

GR QFT

Jan 29

Note Title

2/10/2014

Gauged Lorentz $dx^m = \Lambda^m_{\nu} dx^\nu$
 $g_{\mu\nu}(x)$

Covariant action

$$S = \int d^4x \sqrt{-g} \frac{1}{2} [g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - m^2 \phi^2]$$

$$\frac{2}{\sqrt{g}} \frac{\delta S}{\delta g^{\mu\nu}} = T_{\mu\nu} \quad \leftarrow \text{source}$$

Covariant Deriv

$$V'^\beta = \Lambda^\rho_\alpha V^\alpha$$

$$D_\mu V'^\beta = \Lambda^\nu_\mu \Lambda^\beta_\alpha D_\nu V^\alpha$$

$$D_\mu V^\beta = \underline{\partial_\mu V^\beta} + \underbrace{\Gamma_{\mu\nu}^\beta}_{\Gamma} V^\nu$$

$$\Gamma'_{\mu\nu}^\lambda = \Lambda_\mu^\rho \Lambda_\nu^\sigma \Lambda^\lambda_\tau [\Gamma_{\rho\sigma}^{\tau\lambda} + \Lambda_\alpha^\gamma \partial_\rho \Lambda_\gamma^\lambda]$$

$$\Gamma_{\mu\nu}^\lambda = \frac{1}{2} g^{\lambda\sigma} [\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}]$$

Tensors - transform covariantly

$$T^{\alpha\beta} = \Gamma^\alpha_\gamma \Gamma^\beta_\mu \Gamma^\gamma_\nu \Gamma^\mu_\sigma \Gamma^\nu_\tau T^{\alpha'\beta'} \delta^{\sigma\tau}$$

$$D_\mu T^{\alpha\beta} = \partial_\mu T^{\alpha\beta} + \Gamma^\alpha_{\mu\nu} T^{\nu\beta} + \Gamma^\beta_{\mu\nu} T^{\alpha\nu}$$

$$D_\mu V_\lambda = \partial_\mu V_\lambda - \underline{\Gamma^\alpha_{\mu\nu}} V_\alpha$$

$$(from \partial_\mu (V_\lambda V^\lambda) = D_\lambda (V_\lambda V^\lambda))$$

Note $\Gamma^\lambda_{\mu\nu}$ is not a tensor!
↑ a "connection"

Aside: Photon is not a 4 vector

Weninger QFT I Ch 5.9

$$A_\mu(x) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E}} \left[\epsilon^\lambda(p, \lambda) a(p, \lambda) e^{-ip \cdot x} + h.c. \right]$$

$\nwarrow p_\mu \epsilon^\mu = 0 \quad , \quad \epsilon^0 = 0$

$$\epsilon(p^\mu) = \frac{1}{\sqrt{2}} (0, 1+i, 0)$$

Lorentz trans. $\Lambda_\mu^\nu A_\nu \rightarrow \underbrace{\Lambda_\mu^\nu \epsilon_\nu}_\epsilon$

ϵ' no longer satisfies $p^\mu \cdot \epsilon' = 0$

Need to do gauge trans to put it in proper form

$F_{\mu\nu}$ is a Lorentz tensor , $\mathcal{L} = \dots - A_\mu J^\mu$

Metricity EP

- Free fall no grav force locally \Rightarrow coord $\Rightarrow g_{\mu\nu} = \eta_{\mu\nu}$
- Set up local flat coord $\boxed{+}$.

$$g_{\mu\nu} \rightarrow \underline{\eta_{\mu\nu}} \text{ locally}$$

$$\underline{\Gamma^{\lambda}_{\mu\nu}} \partial_{\lambda} g \rightarrow 0$$

$$\boxed{D_{\alpha} g_{\mu\nu} = 0} \quad \leftarrow EP$$

Physicist's Point particle $\ddot{x}^m + \Gamma_{\lambda\sigma}^m \dot{x}^\lambda \dot{x}^\sigma = 0 \quad \leftarrow \text{geodesic}$

$$\begin{matrix} \Gamma_a \\ \uparrow F/m \end{matrix} \quad \dot{x}^m = \frac{dx^m}{d\tau}$$

Derivation

$$\textcircled{1} \quad D_\alpha g_{\mu\nu} = \partial_\alpha g_{\mu\nu} - \overline{\Gamma}^{\sigma}_{\mu\alpha} g_{\sigma\nu} - \overline{\Gamma}^{\sigma}_{\nu\alpha} g_{\mu\sigma} = 0$$

$$\textcircled{2} \quad D_\mu g_{\nu\alpha} = \partial_\mu g_{\nu\alpha} - \overline{\Gamma}^{\sigma}_{\nu\mu} g_{\sigma\alpha} - \overline{\Gamma}^{\sigma}_{\alpha\mu} g_{\nu\sigma} = 0$$

$$\textcircled{3} \quad D_\nu g_{\alpha\lambda} = \partial_\nu g_{\alpha\lambda} - \overline{\Gamma}^{\sigma}_{\alpha\nu} g_{\sigma\lambda} - \overline{\Gamma}^{\sigma}_{\lambda\nu} g_{\alpha\sigma} = 0$$

Then $\overline{\Gamma}^{\sigma}_{\mu\nu} = \overline{\Gamma}^{\sigma}_{\nu\mu}$ ← symmetric (^{"torsion"})

$$\frac{1}{2}\textcircled{1} - \frac{1}{2}(\textcircled{2} + \textcircled{3})$$

$$\overline{\Gamma}^{\sigma}_{\mu\nu} g_{\sigma\alpha} = \frac{1}{2} (\text{-----}) \quad) \leftarrow \text{from before}$$

Curvatures

$$\psi \rightarrow U\psi$$

$$YM \quad D_\mu \psi = (\partial_\mu + i g \frac{\lambda}{2} A_\mu^a) \psi$$

$$[D_\mu, D_\nu] \psi = i g \frac{\lambda^4}{2} F_{\mu\nu}^a \psi$$

$$D'_\mu = U D_\mu U^\dagger \leftarrow$$

$$D'_\mu \psi' = U D_\mu \psi$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - g [A_\mu, A_\nu]$$

$$A'_\mu = \frac{\lambda^4}{2} A_\mu$$

HERE :

$$[D_\mu, D_\nu] V^\beta \equiv R_{\mu\nu\alpha}^\beta V^\alpha$$

\leftarrow Riemann curvature

$$R_{\mu\nu\alpha}^\beta = \partial_\mu \Gamma_{\nu\alpha}^\beta - \partial_\nu \Gamma_{\mu\alpha}^\beta + \Gamma_{\mu\rho}^\beta \Gamma_{\nu\alpha}^\rho - \Gamma_{\nu\rho}^\beta \Gamma_{\mu\alpha}^\rho$$

\leftarrow Tensor

And $R_{\nu\alpha} = R_{\mu\nu\alpha}^{\mu}$ "Ricci" ($R_{\mu\nu\alpha}^{\alpha} = 0$)

$$g^{\nu\alpha} R_{\nu\alpha} = R$$

$\curvearrowleft \quad \circlearrowright$

$\partial^2 g$

Ricci scalar

Soon $S \propto \int d^4x R$

$$\frac{R_{\mu\nu}}{+3g} \approx T_{\mu\nu}$$

Properties of Riemann

$$R_{\mu\nu\alpha}^{\beta} = -R_{\nu\mu\alpha}^{\beta} = R_{[\mu\nu]\alpha}^{\beta}$$

$$\begin{aligned} [\mu\nu] &\equiv \frac{1}{2} [\mu\nu - \nu\mu] \quad \leftarrow AS \\ (\mu\nu) &= \frac{1}{2} (\mu\nu + \nu\mu) \quad \leftarrow S \end{aligned}$$

$$R_{\mu\nu\alpha\beta} = R_{\mu\nu\alpha}^{\gamma} g_{\beta\gamma}$$

$$\text{Then } R_{\mu\nu\alpha\beta} = R_{\alpha\beta\mu\nu}$$

$$\Rightarrow R_{\mu\nu\alpha\beta} = R_{\mu\nu[\alpha\beta]} = R_{[\mu\nu][\alpha\beta]}$$

$$\text{Less obvious } R_{[\mu\nu\alpha]}^{\beta} = 0 \quad (\text{if } \overline{R}_{\mu\nu}^{\lambda} = \overline{R}_{\nu\mu}^{\lambda})$$

$$\overline{R}_{[\mu\nu]}^{\lambda} = 0$$

Branch 1

even less $D_\lambda R_{\mu\nu\alpha}^\beta = 0$ Branchi 2
 $\wedge \frac{1}{3!} []$

contract B2 $\Rightarrow 2 D_\nu R_{\mu}^{\nu} = D_\mu R \quad \times \times$ contracted Branchi

$$\Rightarrow D_\nu \underbrace{(R_\mu^\nu - \frac{1}{2} \delta_\mu^\nu R)}_G = 0$$

Einstein Equations

$$S_{EH} = \int d^4x \sqrt{g} \left[-\frac{2}{\kappa^2} R \right]$$

Einstein Hilbert

$$\kappa^2 = 32\pi G_N \quad (\text{later})$$

$$\begin{aligned} \delta S_{EH} &= -\frac{2}{\kappa^2} \int d^4x \delta \left(\sqrt{g} g^{uv} R_{uv} \right) = -\frac{2}{\kappa^2} \int d^4x \left(\delta \sqrt{g} R + \sqrt{g} \delta g^{uv} R_{uv} \right. \\ &\quad \left. + \sqrt{g} g^{uv} \delta R_{uv} \right) \\ &= -\frac{2}{\kappa^2} \int d^4x \underbrace{\left[-\frac{1}{2} g_{uv} R + R_{uv} \right]}_{\delta g^{uv}} + \left(\begin{array}{c} \text{total} \\ \text{deriv.} \end{array} \right) \left[\delta g^{uv} \right] \end{aligned}$$

$$= \frac{2}{\kappa^2} \int d^4x \sqrt{g} G_{uv} \delta g^{uv}$$

$$\left(R_{uv} - \frac{1}{2} g_{uv} R \right) \longleftarrow *$$

To show total deriv.

$$\begin{aligned}\delta R_{\mu\nu} &= \delta \left(\partial_\lambda \Gamma_{\mu\nu}^\lambda + \Gamma_{\lambda\rho}^\lambda \Gamma_{\mu\nu}^\rho - (\lambda \leftrightarrow \mu) \right) \\ &= \underline{\partial_\lambda} \underline{\delta \Gamma_{\mu\nu}^\lambda} + (\delta \Gamma_{\lambda\rho}^\lambda) \Gamma_{\mu\nu}^\rho + \Gamma_{\lambda\rho}^\lambda (\delta \Gamma_{\mu\nu}^\rho) - - -\end{aligned}$$

$$D_\lambda (\delta \Gamma_{\mu\nu}^\lambda) = \underline{\partial_\lambda} (\delta \Gamma_{\mu\nu}^\lambda) - \nabla \Gamma \nabla \Gamma,$$

$$\delta R_{\mu\nu} = D_\lambda (\delta \Gamma_{\mu\nu}^\lambda) - D_\nu (\delta \Gamma_{\lambda\nu}^\lambda) \quad \leftarrow \text{contracted Palatini Identity}$$

$$\int d^4x \underbrace{\sqrt{g} g^{\mu\nu}}_{\text{Surface term}} D_\lambda \delta \Gamma \rightarrow 0 \quad \text{because } \underline{\underline{D_\lambda (\sqrt{g} g^{\mu\nu})}} = 0$$

Add side

$$S = S_{EH} + S_{\text{matter}} = \int d^4x \sqrt{-g} \left[-\frac{1}{8\pi G} R + g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \right].$$

$$\frac{\delta S_{\text{matter}}}{\delta g^{\mu\nu}} = \frac{2}{Tg} T_{\mu\nu}$$

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{Tg} T_{\mu\nu}$$

~~XXX~~

Exercise

$$ds^2 = \underbrace{f^2(t) dt}_{\text{time component}} - a^2(t) dx^2$$

$$R = \frac{6}{a^2 f^3} [f(\dot{a}^2 + a\ddot{a}) - a\dot{a}\dot{f}]$$

$$\mathcal{L}_m = \mathcal{L}(t, f, g)$$

Vary w.r.t f

a

\leftarrow FRW eq time component

\leftarrow

space component

$$\underline{f=1},$$

$$\underline{f=a}$$