

GR QFT

Feb 12

Note Title

2/10/2014

No class on Tuesday Feb 17 — Monday class schedule at UMass

Plan: Advanced QFT Technique

- 1) Background Field
- 2) Heat Kernel
- 3) Gauge fixing & ghosts

App A, B

+ G + L

QED vac. pol. story

$$\mathcal{L} = -\frac{1}{4}F^2 + (\bar{\psi}\gamma^\mu\psi)^*$$

$$1) \langle \phi A | \mathcal{L}_+ | \phi \rangle = -e(p+p') \underbrace{\epsilon_1^{\mu\nu\rho\sigma} \frac{1}{2\pi i p' \sqrt{2\omega}}}_{\text{vac. pol. tensor}} \gamma_\mu$$

$\underbrace{\qquad\qquad\qquad}_{\text{vac. pol. tensor}} = -ie(p+p') \quad \text{or} \quad -ie\gamma_\mu$

From

$$\Pi_{\mu\nu} = \int \frac{d^4k}{(2\pi)^4} -ie(2\ell+\gamma)^\mu \frac{1}{k^2} \frac{1-ie(2\ell+\gamma)^\nu}{(\ell+\gamma)^2}$$

$$= \ell^2 (g_{\mu\nu} - g_{\mu\nu} \gamma^2) \# \sim \ell^2 \gamma^2$$

DSM

$$\begin{aligned} iD'_{\mu\nu} &= iD_{\mu\nu} + iD_{\mu\alpha}(i\Pi^{\alpha\beta}) iD_{\beta\nu} + \dots \\ &= \frac{-i}{q^2} \left[\frac{1}{1+\Pi(q)} \left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right) + \xi_0 \frac{q_\mu q_\nu}{q^2} \right], \end{aligned} \quad (1.18)$$

where the proper contribution

$$i\Pi^{\alpha\beta}(q) = (q^\alpha q^\beta - q^2 g^{\alpha\beta}) i\Pi(q) \quad (1.19)$$

is called the *vacuum polarization tensor*. It is depicted in Fig. II-2(a) (along with corrections to the photon-fermion vertex and fermion propagator in Figs. II-2(b)-(c)), and is given to lowest order by

$$i\Pi^{\alpha\beta}(q) = -(-ie_0)^2 \int \frac{d^4p}{(2\pi)^4} \text{Tr} \left[\gamma^\alpha \frac{i}{p-m+i\epsilon} \gamma^\beta \frac{i}{p-q-m+i\epsilon} \right]. \quad (1.20)$$

The self-energy of Eq. (1.20), now expressed as an integral in d dimensions, is

$$\Pi^{\alpha\beta}(q) = 4ie_0^2 \mu^{2\epsilon} \int \frac{d^d p}{(2\pi)^d} \frac{p^\alpha (p-q)^\beta + p^\beta (p-q)^\alpha + g^{\alpha\beta} (m^2 - p \cdot (p-q))}{[p^2 - m^2 + i\epsilon][(p-q)^2 - m^2 + i\epsilon]} \quad (1.22)$$

where we retain the same notation $\Pi^{\alpha\beta}(q)$ as for $d=4$ and we have already computed the trace. Upon introducing the Feynman parameterization, Dirac relations, and integral identities of App. C-5, we can perform the integration over momentum to obtain

$$\Pi^{\alpha\beta}(q) = (q^\alpha q^\beta - q^2 g^{\alpha\beta}) \frac{e_0^2}{2\pi^2} \frac{\Gamma(\epsilon)}{(4\pi)^{-\epsilon}} \mu^\epsilon \int_0^1 dx \frac{x(1-x)}{(m^2 - q^2 x(1-x))^\epsilon}. \quad (1.23)$$

$$e^2 \left[\frac{1}{g^2} + \frac{1}{g^2} e^2 g^2 \frac{\pi/\epsilon^2}{\delta^2} \right] \notag$$

$$= \frac{e^2}{g^2(1-\pi)} \Rightarrow e_R^2 = \frac{e_0^2}{1-\pi(0)} \notag$$

$$\frac{1}{g^2(1-\pi(0))}$$

wavefunction
renorm

$$\begin{aligned} \Pi(q) &= \frac{e_0^2}{12\pi^2} \left[\frac{1}{\epsilon} + \ln(4\pi) - \gamma \right. \\ &\quad \left. - 6 \int_0^1 dx x(1-x) \ln \left(\frac{m^2 - q^2 x(1-x)}{\mu^2} \right) + \mathcal{O}(\epsilon) \right] \\ &= \frac{e_0^2}{12\pi^2} \left\{ \frac{1}{\epsilon} + \ln(4\pi) - \gamma + \frac{5}{3} - \ln \frac{-q^2}{\mu^2} + \dots \right. \\ &\quad \left. (|q^2| \gg m^2), \right. \\ &\quad \left. \frac{1}{\epsilon} + \ln(4\pi) - \gamma - \ln \frac{m^2}{\mu^2} + \frac{q^2}{5m^2} + \dots \right. \\ &\quad \left. (m^2 \gg |q^2|). \right. \end{aligned} \quad (1.26)$$

$$\tilde{e} = \frac{e_0 - q}{2}$$

$$ie_0^2 D_{\mu\nu} = -\frac{i}{q^2} e_0^2 g_{\mu\nu} \rightarrow ie^2 D'_{\mu\nu} = -\frac{i}{q^2} \frac{e_0^2}{1+\Pi(q)} g_{\mu\nu} . \quad (1.29)$$

We display only the $g_{\mu\nu}$ piece since, in view of current conservation, only it can contribute to the full amplitude upon coupling the propagator to electromagnetic vertices. The above suggests that we associate the physical, renormalized charge e with the bare charge parameter e_0 by

$$e^2 = \frac{e_0^2}{1+\Pi(0)} \simeq e_0^2 [1 - \Pi(0)] . \quad (1.30)$$

$$\begin{aligned} \psi &= Z_2^{1/2} \psi^r , & A_\mu &= Z_3^{1/2} A_\mu^r , \\ e_0 &= Z_1 Z_2^{-1} Z_3^{-1/2} e , & m_0 &= m - \delta m , \\ \xi_0 &= Z_3 \xi , \end{aligned}$$

Gravity matter loop



$$\overbrace{\quad}^{\sim} = \frac{K}{2} (p_\mu p'_\nu + p'_\nu p_\mu)$$

$$m \text{---} m' \quad T_{\mu\nu\alpha\beta} = S \frac{d^4 \ell}{(2\pi)^4} \frac{K}{2} (\ell_\mu (\ell + g)_\nu) \frac{1}{\ell^2} \frac{1}{(\ell + g)^2} \frac{K}{2} \delta_\alpha (\ell + g)_\beta$$

$$= K^2 g^4$$

G

$$\overbrace{\quad}^{\{ \} \text{---}} - \overbrace{\quad}^{\{ \} \text{---}} + \dots \quad \frac{K}{2} \frac{1}{g^2} \frac{K}{2} + \frac{K}{2} * \frac{1}{g^2} \frac{K^2 g^4}{g^2} \frac{1}{g^2} \frac{K}{2}$$

$$= K^2 \frac{1}{g} \left[1 + K^2 g^2 + \dots \right]$$

↑ $G = \frac{1}{M_p}$

$$k^2 \frac{1}{g^2 E(1 - k^2 j^2)}$$

$\uparrow \cancel{\star}$

does not renorm k^2

$\prod_{\mu\nu\beta}$

$$T_{\mu\nu\alpha\beta}(k) = \frac{i}{3840\pi^2} \left(\frac{1}{\bar{e}} - \log \left(\frac{-k^2}{\mu^2} \right) \right) [k^4 (6\eta_{\mu\nu}\eta_{\alpha\beta} + \eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\mu\beta}\eta_{\nu\alpha}) + 8k_\mu k_\nu k_\alpha k_\beta - k^2 (6k_\mu k_\nu \eta_{\alpha\beta} + 6k_\alpha k_\beta \eta_{\mu\nu} + k_\mu k_\alpha \eta_{\nu\beta} + k_\mu k_\beta \eta_{\nu\alpha} + k_\nu k_\alpha \eta_{\mu\beta} + k_\nu k_\beta \eta_{\mu\alpha})] \quad (14)$$

and

$$\frac{1}{\bar{e}} \equiv \frac{1}{e} - \gamma + \log 4\pi \quad (15)$$

with $2e = 4 - d$.

But $\frac{1}{e}$ divergence

does not go into k^2 renorm or $h_{\mu\nu}$ renorm!

Hint: not renorm R $\uparrow \cancel{\star}$ but R^2 $\leftarrow \star$

Quick P.I. review

Basic Gaussian

$$\int_{-\infty}^{\infty} dy e^{-\frac{1}{2} \alpha y^2} = \left(\frac{2\pi}{\alpha}\right)^{1/2}$$

$$\int dx e^{-\frac{1}{2} \alpha x^2 - i\sqrt{\lambda} x} = \left(\frac{2\pi i}{\alpha}\right)^{1/2} e^{-\frac{\sqrt{\lambda}^2}{2\alpha}}$$

Then

$$\int dx_1 \dots dx_N e^{-i\left[\frac{1}{2} \sum_i A_{ii} x_i^2 + \sum_i J_i x_i\right]}$$

$$= \frac{\left[2\pi i\right]^{N/2}}{\left[\det A\right]^{1/2}} e^{-J^T A^{-1} J}$$

Proof:

$$X^T A X = \underbrace{(X^T)}_Y \underbrace{(A)}_{A_D} \underbrace{(X)}_Y = (X^T) \underbrace{O}_D A_D^{-1} O (X)$$

$$a_D, a_{D_2}, a_{D_3}, \dots = \det A_D = \det A$$

$$FT \quad x_i \rightarrow \phi(x) \quad , \quad \sum_i \rightarrow \int d^4x$$

$$dx_1 - dx_N = [dx] \rightarrow \int [d\phi]$$

$$\int [d\phi] e^{i \int d^4x [\phi \partial \phi + J\phi]}$$

$$= N \frac{1}{[\det \mathcal{O}]^{1/2}} e^{-\frac{i}{2} \int d^4x d^4y J(x) \mathcal{O}^{-1}(x,y) J(y)}$$

$$\mathcal{O} = \frac{(I+m)}{|\det \mathcal{O}|}$$

$$\mathcal{O}^{-1} = i D_F(x-y)$$

Complex scalar

$$S[d\phi][d\phi^*] e^{-i \int d^4x \phi_m \partial^\mu \phi} = \frac{N}{[\det \partial]}$$

Math identity

$$\det \partial = e^{\text{Tr}(\ln \partial)}$$

Tr includes $\int d^4x$

Overall

$$S[d\phi][d\phi^*] e^{-i \int d^4x \phi \partial^\mu \phi} = [\det \partial]^{-1} = e^{-\text{Tr} \ln \partial} = e^{-\int d^4x \langle x | \ln \partial | x \rangle}$$

Background field

B.F. - Calc P.I. in presence of field A_m

- Change in \mathcal{L}
- Renormalize directly in \mathcal{L}
- Take matrix elements

Photon + rac. pol

$$\mathcal{L} = (\partial_\mu \phi)^* (\partial^\mu \phi) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

\equiv

$$D_\mu = (\partial_\mu + ie A_\mu)$$

Int by parts

$$\mathcal{L} = -\phi^* D_\mu D^\mu \phi \Rightarrow \cancel{\mathcal{L}} = D_\mu D^\mu$$

$$\begin{aligned} D_\mu D^\mu &= \square + ie A_\mu \partial^\mu + \partial^\mu (ie A_\mu) - e^2 A_\mu A^\mu \\ &= \square + 2ie A_\mu \partial^\mu + ie (\partial^\mu A_\mu) - e^2 A_\mu A^\mu \\ &= \square + \sqrt{g} G(\partial) \end{aligned}$$

$$\begin{aligned}
 \text{Tr} \ln D_x P^n &= \text{Tr} \ln (I + n) \\
 \xrightarrow{\quad} &= \text{Tr} \ln \left[I \left(1 + \frac{1}{n} n \right) \right] \\
 &= \text{Tr} \ln I + \text{Tr} \ln \left(1 + \frac{1}{n} n \right) \\
 &= \underbrace{\text{Tr} \ln I}_{\sim} + \text{Tr} \left(\frac{1}{n} n - \frac{1}{2} \frac{1}{n} n \frac{1}{n} n + \dots \right)
 \end{aligned}$$

Drop

First term $\langle x | \frac{1}{n} | y \rangle = i \Delta_F(x-y)$

$\square \Delta_F(x-y) = \delta''(x-y)$

$\square \frac{1}{n} = 1$

$$\Rightarrow \text{Tr} \left(\frac{1}{\Omega} N \right)' = \int d^4x \ i \Delta_F(x-y) N(y)$$

under

$$\Delta(0) = \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 - m^2} = (m^2)^{\frac{1-2}{2}} \xrightarrow[m \rightarrow 0]{} 0$$

2nd order $\downarrow_{y>x}$

$$\frac{1}{2} \text{Tr} \left(\frac{1}{\Omega} N \frac{1}{\Omega} N \right) = \frac{1}{2} \int d^4x d^4y \left(i \Delta_F(x-y) N(y) i \Delta_F(y-x) N(x) \right)$$

~on

$\uparrow \text{ keep } (A)' \text{ in } N$

$$N = i \epsilon (\partial^\mu A_\mu) + i \epsilon \underset{=}{{\bar A}^\mu} \partial_\mu$$

Aiming for

$$\frac{e^2}{2} \int d^4x d^4y A_\mu(x) M_{\mu\nu}(x-y) A_\nu(y)$$

Do by int by part , $\partial^\mu_x \Delta_F(x-y) = \partial^\mu_y \Delta_F(x-y)$

$$M_{\mu\nu}(x-y) = \partial_\mu \Delta_F(x-y) \partial_\nu \Delta_F(x-y) - \Delta_F(x-y) \partial_\mu \partial_\nu \Delta_F(x-y)$$

Claim $\boxed{M_{\mu\nu}(x-y) = \left(g_{\mu\nu} \square - \partial_\mu \partial_\nu\right) \frac{\Delta_F^2(x-y)}{d-1}}$ \hookrightarrow reg. dim.

$$1) \Delta_F^2(x) \partial_\mu \Delta_F(x) = \frac{1}{2} \partial_\mu \Delta^2(x)$$

$$2) \Delta_F^2(x) \partial_\mu \partial_\nu \Delta_F(x) = (d \partial_\mu \partial_\nu - g_{\mu\nu} \square) \frac{\Delta_F^2(x)}{4(d-1)}$$

$$\begin{aligned} \text{F.T of LHS} &= \int d^4x e^{-ih \cdot x} \int \frac{d^4p}{(2\pi)^4} e^{-ip \cdot x} \int \frac{d^4p'}{(2\pi)^4} e^{-ip' \cdot x} \frac{p_\mu p'_\nu}{p'^2} \\ &= \int \frac{d^4p}{(2\pi)^4} \frac{1}{p^2} \frac{1}{(p+h)^2} \frac{(p+\lambda)_\mu (p+\lambda)_\nu}{p'^2} = a \delta_{\mu\nu} + b g_{\mu\nu} h^2 \\ &= \text{F.T. RHS} \quad \text{match } a+b \end{aligned}$$

$$3) \partial_\mu \Delta_F(x) \partial_\nu \Delta_F(x) = ((d-2) \partial_\mu \partial_\nu + g_{\mu\nu} \square) \frac{\Delta_F^2(x)}{4(d-1)}$$

Next trick $\partial_x \rightarrow \partial_y$, unit by parts

$$\Rightarrow = \left[-i \int d^4x d^4y F_{\mu\nu}(x) \frac{\Delta_F^2(x-y)}{4(d-1)} F^{\mu\nu}(y) \right] \quad \leftarrow$$

in J_{μν}

Next - use this