

GR QFT Feb 24

Note Title

2/10/2014

Background field exercises Instability in Higgs $V(h)$

1) ground field $h(x) = v + H(x)$. In the following we take H (and hence h) as constant and thus omit any spacetime dependence,

$$-\mathcal{L}_t = \frac{\Gamma_t}{\sqrt{2}}(v + H)\bar{t}t \equiv \frac{\Gamma_t}{\sqrt{2}}h\bar{t}t \equiv m_t(h)\bar{t}t , \quad (5.1)$$

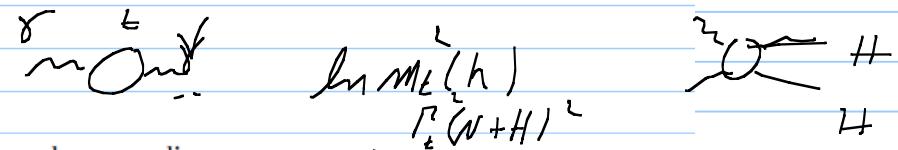
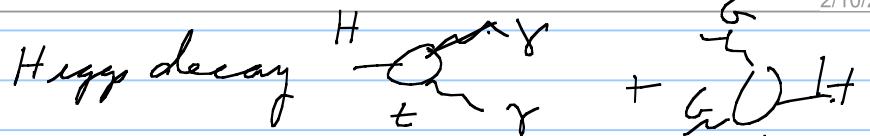
where $m_t(h) = \Gamma_t h / \sqrt{2}$ is the field dependent mass. We then calculate the vacuum energy as a function of $m_t(h)$. This can be done relatively simply by studying the $t\bar{t}$ contribution to the vacuum matrix element of the energy momentum tensor $\mathcal{T}_{\mu\nu}$,

$$\begin{aligned} \langle 0 | \mathcal{T}_{\mu\nu} | 0 \rangle_{\text{top}} &= -N_c \int \frac{d^d p}{(2\pi)^d} \frac{1}{2} \text{Tr}[(\gamma_\mu p_\nu + \gamma_\nu p_\mu) \frac{i}{p - m_t(h) + i\epsilon}] \\ &= -12 \int \frac{d^d p}{(2\pi)^d} p_\mu p_\nu \frac{i}{p^2 - m_t^2(h) + i\epsilon} \\ &= \delta V(h) g_{\mu\nu} , \end{aligned} \quad (5.2)$$

where the important minus sign comes from the Feynman rule for a closed fermion loop. This leads to a result

$$\delta V(h) = \frac{3m_t^4(h)}{16\pi^2} \left[\frac{2}{4-d} - \gamma + \ln 4\pi - \ln \frac{m_t^2(h)}{\mu_d^2} + \frac{3}{2} \right] , \quad (5.3)$$

$$V(h) = -\frac{1}{2}\mu^2 h^2 + \frac{1}{4}\lambda(\mu_d)h^4 - \frac{3m_t^4(h)}{16\pi^2} \left[\ln \frac{m_t^2(h)}{\mu_d^2} - \frac{3}{2} \right]$$



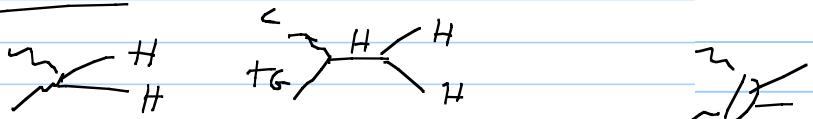
2) Higgs-gluon coupling

In the text we used the background field method to show that, at lowest order in the momenta, the effective Higgs coupling to gluons is

$$\mathcal{L}_{eff} = \frac{\alpha_s}{24\pi} \ln \left(\frac{h^2}{v^2} \right) F_{\mu\nu}^a F_a^{\mu\nu} ,$$

with $h = v + H$. As mentioned briefly in the text, this coupling implies a cancellation in the Standard Model prediction for the reaction in which two gluons produce two Higgs bosons, which makes the residual effect small. In addition to the direct coupling from the above effective

lagrangian, there is a pole diagram of $GG \rightarrow H \rightarrow HH$, which utilizes the triple Higgs coupling. Show that these two contributions cancel exactly at threshold.



Heat Kernel (General)

$$K(x, y, \tau, D) = \langle x | e^{-\tau D} | y \rangle$$

Satisfies

$$(\partial_\tau + D_x) K(x, y, \tau, D) = 0$$

with

$$K(x, y, 0, D) = \delta^d(x - y)$$

τ coef of heat diffusion

$$\text{if } D = -\alpha \vec{\nabla}^2$$

heat eq green function

History

→ 1800's Math literature

- Fock, Schwinger for QFT → Schwinger proper time

- Gravity De Witt (Seeley, Gilkey(1975) - ...)

If $D = \square + m^2$

$$K(x, y, \tau, \square + m^2) = \frac{1}{(4\pi\tau)} e^{-\left[\frac{(x-y)^2}{4\tau} + \tau m^2\right]}$$

Also propagator

$$\frac{i}{A+i\varepsilon} = \int_0^\infty d\tilde{\tau} e^{+i\tilde{\tau}(A+i\varepsilon)}$$

$$iD_F(x-y) = \langle x | \frac{i}{\square + m^2 + i\varepsilon} | y \rangle = -i \int_0^\infty d\tilde{\tau} \frac{1}{16\pi^2 \tilde{\tau}^2} e^{-i\left[\frac{(x-y)^2}{4\tilde{\tau}} + \tau(m^2 + i\varepsilon)\right]}$$
$$= \frac{-i}{4\pi^2} \frac{1}{(x-y)^2 - i\varepsilon} \quad \text{if } m=0$$

Schwartz
QFT

Example in Ch 8

$$\text{Tr } \ln(D_\mu D^\mu + m^2) \quad D_\mu = \partial_\mu + ieA_\mu$$

$$= \int d^4x \langle x | \quad | x \rangle$$

$$\text{Use } \ln b/a = \int_0^\infty \frac{dt}{t} \left(e^{-ta} - e^{-tb} \right)$$

$$\begin{aligned} \text{Tr } \ln(D_\mu D^\mu + m^2) &= \int d^4x \int_0^\infty dt \langle x | e^{-t(D_\mu D^\mu + m^2)} | x \rangle \\ &\quad \underbrace{\qquad\qquad\qquad}_{\hat{C}_H} \underbrace{e^{-t(D_\mu D^\mu + m^2)}}_{K(x, t)} \end{aligned}$$

To evaluate K

completeness with $\langle p|x \rangle = e^{ip \cdot x}$

$$K(x, \tau) = \int \frac{d^4 p}{(2\pi)^4} e^{-ip \cdot x} e^{-\tau D} e^{ip \cdot x} \quad \leftarrow \text{expand in } \tau$$

Then $D_m e^{ip \cdot x} = e^{ip \cdot x} (i p_m + D_m)$

$$D_m D^m e^{ip \cdot x} = e^{ip \cdot x} (i p_m + D_m) (i p^m + D^m)$$

$$K(x, \tau) = \int \frac{d^4 p}{(2\pi)^4} e^{-i\tau ([ip_m + D_m]^2 + m^2)} e^{i p \cdot (x - \eta)} \quad \text{in general} \quad e^{\tau (p^2 - m^2)} e^{-\tau [D_m D^m + 2 i p_m D^m]}$$

$$\begin{aligned}
 K(n, \tilde{\gamma}) &= \int \frac{d^d p}{(2\pi)^d} e^{-\tilde{\gamma}(p^2 - m^2)} \left[1 - \tilde{\gamma} (D_\mu D^\mu) \right. \\
 &\quad + \frac{\tilde{\gamma}^2}{2} \left(D_\mu D^\mu - 4(p \cdot D)(p \cdot D) \right) \\
 &\quad + \frac{4\tilde{\gamma}^3}{3!} (p \cdot D p \cdot D p \cdot D \dots) \\
 &\quad \left. + \frac{16\tilde{\gamma}^4}{4!} (p \cdot D p \cdot D p \cdot D p \cdot D) \dots \right]
 \end{aligned}$$

$$\begin{aligned}
 \int \frac{d^d p_E}{(2\pi)^d} e^{-\tilde{\gamma}(p_E^2 + m^2)} &= \int \frac{dR_4}{(2\pi)^d} \int dp_E p_E^{d-1} e^{-\tilde{\gamma}(p_E^2 + m^2)} \quad P \rightarrow (p_1, p_2, p_3, -ip_0) \\
 &= p_E \\
 &= \frac{2\pi^{d/2}}{\Gamma(d/2)} \frac{1}{(2\pi)^d} e^{-m^2 \tilde{\gamma}} \frac{\Gamma(d/2)}{2\pi^{d/2}} = \frac{1}{(4\pi)^{d/2}} e^{-m^2 \tilde{\gamma}}
 \end{aligned}$$

$$\int \frac{d^d p_E}{(2\pi)^d} e^{-(p_E^2 + m^2)\tilde{\tau}} P_E^\mu P_E^\nu = \frac{\delta^{uv}}{d} \frac{1}{(4\pi)^{d/2}} \frac{1}{\tilde{\tau}^{d/2+1}} e^{-\frac{m^2}{\tilde{\tau}}} \frac{\Gamma(d/2+1)}{\Gamma(d/2)}$$

$$K(x, \tilde{\tau}) = \frac{e^{-m^2 \tilde{\tau}}}{(4\pi)^{d/2} \tilde{\tau}^{d/2}} \left[1 + \underbrace{\frac{1}{12} [D_u, D_v] [D^u, D^v]}_{\propto F_{uv} F^{uv}} \tilde{\tau}^2 + \dots \right]$$

$$\int_0^\infty \frac{d\tilde{\tau}}{\tilde{\tau}} \frac{e^{-m^2 \tilde{\tau}}}{(4\pi)^{d/2}} \frac{\tilde{\tau}^2}{\tilde{\tau}^{d/2}} = -\frac{1}{(4\pi)^{d/2}} (m^2)^{\frac{d-4}{2}} \underbrace{\Gamma(2 - \frac{d}{2})}_{\propto} \frac{1}{\varepsilon} = \frac{2}{4-d}$$

Renormalization constant m_0

$$\Delta Z = \int d^4x F_{\mu\nu} F^{\mu\nu} \frac{e^2}{16\pi^2} \frac{1}{12} \left(\frac{2}{4-d} + \dots \right)$$

Feynman diagram ?

In general

$$K(x, \tilde{x}) = \langle x | e^{-\tilde{x} \partial_x} | x \rangle$$
$$= \frac{i}{(4\pi)}^{d_1} \frac{e^{-\tilde{x} m^2}}{\tilde{x}^{d_1}} [a_0(x) + \tilde{x} a_1(x) + \tilde{x}^2 a_2(x) + \dots]$$

$$\text{If } \partial = (d_m d^m + \Gamma)_{+m^2} \quad d_m = \partial_m + \Gamma_m$$

$$a_0(x) = 1 \quad , \quad a_1(x) = -\Gamma(x)$$

$$a_2(x) = \underbrace{\frac{1}{2} \Gamma^2 + \frac{1}{12} [d_m, d_n] [d^m, d^n]}_{\star \star} + \frac{1}{6} [d_m, [d_m, \Gamma]]$$

$\stackrel{\text{total deriv}}{\sim}$
 $\int d^4x a_2(x)$

And

$$\int d^4x \langle N | \ln \mathcal{O} | N \rangle = \frac{-i}{4\pi} \zeta_2 \sum_{m=0}^{\infty} m^{d-2m} \Gamma(\eta - d_m) \int d^4x \alpha_m(x)$$

\nwarrow
 α_m contains
divergences of theory

$\alpha_m(\eta)$ = "Seeley DeWitt coeff."

GILKEY TS

Web page "User Manual"
Vassilivitch

BYD, Parker Toms

For gravity

(Should work - but . . .)

$$S = \int d^4x \sqrt{-g} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi = \int d^4x \underbrace{\phi \partial^\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu)}_{\mathcal{O}} \phi$$

Asy. Riemann normal coord

- locally flat at x , $y = x' - x$

$$g_{\mu\nu}(x') = \eta_{\mu\nu} - \frac{1}{3} R_{\alpha\beta\gamma\delta} y^\alpha y^\beta + \mathcal{O}(y^3) + \mathcal{O}(y^4)$$

$$\sqrt{-g} = 1 - \frac{1}{3} R_{\alpha\beta} y^\alpha y^\beta \quad \text{evaluated at } x$$

B+D

Exercise obtain R_{μναβ} in R.N.C.

$$\Gamma_{\mu\nu}^{\lambda} = \frac{1}{2} g^{\lambda\alpha} \left[\partial_\mu g_{\nu\alpha} - \partial_\nu g_{\mu\alpha} + \dots \right] = -\frac{g^{\lambda\alpha}}{3} [R_{\mu\alpha\nu\sigma} + R_{\nu\alpha\mu\sigma}] g^\sigma_\lambda$$

$$R = \partial_\mu \overline{R} + \cancel{\partial^2} + \text{Bianchi I } R_{[\mu\nu\alpha\beta]} = 0$$

Using this $\mathcal{O} = d_\mu d^{m+1} \sigma$ form

Seeger DeWitt

Scalar $\mathcal{L} = \partial_\mu \phi \partial^\mu \phi + m^2 \phi^2 + \zeta R \phi^2$

$\zeta = 0$ minimally coupled
 $\zeta = \frac{1}{2}$ conformal

$$g_{\mu\nu}^{(0)} = e^{\frac{2R\phi}{m^2}} \tilde{g}_{\mu\nu}$$

$$\alpha_1 = \left(\frac{1}{6} - \zeta\right) R$$

$$\alpha_2 = \frac{1}{180} \left[R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - R_{\mu\nu} R^{\mu\nu} + \frac{5}{2} (6\zeta - 1)^2 R^2 - 6 \square R \right]$$

Photons:

$$\alpha_1 = 0$$

$$\alpha_2 = -\frac{1}{180} \left(R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 86 R_{\mu\nu} R^{\mu\nu} + 20 R^2 + ? \square R \right)$$

Notes from earlier class Feb 12

Gravity matter loop

$$= \frac{k}{2} (P_\mu P'_\nu + P'_\nu P_\mu)$$



$$T_{\mu\nu\rho\sigma} = S \frac{d^d g}{(2\pi)^4} \frac{k}{2} (l + g)^\nu \frac{1}{l^2} \frac{1}{(l+g)^2} \frac{k}{2} l \otimes (l+g)_\rho$$

$$= \frac{k^2}{8} g^4$$

G

$$\begin{aligned} I &= \underline{\underline{I}} + \dots \\ &= k^2 \frac{1}{8} \left[1 + k^2 g^2 \right] \end{aligned}$$

$$G = \frac{1}{8} g^2$$

$$k^2 \frac{1}{g^2} \cancel{\frac{1}{E(1-k^2 g^2)}}$$

does not renorm k^2

$\cancel{T_{\mu\nu}}$

But $\frac{1}{\epsilon}$ diverges

does not go into k^2 renorm or $T_{\mu\nu}$ renorm!

Hint: not renorm R but $\frac{R^2}{\delta} \cancel{\int d^d k}$

$$T_{\mu\nu\rho\beta}(k) = \frac{k^2}{3840\pi^2} \left(\frac{1}{\epsilon} - \log \left(\frac{-k^2}{\mu^2} \right) \right) [k^4 (6\eta_{\mu\nu}\eta_{\alpha\beta} + \eta_{\mu\alpha}\eta_{\nu\beta}) + \eta_{\mu\beta}\eta_{\nu\alpha}) + 8k_\mu k_\nu k_\alpha k_\beta - k^2 (6k_\mu k_\nu \eta_{\alpha\beta} + 6k_\alpha k_\beta \eta_{\mu\nu} + k_\mu k_\alpha \eta_{\nu\beta} + k_\mu k_\beta \eta_{\nu\alpha} + k_\nu k_\alpha \eta_{\mu\beta} + k_\nu k_\beta \eta_{\mu\alpha})]$$

and

$$\frac{1}{\epsilon} = \frac{1}{\epsilon} - \gamma + \log 4\pi \quad (15)$$

with $2\epsilon = 4 - d$.

Recall Feynman rules gravity

$$\overbrace{p \quad p'}^{\{ } = -i \frac{k}{2} (p_\mu p'_\nu + p'_\mu p_\nu)$$

$$m \nu m \partial^\mu \partial_\mu \sim k^2 \left[\frac{1}{\epsilon} + \dots \right] [g_{\mu\nu} g_{\rho\sigma} \partial^\mu \partial^\sigma + \dots]$$

We see this better using heat kernel

$$\int d^4x \langle N | h \theta | N \rangle = \frac{1}{16\pi^2} \int \frac{1}{\epsilon} \int \frac{1}{180} \int d^4x \sqrt{g} (R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - R_{\mu\nu} R^{\mu\nu} + \frac{5}{2} R^2)$$

$$R \sim \partial^2 h \quad R^2 \sim g^4 \quad \text{at } \star = m \Omega_m \quad \text{at } \mathcal{O}(h^2)$$

But 1) easy

2) also $\sim \text{O}^2 + 3\text{O} + \text{O}^3 + \dots$

All divergences

constrained by symmetry - general covariance

3) Make general covariance obvious !

4) Shows us how to renormalize

$$S = \int d^4x \left[\frac{2}{k^2} R + c_1 R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} + c_2 R_{\mu\nu} R^{\mu\nu} \right]$$

\uparrow \uparrow
div go here