

GRQFT April 16

Note Title

2/10/2014

Anomalies

\mathcal{L} invariant but P.I. is not

$$S[d^4\bar{\psi}][d^4\psi] \rightarrow S d^4\bar{\psi} d^4\psi \xrightarrow{\text{e}^{i \int d^4x \alpha(x) A_2(x)}} \text{parameter } \frac{1}{16\pi^2 g_2^2} A_2(x)$$

$$\partial_\mu J^\mu = A_2(x)$$

Feynman diagram

$$\sim \frac{1}{d-4} \times d-4 = \text{finite}$$

- UV physics

- IR derivation - EFT \rightarrow Non local \mathcal{L}

\rightarrow dispersion relation

Non-local

$$g \ln \square \leftarrow \frac{1}{\varepsilon} + \underline{\ln g^2}$$

Rescaling $A(\lambda x) = \frac{1}{\lambda} A(\cancel{\lambda x})$

Rescale $\ln \square$

No longer invariant

$$\partial_\mu T^M = \theta_M^M = \frac{b}{2} F_{\mu\nu} F^{\mu\nu}$$

$$= \frac{be^2}{\lambda^2} F_{\mu\nu} F^{\mu\nu} \quad \text{normal norm}$$

β function QED trans

$$A \rightarrow \frac{1}{\ell} A$$

$$S = \int d^4x - \frac{1}{4} F_{\rho\sigma} \left[\frac{1}{e^2(\mu)} + b_i \ln (\square/\mu^2) \right] F^{\rho\sigma}$$

$$\ln \square \rightarrow \ln \square - \ln \cancel{\ell}^2$$

$$\langle x | \ln \left(\frac{\square}{\mu^2} \right) | y \rangle \equiv L(x-y) = \int \frac{d^4q}{(2\pi)^4} e^{-iq \cdot (x-y)} \ln \left(\frac{-q^2}{\mu^2} \right)$$

However, under rescaling, this behaves in the same way as described above with a local term

$$L(x-y) \rightarrow \lambda^{-4} (L(x-y) - \ln \lambda^2 \delta^4(x-y))$$

Sometimes see (Nonrigorous) (DSM)

$$S = \int d^4x - \frac{1}{4e^2(\lambda)} F_{\mu\nu} F^{\mu\nu}$$

$$\frac{\delta S}{\delta \lambda} = \int d^4x \underbrace{\frac{\partial}{\partial \lambda} \left(\frac{-1}{4e^2(\lambda)} \right)}_{\frac{1}{2e^3} \frac{\partial e(\lambda)}{\partial \lambda}} F^2 = \int d^4x \frac{\beta_0}{2e} F^2$$

Tiri Hor\v{e}j\v{s}

Dispersion relations

$$\left(2 - p \cdot \frac{\partial}{\partial p}\right) \Pi_{\mu\nu}(p) = \Delta_{\mu\nu}(p) \quad (9)$$

Here the $\Pi_{\mu\nu}$ stands for the vacuum polarization tensor

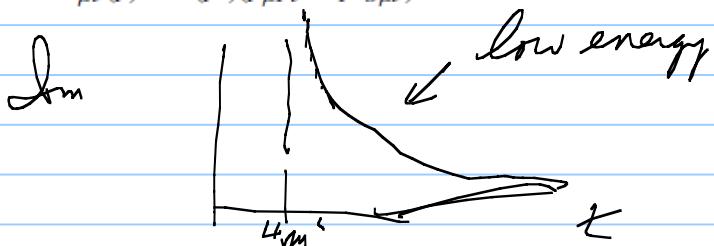
$$\Pi_{\mu\nu}(p) = i \int d^4x e^{ipx} \langle 0 | T(J_\mu(x) J_\nu(0)) | 0 \rangle \quad (10)$$

and the $\Delta_{\mu\nu}$ represents the three point vertex function

$$\Delta_{\mu\nu}(p) = \int d^4x d^4y e^{ipy} \langle 0 | T(\theta_\alpha^\alpha(x) J_\mu(y) J_\nu(0)) | 0 \rangle \quad (11)$$

$$\Pi_{\mu\nu}(p) = \Pi(p^2)(p_\mu p_\nu - p^2 g_{\mu\nu})$$

$$\Delta_{\mu\nu}(p) = \Delta(p^2)(p_\mu p_\nu - p^2 g_{\mu\nu})$$



employ the identity (20) to get, after a simple manipulation

$$\begin{aligned} -2p^2 \frac{\partial}{\partial p^2} \Pi(p^2) &= p^2 \frac{1}{\pi} \int_{4m^2}^{\infty} \frac{\text{Im } \Delta(t)}{t(t-p^2)} dt \\ &= \frac{1}{\pi} \int_{4m^2}^{\infty} \frac{\text{Im } \Delta(t)}{t-p^2} dt - \frac{1}{\pi} \int_{4m^2}^{\infty} \frac{\text{Im } \Delta(t)}{t} dt \end{aligned} \quad (24)$$

Taking into account the definition (19) one can thus write finally

$$-2p^2 \frac{\partial}{\partial p^2} \Pi(p^2) = \Delta(p^2) - \frac{1}{\pi} \int_{4m^2}^{\infty} \frac{\text{Im } \Delta(t)}{t} dt \quad (25)$$



Having reproduced the form of the anomalous Ward identity (16), one should check that the correct value of the trace anomaly is indeed recovered in (25). Using the expression (22) and performing the integral one obtains

$$\frac{1}{\pi} \int_{4m^2}^{\infty} \frac{\text{Im } \Delta(t)}{t} dt = -\frac{1}{6\pi} e^2 \quad (26)$$

so that the anticipated result (16) is confirmed. From (22) and (26) it is also clear that

$$\lim_{m \rightarrow 0} \frac{1}{t} \text{Im } \Delta(t; m^2) = -\frac{e^2}{6\pi} \delta(t) \quad (27)$$

Thus, within the dispersion relation approach the trace anomaly is tantamount to the sum rule (26) for the imaginary part of the classical symmetry-breaking term $\Delta(t; m^2)$.

$$\boxed{\text{Im } \Delta(t; m^2) = -\frac{2e^2}{\pi} \frac{m^4}{t^2} \frac{1}{\sqrt{1 - \frac{4m^2}{t}}}}$$

Gravitational anomalies

Note: scale and conformal symmetry

- variations, not exactly same

- anomaly same

- Scale rescale $g_{\mu\nu}(x') = \lambda^0 g_{\mu\nu}(\lambda x)$

- conformal $g_{\mu\nu} \rightarrow e^{2\phi} g_{\mu\nu}$ related $\phi = \text{const}$

Scale already done a_2 coeff

$$T_m^{\mu} = -\frac{1}{16\pi^2} \text{tr} a_2$$

$$= -\frac{1}{16\pi^2} \frac{1}{180} \left[R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - R_{\mu\nu} R^{\mu\nu} - 1/2 R^2 \right] \text{ scalar } \Sigma = \frac{1}{6}$$

$$\begin{aligned} ds^2 &= e^{2\phi} g_{\mu\nu} dx^\mu dx^\nu \\ &= g_{\mu\nu} d(e^{\phi} dx^\mu) d(e^{\phi} x^\nu) \end{aligned}$$

Non local Action (EFT)

- like $F \ln D F$ discussion

$$\text{non} \quad \frac{1}{\varepsilon} + \ln g^2$$

coeff determined by α_2

$$S_{QL} = \int d^4x \sqrt{g} \left(\alpha R \log \left(\frac{\square}{\mu_\alpha^2} \right) R + \beta R_{\mu\nu} \log \left(\frac{\square}{\mu_\beta^2} \right) R^{\mu\nu} + \gamma R_{\mu\nu\alpha\beta} \log \left(\frac{\square}{\mu_\gamma^2} \right) R^{\mu\nu\alpha\beta} \right)$$

	α	β	γ	$\bar{\alpha}$	$\bar{\beta}$	$\bar{\gamma}$
Scalar	$5(6\xi - 1)^2$	-2	2	$5(6\xi - 1)^2$	3	-1
Fermion	-5	8	7	0	18	-11
Vector	-50	176	-26	0	36	-62
Graviton	430	-1444	424	90	126	298

TABLE I: Coefficients of different fields. All numbers should be divided by $11520\pi^2$.

Conformal symmetry $\leftarrow \sigma = \sigma(x)$

$$g'(x) = e^{\frac{2\sigma(x)}{n}} g_{\mu\nu}(x)$$

$$\phi' \Rightarrow e^{P\sigma(x)} \phi$$

$$S_m(\phi, g)$$

If symmetry

$$\delta S = \int d^nx \left[\underbrace{\frac{\delta S_m}{\delta \phi} \delta \phi}_{EL \text{ eq} = 0} + \frac{\delta S_m}{\delta g_{\mu\nu}} \delta g_{\mu\nu} \right]$$

$$\delta g_{\mu\nu} = 2\sigma g_{\mu\nu}$$

$$| g_{\mu\nu} T^{\mu\nu} = 0 = T^m_m \rangle$$

Scalar fields

$$g_{\mu\nu} \rightarrow e^{\frac{2\sigma}{\Omega}} g_{\mu\nu} \quad \rightarrow \quad \tilde{g}^{\mu\nu} \rightarrow e^{-\frac{2\sigma}{\Omega}} \tilde{g}_{\mu\nu}, \quad \sqrt{g} \Rightarrow e^{\frac{4\sigma}{\Omega}} \sqrt{\tilde{g}}$$

$$R \rightarrow e^{-2\sigma} [R + 6\Box\sigma]$$

$$\Box = \frac{1}{\sqrt{g}} \partial_\mu \left(\sqrt{g} \tilde{g}^{\mu\nu} \partial_\nu \right) \rightarrow e^{-2\sigma} [\Box + 2(\partial_\mu \sigma) \partial^\mu]$$

$$\phi \rightarrow e^{-\sigma} \phi$$

$$\begin{aligned} (\Box + \frac{1}{6} R) \phi &\rightarrow e^{-2\sigma} \left[\Box + 2(\partial_\mu \sigma) \partial^\mu + \frac{1}{6} R + \Box \sigma \right] e^{-\sigma} \phi \\ &\cdot e^{-2\sigma} e^{-\sigma} \left[\Box - \Box \sigma - 2(\partial_\mu \sigma) \partial^\mu + 2(\partial_\mu \sigma) \partial^\mu + \frac{1}{6} R \right] \phi \\ &+ \Box \sigma \end{aligned}$$

$$= e^{-3\sigma} \left[D + \frac{1}{6} R \right] \phi$$

So that

$$\int d^4x \sqrt{g} \phi \left(D + \frac{1}{6} R \right) \phi \quad \text{invariant} \quad \checkmark \quad \left(\begin{array}{l} \xi R \\ \text{if } \xi = \frac{1}{6} \text{ conformally} \\ \text{invariant} \end{array} \right)$$

$$\int g^{mn} \partial_\mu \phi \partial_\nu \phi \rightarrow (\text{invariant}) + \frac{1}{2} \partial_\mu (\sqrt{g} g^{mn} \phi^2 \partial_\nu \phi)$$

\curvearrowright total deriv

Photon

$$-\frac{1}{4} \int d^4x \sqrt{g} g^{mn} g^{\alpha\beta} F_{\mu\alpha} F_{\nu\beta}$$

$$\underline{F_{\mu\nu} \rightarrow \underline{F}_{\mu\nu}}$$

Conformal anomaly

$$g_{\mu\nu} \rightarrow e^{2\sigma} g_{\mu\nu}$$

$$\phi \rightarrow e^{-\sigma} \phi$$

$$\mathcal{J} = \det \begin{bmatrix} e^{-\sigma} \end{bmatrix}$$

$\nwarrow \alpha_2 \text{ coeff}$

Same trace $T_m^m = \frac{1}{16\pi^2} \cdot \frac{1}{188} [R_{\mu\nu\rho\lambda} R^{\mu\nu\rho\lambda} - \dots]$

GR is not conformally invariant

$$S = \int d^4x F_g \frac{-2}{K^2} R \rightarrow \int d^4x F_g \underbrace{e^{4\sigma}}_{\uparrow} \underbrace{e^{-2\sigma}}_{\uparrow} [R + 6 \square \sigma] \frac{-2}{K^2}$$

Weyl Tensor $g_{\mu\nu} \rightarrow e^{\frac{2\phi}{d}} g_{\mu\nu}$, $C_{\mu\nu\rho\sigma} \rightarrow e^{\frac{2\phi}{d}} C_{\mu\nu\rho\sigma}$

In d dimensions

$$C_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} - \frac{1}{d-2} (g_{\mu\alpha} R_{\nu\rho} - g_{\mu\rho} R_{\nu\alpha} - g_{\nu\alpha} R_{\mu\rho} + g_{\nu\rho} R_{\mu\alpha}) \\ + \frac{1}{(d-1)(d-2)} (g_{\mu\alpha} g_{\nu\rho} - g_{\nu\alpha} g_{\mu\rho}) R$$

Squared

$$C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 2R_{\mu\nu} R^{\mu\nu} + \frac{1}{3} R^2 \quad (4d)$$

Conformally invariant action

$$\int g \underbrace{g^{mn} g^{vv'} g^{dd'} g^{\beta\beta'}}_{e^{4\alpha} \overline{(e^{-2\alpha})^4} (\partial^{2\alpha})^2} C_{\mu\nu\rho\sigma} C_{v'd'w'p'} = \int g C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} \text{ invariant Marnheim}$$

Contractions gave zero

$$g^{ad} C_{\mu\nu\rho\sigma} = 0$$

$$\text{If } g_{\mu\nu} = e^{2\alpha} \eta_{\mu\nu} \Rightarrow C_{\mu\nu\rho\sigma} = 0 \quad \text{Flat FLRW}$$

Anomaly $\propto C^2$

Non local L



$$S_{QL} = \int d^4x \sqrt{g} \left[\bar{\alpha} R \log\left(\frac{\square}{\mu_1^2}\right) R + \bar{\beta} C_{\mu\nu\alpha\beta} \log\left(\frac{\square}{\mu_2^2}\right) C^{\mu\nu\alpha\beta} + \bar{\gamma} (R_{\mu\nu\alpha\beta} \log(\square) R^{\mu\nu\alpha\beta} - 4R_{\mu\nu} \log(\square) R^{\mu\nu}) \right].$$

For conformal fields

$$\rightarrow R_{\mu\nu\rho\sigma}^2 - 4R_{\mu\nu}^2 + R^2 = \text{Total derm}$$

G.B.

$$T_M^M = \# C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma}$$

Axial Anomaly

$$J_\mu = \bar{\psi} \gamma_\mu \gamma_5 \psi$$

$$QCD: J_\mu^0 = \bar{u} \gamma_\mu \gamma_5 u + \bar{d} \gamma_\mu \gamma_5 d + \bar{s} \gamma_\mu \gamma_5 s \quad \leftarrow \text{anomalous}$$

$$J_\mu^8 = \frac{1}{F_3} \left(\begin{array}{ccc} " & " & -2 \bar{u} \end{array} \right) \quad \leftarrow \text{not anomalous}$$

$$QED \quad J_\mu^3 = (\bar{u} \gamma_\mu \gamma_5 u - \bar{d} \gamma_\mu \gamma_5 d) \quad \leftarrow \text{No QCD anomaly}$$

$\leftarrow \text{QED anomaly}$

$$\text{Symmetry } \psi \rightarrow e^{i\alpha \gamma_5} \psi \Rightarrow \partial^\mu J_{5\mu} = 0 \quad \gamma_5 = \gamma_0 \gamma_1 \gamma_2 \gamma_3$$

$\bar{\psi} i \not{D} \psi$ invariant, $m \bar{\psi} \psi$ not invariant

Calculation of anomaly

$$\psi \rightarrow e^{-\beta \delta_5} \psi, \bar{\psi} \rightarrow e^{-\beta \delta_5} \bar{\psi}$$

$$\int d\bar{\psi} d\bar{\psi}' = \int d\psi d\bar{\psi}' f$$

$$f = e^{-2\text{tr} \beta \delta_5} \rightarrow \lim_{M \rightarrow \infty} e^{-2 \cdot \ln(\beta \delta_5) (e^{-D/M})^2}$$

heat kernel expansion

$$\frac{d\alpha}{dt} + \frac{\partial}{\partial t} T_{\mu\nu} F^{\mu\nu}$$

QED (single fermion)

$$J^\mu = \frac{3\alpha}{8\pi} F_{\mu\nu} \tilde{F}^{\mu\nu} \approx \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}$$

$$\text{Tr} (\gamma_5 \sigma^2) =$$

$$\text{Tr} (\gamma_5 \sigma_{\mu\nu} \tau_{\alpha\beta}) = 4\pi \Sigma_{\mu\nu\alpha\beta}$$

1) $\pi^0 \rightarrow \gamma\gamma$

QED

$$2^n \bar{J}_{3,n} = \alpha F^2$$

$\langle \pi / J | 0 \rangle$

2) $U(1)_A$ not symmetry

Anomalies + Gauge Currents

Anomalies in global current OK

in gauge currents make theory inconsistent

Anomaly condition

$$\int \text{Tr} (Q_A^\dagger Q_B^\nu Q_C^\ell) = 0$$

Axial Gravitational anomaly

$$\partial^\mu J_{A\mu} = \frac{1}{384\pi^2} \epsilon^{\mu\nu\alpha\beta} R_{\mu\nu\alpha\sigma} R_{\alpha\beta}^{\sigma\tau} \quad \leftarrow \star\star$$

