

GR QFT Feb 10

Gravitors Day

Note Title

2/10/2014

$$[R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R] = 8\pi G T_{\mu\nu}$$

$$g_{\mu\nu} = \eta_{\mu\nu} + k h_{\mu\nu}$$

$$T_{\mu\nu} = \frac{2}{\Gamma g} \frac{\delta S}{\delta g^{\mu\nu}} \quad \text{for matter}$$

$$= \frac{4}{\Gamma g} [R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R] + \text{total deriv}$$

↑  
 $t_{\mu\nu}$  for gravity?

Expand to  $O(h^2)$

$$\left[ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right]^{(1)} + \underbrace{\left[ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right]^{(2)}}_{\dots} = 8\pi G T_{\mu\nu}$$

Define  $-\frac{1}{8\pi G} \left[ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right]^{(2)} = T_{\mu\nu}$  on  $h \partial \partial h$

$\Rightarrow$  Harmonic gauge

$$\square h_{\mu\nu} = -8\pi G (T_{\mu\nu} + T_{\nu\mu})$$

↑ gravity E, P act  
as source of gravity

Calculate

(Wainberg)

also generates  
 $\alpha\beta$  of  
 $t_{\mu\nu}$

$$t_{\mu\nu} =$$

$$\begin{aligned} T_{\mu\nu} = & -\frac{1}{4}h_{\alpha\beta}\partial_\mu\partial_\nu h^{\alpha\beta} + \frac{1}{8}h\partial_\mu\partial_\nu h \\ & + \frac{1}{8}\eta_{\mu\nu}\left(h^{\alpha\beta}\square h_{\alpha\beta} - \frac{1}{2}h\square h\right) \\ & - \frac{1}{4}\left(h_{\mu\rho}\square h^\rho_\nu + h_{\nu\rho}\square h^\rho_\mu - h_{\mu\nu}\square h\right) \\ & + \frac{1}{8}\partial_\mu\partial_\nu\left\{h_{\alpha\beta}h^{\alpha\beta} - \frac{1}{2}hh\right\} - \frac{1}{16}\eta_{\mu\nu}\square\left\{h_{\alpha\beta}h^{\alpha\beta} - \frac{1}{2}hh\right\} \\ & - \frac{1}{4}\partial_\alpha\left[\partial_\nu\left\{h_{\mu\beta}h^{\alpha\beta}\right\} + \partial_\mu\left\{h_{\nu\beta}h^{\alpha\beta}\right\}\right] \\ & + \frac{1}{2}\partial_\alpha\left[h^{\alpha\beta}(\partial_\nu h_{\mu\beta} + \partial_\mu h_{\nu\beta})\right] \end{aligned}$$

most important

## Gravity (GR) from self interactions

Deser 1970 reprinted gr-qc/0411023

- 1) Start with massless spin 2  $\underline{h_{\mu\nu}}$  + couple to energy & momentum
  - 2) But,  $\mathcal{L} = h \partial \partial h \rightarrow t_{\mu\nu}$  for gravity  $t_{\mu\nu} = h \partial \partial h$
  - 3) This couples to  $h$  also,  $\mathcal{L} = h \underline{h \partial \partial h}$
  - 4) This changes  $t_{\mu\nu}$  — new piece  $h(h h \partial \partial h)$
  - 5) Go to 4
- $\Rightarrow$  GR nonlinear

## Gravitational waves

$$\text{if } T_{\mu\nu} = 0 \implies \square h_{\mu\nu} = 0$$

$$\text{in harmonic gauge } \partial^\mu h_{\mu\nu} = \frac{1}{2} \partial_\nu h \quad h = h^\lambda \chi$$

Symmetric  $h_{\mu\nu} \Rightarrow 10$  components

harmonic 4 constraints  $\Rightarrow \underline{6 \text{ components}}$

Still another gauge trans in empty space - stay in harmonic

$$\partial^\mu (h_{\mu\nu} - 2\partial_\nu \xi_\mu - 2\partial_\mu \xi_\nu) = \frac{1}{2} \partial_\nu (h^\lambda_\lambda - 2\partial^\lambda \chi)$$

$$\partial^\mu h_{\mu\nu} - \underbrace{\left( \partial_\nu \xi_\mu - \partial_\mu \xi_\nu \right)}_{} \quad \boxed{}$$

$$\Rightarrow \square \xi_\nu = 0 \quad \text{stay in harmonic gauge}$$

~ 4 more constraints

Choose to make  $\underline{h}_{\nu 0} = 0$

$$h'_{\nu 0} = h_{\nu 1} - \partial_\nu \xi_0 - \partial_0 \xi_\nu$$

$$- \quad \square h = d, \quad \square \xi_\nu = 0$$

$\Rightarrow$  2 components

motion in  $\mathbb{Z}$  direction  $h_{\mu\nu} = h_{\mu\nu}(z-t)$

$$\partial^m h_{\mu\nu} = \partial^0 h_{\mu\nu} + \partial^z h_{\nu z} = \frac{1}{2} \partial_\nu h$$

if  $v=0$

$$\partial^0 h_{\mu z} = \frac{1}{2} \partial_\mu h \Rightarrow h = h^\lambda \lambda = 0$$

Then

$$\partial^z h_{\nu z} = 0 \Rightarrow h_{\nu z} = \text{const} \rightarrow 0$$

## Transverse Traceless (TT)

$$\square h_{\mu\nu} = 0, \quad \partial^\mu h_{\mu\nu} = 0, \quad h = 0, \quad h_{0\nu} = 0$$

Construct using photons polarization

Photon  $A_\mu \sim \epsilon_n e^{-i k \cdot x}$   $k^2 = 0, \quad k \cdot E = 0$

helicity  $\pm 1 = \lambda$   $\epsilon_n^{(\lambda)} = \frac{1}{\sqrt{2}} (0, 1, \pm i, 0)$   $\leftarrow$  motion in z direction  
 $\epsilon_n^{(\lambda)} \epsilon_{n'}^{(\lambda')} = \delta_{\lambda \lambda'}, \quad \epsilon_n(\lambda) \epsilon^{*(\lambda)} = \frac{1}{2} (0 + 1 - (i)^2 - 0) = \underline{\underline{0}}$

$$\underline{\underline{\epsilon_n(\lambda) \epsilon_{n'}^{(\lambda')}}} = \delta_{\lambda \lambda'}, \quad \underline{\underline{\epsilon_n(\lambda) \epsilon^{*(\lambda)}}} = \frac{1}{2} (0 + 1 - (i)^2 - 0) = \underline{\underline{0}}$$

Construction :

$$h_{\mu\nu} = A \sum_{\nu} e^{kh_{\mu\nu}} \quad k^2 = 0 \quad \checkmark$$

↖ photon pol.

$$\sum_{\nu} (\mu\nu) = \underbrace{\sum_{\mu} (+) \sum_{\nu} (+)}_{\text{, }} \quad , \quad \sum_{\nu} (\mu\nu) = \underbrace{\sum_{\mu} (-) \sum_{\nu} (-)}$$

$$\square h_{\mu\nu} = 0$$

$$\partial^{\mu} h_{\mu\nu} = 0 \quad \text{from } k^{\mu} \sum_{\nu} = 0$$

$$h^{\lambda}_{\lambda} = 0 \quad \sum_{\mu} \sum^{\mu} = 0$$

$$h_{\sigma\nu} = 0$$

2 solution

$$E_{\mu\nu}(\pm 2) = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & \pm i & 0 \\ 0 & \pm i & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \leftarrow \text{like circular pol.}$$

↑

motion in  
2 directions

GR book  
+, x

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & a_+ & a_x & 0 \\ 0 & a_x & -a_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \leftarrow \text{like linear pol.}$$

for photons

## Gravitons - second quantization

$$E+M \quad A_\mu = \sum_{\lambda} \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2\omega}} [a(p, \lambda) \epsilon_\mu(p, \lambda) e^{-ip\cdot x} + h.c.]$$

$$\text{with } [a(p, \lambda), a^\dagger(p', \lambda')] = \delta_{\lambda \lambda'} (2\pi)^3 \delta^3(p - p')$$

$$\text{Then } H = \frac{1}{2} \int d^3 x (E^2 + B^2) = \sum_{\lambda} \int \frac{d^3 p}{(2\pi)^3} \hbar \omega_p [a^\dagger(p, \lambda) a(p, \lambda) + \frac{1}{4}]$$

$$\text{States } |p, \lambda\rangle = a^\dagger(p, \lambda) |0\rangle$$

$$|+\rangle |p, \lambda\rangle = \omega_p |p, \lambda\rangle$$

Same for gravity

↓ above

$$h_{\mu\nu}(x) = \sum_{\lambda=\pm} \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2\omega}} \left[ a(p, \lambda) \epsilon_{\mu\nu}(p, \lambda) e^{-ip \cdot x} + h.c. \right]$$

$$t_{\mu\nu} = -\frac{1}{4} \underline{h}_{\alpha\beta} \underline{\partial}_\mu \underline{\partial}_\nu h^{\alpha\beta} + \frac{1}{8} h \partial_\mu \partial_\nu h^\alpha \partial_\alpha h^\beta + \dots$$

$$H = \int d^3x t_{00}$$

$$= \sum_{\lambda} \int \frac{d^3 p}{(2\pi)^3} \omega \left[ a^+(p, \lambda) a(p, \lambda) + \frac{1}{2} \right] \quad \sum_{\mu\nu} \epsilon^{\mu\nu} = 0 \quad \checkmark \quad a^+ a^\perp$$

$$\int d^3x t_{\mu\nu} = \sum_{\lambda} \int \frac{d^3 p}{(2\pi)^3} \frac{p_\mu p_\nu}{\omega} a^+ a$$

## Weak field QFT

I }

Start with

$$S = \int d^4x \sqrt{-g} \left[ -\frac{2}{K^2} R + L_m \right] \quad , \quad \text{expand } g_{\mu\nu} = \eta_{\mu\nu} + k h_{\mu\nu}$$

$$-\frac{2}{K^2} \sqrt{g} R = -\frac{2}{K} [\underbrace{\partial_\lambda \partial^\lambda h}_{\text{Total deriv}} - \square h] + \frac{1}{2} [\underbrace{\partial_\lambda h_{\mu\nu} \partial^\lambda \bar{h}^{\mu\nu}}_{\text{action}} - 2 \partial^\lambda h_{\mu\lambda} \partial_\sigma \bar{h}^{\mu\sigma}]$$
$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} h$$

Fix gauge

harmonic gauge  $\partial_\mu \bar{h}^{\mu\nu} = 0$

$$\mathcal{L}_{gf} = + \left\{ \partial_\mu \bar{h}^{\mu\nu} \partial^\lambda \bar{h}_{\lambda\nu} \right\} = / \Rightarrow \text{harmonic gauge}$$

$$Rg \mathcal{L} = \frac{1}{2} \partial_\lambda h_{\mu\nu} \partial^\lambda h^{\mu\nu} - \frac{1}{4} \partial_\lambda h \partial^\lambda h - \frac{k}{2} h^{\mu\nu} T_{\mu\nu} \xrightarrow{\text{matter}} + \mathcal{L}_{m\partial}$$

Integrate by parts

$$\mathcal{L} = \frac{1}{2} h_{\mu\nu} \left[ \square \left( I^{\mu\nu\alpha\beta} - \frac{1}{2} \gamma^{\mu\nu} \gamma^{\alpha\beta} \right) \right] h_{\alpha\beta} - \frac{k}{2} h^{\mu\nu} T_{\mu\nu}$$

$I^{\mu\nu\alpha\beta} = \frac{1}{2} (\gamma^{\mu\alpha} \gamma^{\nu\beta} + \gamma^{\mu\beta} \gamma^{\nu\alpha})$

To get propagator

Invert  $\left[ I^{\mu\nu\alpha\beta} - \frac{1}{2} \gamma^{\mu\nu} \gamma^{\alpha\beta} \right] \left[ a T_{\alpha\beta,\gamma\delta} + b \gamma_{\alpha\beta} \gamma_{\gamma\delta} \right] = I^{\mu\nu}{}_{\gamma\delta}$

$$= a I^{\mu\nu}{}_{\gamma\delta} - \frac{1}{2} a \gamma^{\mu\nu} \gamma_{\gamma\delta} + b \gamma^{\mu\nu} \gamma_{\gamma\delta} - 2b \gamma^{\mu\nu} \gamma_{\gamma\delta}$$

$\underbrace{\hspace{10em}}$

$$a=1, b=-\frac{1}{2}$$

Then

$$i D^{\alpha\beta\gamma\delta}(x) = S \frac{d^4 p}{(2\pi)^4} \times \frac{e^{-ip \cdot x}}{q^2} P^{\alpha\beta\gamma\delta}$$

$$P^{\alpha\beta\gamma\delta} = \frac{1}{2} \left[ \gamma^{\alpha\gamma} \gamma^{\beta\delta} + \gamma^{\alpha\delta} \gamma^{\beta\gamma} - \gamma^{\alpha\beta} \gamma^{\gamma\delta} \right]$$

$$h^{\alpha\beta}(x) = \int d^4y D^{\alpha\beta\gamma\delta}(x-y) \frac{k}{2} T_{\gamma\delta}(y)$$

Satisfies  $\overline{J} P^{\mu\nu\alpha\beta} h_{\alpha\beta} = \frac{k}{2} \overline{J}^{\mu\nu}$

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Feynman rule

$$\begin{array}{c} \alpha\beta \\ \text{---} \\ g \end{array} = \frac{i}{\epsilon_0^2} P^{\alpha\beta\gamma\delta}$$

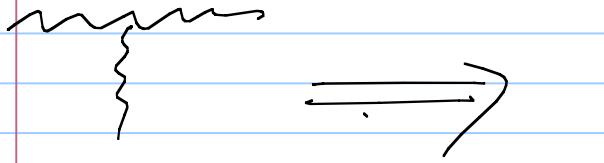
$$\overbrace{\hspace{10em}}^L$$

=

$$\frac{k}{2} h_{\mu\nu} T^{\mu\nu}$$

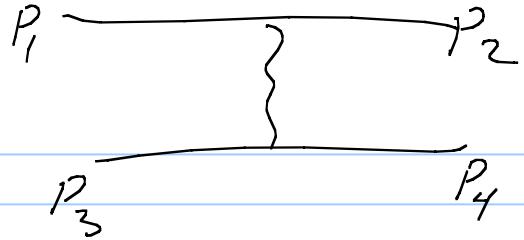
$$\langle p' | T_{\mu\nu}^{(w)} | p \rangle = \frac{e^{-i(p-p') \cdot \gamma}}{(2\pi)^3 \sqrt{2w_1 2w_2}} \left[ (p^\mu p'^\nu + p'^\mu p^\nu) - g^{\mu\nu} (p \cdot p' - m^2) \right]$$

$$\overbrace{\hspace{10em}}^L = i \frac{k}{2} \left[ (p_r p'_v + p'_r p_v) - g_{\mu\nu} (p \cdot p' - m^2) \right]$$



$$\begin{aligned}
 \tau_{\alpha\beta,\gamma\delta}^{\mu\nu} = & \frac{i\kappa}{2} \left( P_{\alpha\beta,\gamma\delta} \left[ k^\mu k^\nu + (k-q)^\mu (k-q)^\nu + q^\mu q^\nu - \frac{3}{2} \eta^{\mu\nu} q^2 \right] \right. \\
 & + 2q_\lambda q_\sigma \left[ I^{\lambda\sigma,}_{\alpha\beta} I^{\mu\nu,}_{\gamma\delta} + I^{\lambda\sigma,}_{\gamma\delta} I^{\mu\nu,}_{\alpha\beta} - I^{\lambda\mu,}_{\alpha\beta} I^{\sigma\nu,}_{\gamma\delta} - I^{\sigma\nu,}_{\alpha\beta} I^{\lambda\mu,}_{\gamma\delta} \right] \\
 & + \left[ q_\lambda q^\mu \left( \eta_{\alpha\beta} I^{\lambda\nu,}_{\gamma\delta} + \eta_{\gamma\delta} I^{\lambda\nu,}_{\alpha\beta} \right) + q_\lambda q^\nu \left( \eta_{\alpha\beta} I^{\lambda\mu,}_{\gamma\delta} + \eta_{\gamma\delta} I^{\lambda\mu,}_{\alpha\beta} \right) \right. \\
 & \left. - q^2 \left( \eta_{\alpha\beta} I^{\mu\nu,}_{\gamma\delta} + \eta_{\gamma\delta} I^{\mu\nu,}_{\alpha\beta} \right) - \eta^{\mu\nu} q^\lambda q^\sigma \left( \eta_{\alpha\beta} I_{\gamma\delta,\lambda\sigma} + \eta_{\gamma\delta} I_{\alpha\beta,\lambda\sigma} \right) \right] \\
 & + \left[ 2q^\lambda \left( I^{\sigma\nu,}_{\alpha\beta} I_{\gamma\delta,\lambda\sigma} (k-q)^\mu + I^{\sigma\mu,}_{\alpha\beta} I_{\gamma\delta,\lambda\sigma} (k-q)^\nu \right. \right. \\
 & \left. \left. - I^{\sigma\nu,}_{\gamma\delta} I_{\alpha\beta,\lambda\sigma} k^\mu - I^{\sigma\mu,}_{\gamma\delta} I_{\alpha\beta,\lambda\sigma} k^\nu \right) \right. \\
 & \left. + q^2 \left( I^{\sigma\mu,}_{\alpha\beta} I_{\gamma\delta,\sigma}^\nu + I_{\alpha\beta,\sigma}^\nu I^{\sigma\mu,}_{\alpha\delta} \right) + \eta^{\mu\nu} q^\lambda q_\sigma \left( I_{\alpha\beta,\lambda\rho} I^{\rho\sigma,}_{\gamma\delta} + I_{\gamma\delta,\lambda\rho} I^{\rho\sigma,}_{\alpha\beta} \right) \right] \\
 & + \left\{ \left( k^2 + (k-q)^2 \right) \left( I^{\sigma\mu,}_{\alpha\beta} I_{\gamma\delta,\sigma}^\nu + I^{\sigma\nu,}_{\alpha\beta} I_{\gamma\delta,\sigma}^\mu - \frac{1}{2} \eta^{\mu\nu} P_{\alpha\beta,\gamma\delta} \right) \right. \\
 & \left. - \left( k^2 \eta_{\gamma\delta} I^{\mu\nu,}_{\alpha\beta} + (k-q)^2 \eta_{\alpha\beta} I^{\mu\nu,}_{\gamma\delta} \right) \right\} \quad (53)
 \end{aligned}$$

## Gravitational scattering



$$-i \mathcal{M}_\mu = i \frac{k}{2} \left[ (\hat{p}_1^\mu \hat{p}_2^\nu + \hat{p}_1^\nu \hat{p}_2^\mu) - \dots \right] \left[ \frac{i}{g^2} P_{\mu\nu\rho\sigma} \frac{k}{2} \left[ (p_3^\rho \hat{p}_4^\sigma + p_3^\sigma \hat{p}_4^\rho) + \dots \right] \right]$$

Non Rel       $p_\mu = (m, \vec{0})$

$$\mathcal{M} = -\frac{k^2}{g^2} \frac{m_1^2 m_2^2}{g^2} = -16\pi G \frac{m_1^2 m_2^2}{r^2}$$

F.T.     $\frac{1}{g^2} \rightarrow \frac{1}{4\pi G r^2}$    ,   norm    $\frac{1}{2E} \frac{1}{2E'} = \frac{1}{4m_1 m_2} \Rightarrow \boxed{V(r) = -G \frac{m_1 m_2}{r}}$