pi a$ in (11)); a result which spawned the whole industry of *conformal field theory* in the context of strings. See, for example, Alvarez-Gaumé [95]. In fact, when writing (41), one usually assumes that T_{ij} is traceless, which forces the anomaly to show up as a diffeomorphism anomaly, but the results are entirely equivalent [96].

Polyakov went on to describe what happens in *non-critical* string theory when the Weyl invariance is lost, and the metric conformal mode propagates. In this case, the two-dimensional effective action is given by

$$S_{\text{eff}} \sim \int d^2\xi \sqrt{\gamma} [R \,\Box^{-1} R + \mu] \tag{42}$$

where we have allowed for a worldsheet cosmological term produced by quantum corrections. If we now separate out the conformal mode σ and let $\gamma_{ij} \to e^\sigma \gamma_{ij}$, we obtain the Liouville action

$$S_{\text{L}}[\sigma] = \int d^2\xi \sqrt{\gamma} \left(\tfrac{1}{2} \gamma^{ij} \partial_i \sigma \partial_j \sigma + R\sigma + \mu e^\sigma \right). \tag{43}$$

This is the starting point for much of non-critical string theory.

The role of the Weyl anomaly becomes even more interesting when we allow for the presence of the spacetime background fields, as shown by Callan *et al* [97] and Fradkin and

Tseytlin [98]. In the case of the bosonic string, for example, the worldsheet action takes the form

$$S = \frac{1}{2\pi\alpha'} \int d^2\xi \frac{1}{2} [\sqrt{\gamma}\gamma^{ij}\partial_i X^\mu \partial_j X^\nu G_{\mu\nu}(X) + \epsilon^{ij}\partial_i X^\mu \partial_j X^\nu B_{\mu\nu}(X)]$$
$$+ \frac{1}{4\pi} \int d^2\xi \sqrt{\gamma} R(\gamma)\Phi(X) \tag{44}$$

corresponding to background fields $G_{\mu\nu}(X)$, $B_{\mu\nu}(X)$ and $\Phi(X)$. Now the anomaly may be written as

$$\frac{1}{2\pi\alpha'}\gamma^{ij}\langle T_{ij}\rangle = \beta^\Phi \sqrt{\gamma} R(\gamma) + \beta^G_{\mu\nu}\sqrt{\gamma}\gamma^{ij}\partial_i X^\mu \partial_j X^\nu + \beta^B_{\mu\nu}\epsilon^{ij}\partial_i X^\mu \partial_j X^\nu. \tag{45}$$

The absence of the Weyl anomaly thus means the vanishing of the β functions, which to lowest order turn out to be [97]

$$0 = \beta^G_{\mu\nu} = R_{\mu\nu} - \tfrac{1}{4}H_\mu{}^{\lambda\sigma}H_{\nu\lambda\sigma} + 2\nabla_\mu\nabla_\nu\Phi + O(\alpha') \tag{46}$$

$$0 = \beta^B_{\mu\nu} = \nabla_\lambda H_{\mu\nu}{}^\lambda - 2\nabla_\lambda\Phi H^\lambda{}_{\mu\nu} + O(\alpha') \tag{47}$$

$$0 = 16\pi^2\beta^\Phi = \frac{D-26}{3\alpha'} + \left[4(\nabla\Phi)^2 - 4\nabla^2\Phi - R - \tfrac{1}{12}H^2\right] + O(\alpha'^2). \tag{48}$$

But these are nothing but the Einstein-matter field equations that result from the action

$$\Gamma_{\text{eff}} \sim \int d^D x \sqrt{-G} e^{-2\Phi} \left[\frac{D-26}{3\alpha'} - R - 4(\nabla\Phi)^2 - \tfrac{1}{12}H^2\right] + \cdots. \tag{49}$$

The common factor $e^{-2\Phi}$ reveals that these terms are tree-level in a string loop perturbation expansion. If we denote the dilaton vacuum expectation value by Φ_0, then from (18) the classical action (44) yields a term

$$e^{-\chi\Phi_0} = e^{-2(1-L)\Phi_0} \tag{50}$$

in the functional integral, where L counts the number of holes in the two-dimensional Riemann surface, i.e. the number of loops.

That the Einstein equations in spacetime should originate from the vanishing of the worldsheet Weyl anomaly is perhaps the most remarkable result in our story.

7. Current problems

New techniques for calculating Weyl anomalies continue to appear. Fujikawa [99, 100] has pioneered the functional-integral approach where the origin of the lack of quantum Weyl invariance in a classically-invariant theory may be attributed to a non-invariant measure in the functional integral. Ceresole *et al* [101] have reproduced these results using flat-space plane waves. Bastianelli and van Nieuwenhuizen [102, 103] have applied to Weyl anomalies the quantum mechanical approach, first used by Alvarez-Gaumé and Witten [6] in the context of axial anomalies.

Much of the current interest in the Weyl anomaly resides not only in high-energy physics and general relativity but in statistical mechanics. See, for example [95, 104–109]. A particularly powerful result is Zamolodchikov's c-theorem [110], which states that there exists a function defined on the space of two-dimensional conformal field theories which is decreasing along renormalization group (RG) trajectories, and is stationary only at RG fixed points, where its value equals the Virasoro central charge c. Recently, there has been a good deal of activity by Cardy [111], Osborn [112], Jack and Osborn [113], Cappelli *et al* [59],

Shore [114, 115], Antoniadis *et al* [89] and Osborn and Petkos [117] attempting to generalize this theorem to higher dimensions. Whereas from (11) the two-dimensional gravitational anomaly depends on only one number, however, from (12) the four-dimensional one depends on two: the Euler term and the Weyl term. In higher dimensions, there will always be one Euler term, but the number of Weyl invariants grows with dimension [118]. Consequently, it is not clear whether there is a unique way to generalize the theorem nor how useful such a generalization might be. Cardy and Jack and Osborn focused on the Euler number term, whereas Cappelli *et al* pointed to the positivity of the Weyl tensor term (30) as a more likely guide, but the analysis is still inconclusive.

On the subject of the Euler term, the results of section 5 are, in fact, still controversial because the anomaly inequivalence between different field representations, discussed by van Nieuwenhuizen and myself, was challenged at the time by Siegel [119] and by Grisaru *et al* [120]. They found that the traces of the two stress tensors were equivalent. Yet the recent string results of Antoniadis *et al* [89] would seem to support our interpretation. Moreover, if it were incorrect, the vanishing of the anomalies for the $N = 4$ and $N = 8$ multiplets would seem to be a gigantic coincidence.

Nevertheless, to tell the truth (in accordance with Salam's maxim), I am still uneasy about the whole thing. The numerical coefficients quoted in table 1 were calculated using the b_4 coefficients discussed in section 3. The claim that these correctly describe the trace anomaly is in turn based on an identity which everyone used to take for granted. See, for example, Hawking [38]. The identity says that if $S[g]$ is a functional of the metric $g_{\mu\nu}$, then

$$\int d^n x \; g^{\mu\nu} \frac{\delta S[g]}{\delta g_{\mu\nu}} \equiv \left. \frac{\partial S[\lambda g]}{\partial \lambda} \right|_{\lambda=1} \tag{51}$$

where λ is a constant. If true, it would mean that the integrated trace of the stress tensor arises exclusively from the non-invariance of the action under constant rescalings of the metric. However, the *Polyakov* action provides an obvious counterexample:

$$S_{\text{eff}} \sim \int d^2 x \sqrt{g} R \, \Box^{-1} R . \tag{52}$$

This action is scale invariant, but gives rise to an anomaly proportional to R! Of course, the integrated anomaly is a purely topological Euler number term, and it is the topological nature of the action which provides the exception to the rule. However, since the entire debate over equivalence versus inequivalence devolved precisely on Euler number terms, the use of the identity (51) in this context makes me feel very uneasy†.

The whole question of whether the dual formulations of supergravity yield the same Weyl anomalies has recently been thrust into the limelight with the string/fivebrane duality conjecture [122, 123] which states that in their critical spacetime dimension $D = 10$, superstrings (extended objects with one spatial dimension) are dual to superfivebranes (extended objects with five spatial dimensions [124]). Whereas the two-dimensional worldsheet of the string couples to the rank-2 field $\phi_{\mu\nu}$, the six-dimensional world-volume of the fivebrane couples to the rank-6 field $\phi_{\mu\nu\rho\lambda\sigma\tau}$. The usual rank-2 formulation of $D = 10$ supergravity dimensionally reduces to the $N = 4$ field content of table 1, but the dual rank-6 formulation reduces to a different field content with $20\phi_{\mu\nu\rho}$ and with 14ϕ replaced

† Of course, there is no real contradiction provided one is sufficently careful about handling the zero modes of the Laplacian. I am grateful to Stuart Dowker, Arkady Tseytlin and Ed Witten for correspondence on this point. A rigorous treatment of this problem may be found in Dowker's recent paper [121]. It was precisely this issue of how to handle the zero modes which separated [87] from [119] and [120]. Unfortunately, merely knowing that one must be careful about zero modes has not yet enabled me to decide who has been the more careful!

by $14\phi_{\mu\nu}$. If table 1 is to be believed, $A(\phi_{\mu\nu}) - A(\phi) = 1$ and $A(\phi_{\mu\nu\rho}) = -2$. Therefore the dual version has non-vanishing coefficent $A = 14 - 40 = -26$. This increases my uneasiness.

A possible resolution of this problem may perhaps be found in the recent paper by Deser and Schwimmer [118] who have re-examined the different origins of the topological versus Weyl tensor contributions to the anomaly (which they call type A and type B, respectively).

I certainly believe that the final word on this subject has still not been written.

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† Historical footnote: at the time Salam had to change the gravitational constant to match the hadronic scale, an idea which spawned his *strong gravity*; today the fashion is the reverse and we change the Regge slope to match the Planck scale!

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