

GR QFT March 5

Note Title

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P.I Quantization of General Relativity

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- 1) Background field
- 2) Gauge fixing & ghost
- 3) One loop divergences

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + K h_{\mu\nu}$$

$$g^{\mu\nu} = \bar{g}^{\mu\nu} - K h^{\mu\nu} + K^2 h^{\mu\lambda} h_\lambda^\nu$$

(drop K)
(back L)
 $K^2 = 32\pi G$

Need Γ, R

$$\text{Work on } \bar{\Gamma}_{\mu\nu}^\beta = \bar{\Gamma}_{\mu\nu}^\beta + \underbrace{\bar{\Gamma}_{\mu\nu}^\beta}_{=0} + \underbrace{\bar{\Gamma}_{\mu\nu}^\beta}_{O(h^2)}$$

$$\bar{\Gamma}_{\mu\nu}^\beta = \frac{1}{2} \bar{g}^{\beta\alpha} \left[\partial_\mu \bar{g}_{\nu\alpha} + \partial_\nu \bar{g}_{\mu\alpha} - \partial_\alpha \bar{g}_{\mu\nu} \right]$$

$$\bar{\Gamma}_{\mu\nu}^\beta = \frac{1}{2} \bar{g}^{\beta\alpha} \left[\partial_\mu h_{\nu\alpha} + \partial_\nu h_{\mu\alpha} - \partial_\alpha h_{\mu\nu} \right]$$

$$- \frac{1}{2} h^{\beta\alpha} \left[\partial_\mu \bar{g}_{\nu\alpha} + \partial_\nu \bar{g}_{\mu\alpha} - \partial_\alpha \bar{g}_{\mu\nu} \right]$$

$$= \frac{1}{2} \bar{g}^{\beta\alpha} \left\{ \left[\partial_\mu h_{\nu\alpha} + \partial_\nu h_{\mu\alpha} - \partial_\alpha h_{\mu\nu} \right] - h_{\alpha\gamma} \bar{g}^{\gamma\sigma} \left[\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu} \right] \right\}$$

- $\bar{g}_{\mu\nu} \bar{g}^{\gamma\sigma}$

$$= \frac{1}{2} \bar{g}^{\beta\alpha} \left\{ \partial_\mu h_{\nu\alpha} - \bar{P}_{\mu\nu}^\gamma h_{\gamma\alpha} + \textcircled{4} \right.$$

$$\partial_\nu h_{\mu\alpha} - \bar{P}_{\nu\mu}^\gamma h_{\gamma\alpha} \quad \left| \begin{array}{l} \\ + \textcircled{2} \end{array} \right.$$

$$- \partial_\alpha h_{\mu\nu} \quad \left| \begin{array}{l} + \textcircled{1} \\ + \textcircled{3} \end{array} \right.$$

$$\left[\bar{P}_{\alpha\nu}^\gamma h_{\gamma\mu} - \bar{P}_{\alpha\nu}^\gamma h_{\gamma\alpha} \right] + \left[\bar{P}_{\mu\alpha}^\gamma h_{\gamma\nu} - \bar{P}_{\mu\alpha}^\gamma h_{\gamma\nu} \right] \quad \left. \begin{array}{c} \\ \\ \textcircled{4} \end{array} \right]$$

$$D_{\mu\nu}^\beta = \frac{1}{2} \bar{g}^{\beta\alpha} \left[\bar{D}_\mu h_{\nu\alpha} + \bar{D}_\nu h_{\mu\alpha} - \bar{D}_\alpha h_{\mu\nu} \right]$$

\curvearrowleft covariant w.r.t. \bar{g}

$$\Gamma_{\mu\nu}^{\beta} = \frac{1}{2} h^{\beta\alpha} \left[\overleftarrow{D}_\mu h_{\nu\alpha} + \overrightarrow{D}_\nu h_{\mu\alpha} - \overleftarrow{D}_\alpha h_{\mu\nu} \right]$$

Gauge invariance in B.F

$$X'^m = X^m + \zeta^m(x)$$

$$dx'^m = dx^m + \partial_\nu \zeta^m dx^\nu = (\delta_\nu^m + \partial_\nu \zeta^m) dx^\nu = \Lambda^m_\nu dx^\nu$$

$$g_{\mu\nu}(x) \rightarrow \Lambda_\mu^\alpha g_{\alpha\beta}(x) \Lambda_\nu^\beta = \underbrace{g_{\mu\nu}(x')}_{g_{\mu\nu}(x) + \zeta^\alpha \partial_\alpha g_{\mu\nu}} + g_{\alpha\nu} \partial_\mu \zeta^\alpha + g_{\mu\alpha} \partial_\nu \zeta^\alpha$$

Inf gauge trans

$$g_{\mu\nu} \rightarrow g_{\mu\nu} + g_{\alpha\nu} \partial_\mu \xi^\alpha + g_{\mu\alpha} \partial_\nu \xi^\alpha + \xi^\alpha \partial_\alpha g_{\mu\nu}$$

Now can replace ∂_μ by \bar{D}_μ

$$\begin{aligned} g_{\mu\nu} + g_{\alpha\nu} (\bar{D}_\mu \xi^\alpha + \bar{\Gamma}_{\mu\sigma}^\alpha \xi^\sigma) + g_{\mu\alpha} (\bar{D}_\nu \xi^\alpha + \bar{\Gamma}_{\nu\sigma}^\alpha \xi^\sigma) \\ + \xi^\alpha [\bar{D}_\alpha g_{\mu\nu} - \bar{\Gamma}_{\alpha\mu}^\sigma \bar{g}_{\nu\sigma} - \bar{\Gamma}_{\alpha\nu}^\sigma \bar{g}_{\mu\sigma}] \end{aligned}$$

$\bar{\Gamma}$ cancel -

$$g_{\mu\nu} \rightarrow g_{\mu\nu} + g_{\mu\alpha} \bar{D}_\nu \xi^\alpha + g_{\alpha\nu} \bar{D}_\mu \xi^\alpha + \xi^\alpha \bar{D}_\alpha g_{\mu\nu}$$

$$\bar{D}_\alpha \bar{g}_{\mu\nu} = 0$$

$$\bar{g}_{\mu\nu} + \bar{h}_{\mu\nu} \rightarrow \bar{g}_{\mu\nu} + h_{\mu\nu} + (\bar{g}_{\mu\alpha} + h_{\mu\alpha}) \bar{D}_\nu \zeta^\alpha + (\bar{g}_{\alpha\nu} + h_{\alpha\nu}) \bar{D}_\mu \zeta^\alpha + \zeta^\alpha \bar{D}_\alpha h_{\mu\nu}$$

Assign transformation to h (use $\bar{g}_{\mu\alpha} \bar{D}_\nu \zeta^\alpha = \bar{D}_\nu \bar{g}_{\mu\alpha} \zeta^\alpha = \bar{D}_\nu \zeta_\alpha$)

$$h_{\mu\nu} \rightarrow h_{\mu\nu} + \bar{D}_\mu \zeta_\nu + \bar{D}_\nu \zeta_\mu + \underbrace{\mathcal{O}(h, \zeta)}$$

$\xrightarrow{\text{covariant w.r.t. } \bar{g}_{\mu\nu}}$

$$\text{Curvatures} \quad R_{\mu\nu\alpha}^{\beta} = \bar{R}_{\mu\nu\alpha}^{\beta} + \underline{R} + R$$

$$\underline{R}_{\mu\nu\alpha}^{\beta} = \bar{D}_\mu \bar{\Gamma}_{\nu\alpha}^{\beta} - \bar{D}_\nu \bar{\Gamma}_{\mu\alpha}^{\beta}$$

$$\underline{R}_{\mu\nu\alpha}^{\beta} = \bar{D}_\mu \bar{\Gamma}_{\nu}^{\beta} - \bar{D}_\nu \bar{\Gamma}_{\mu}^{\beta} + \bar{\Gamma}_\mu \bar{\Gamma}_\nu^\beta - \bar{\Gamma}_\nu \bar{\Gamma}_\mu^\beta$$

But need $S(h)$ not $S(\bar{\Gamma}) \Rightarrow$ more work

$$\bar{D}_\mu \bar{g}_{\alpha\beta} = 0$$

$$[\bar{D}_\mu, \bar{D}_\nu] h_\alpha^\beta = \bar{R}_{\mu\nu\gamma}^\beta h_\alpha^\gamma + \bar{R}_{\mu\nu\alpha}^\gamma h_\gamma^\beta \quad *$$

Result

$$\begin{aligned} \mathcal{L} = \int g^{-1} \frac{-2}{K^2} R &= \int g^{-1} \left[-\frac{2}{K^2} \bar{R} - \frac{1}{K} \left[h^\alpha_\alpha \bar{R} - 2 \bar{R}^\alpha_\nu h^\nu_\alpha \right] \right. \\ &\quad + \frac{1}{2} \bar{D}_\alpha h_{\mu\nu} \bar{D}^\alpha h^{\mu\nu} - \frac{1}{2} \bar{D}_\alpha h^\lambda_\lambda \bar{D}^\alpha h^\sigma_\sigma \left. \right] \quad \left. \begin{array}{l} \text{weak field} \\ \partial \rightarrow D \end{array} \right] \\ &\quad + \bar{D}_\nu h^\lambda_\lambda D^\nu h^\sigma_\sigma - \bar{D}_\nu h_{\alpha\beta} D^\alpha h^{\nu\beta} \\ &\quad \left. - \bar{R} \left[\frac{1}{4} (h^\lambda_\lambda)^2 - \frac{1}{2} h^\alpha_\mu h^\mu_\alpha \right] \rightarrow h^\lambda_\lambda h^\alpha_\nu \bar{R}^\nu_\alpha + 2 h^\nu_\beta h^\sigma_\alpha \bar{R}^\alpha_\nu \right] \end{aligned}$$

First term $\frac{2}{K} h^{\alpha\beta} \left[\bar{R}_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} \bar{R} \right] = 0$ $\stackrel{\frac{1}{4} K^2 T^{\mu\nu}}{\leftarrow}$ from matter L
Equation of motion

Gauge fixing

Weak field $\partial_\mu h_{\nu\rho} - \frac{1}{2} \partial_\nu h^\lambda_\lambda = 0 \quad \leftarrow \text{harmonic gauge}$

Added gauge fixing term:

$$L_{gf} = \frac{1}{2} [\partial_\mu h^{\mu\nu} - \partial^\nu h^\lambda_\lambda]^2$$

Simplified L

$$L = \frac{1}{2} \partial_\mu h_{\nu\rho} \partial^\rho h^{\mu\nu} - \frac{1}{4} \partial_\mu h \partial^\mu h - \frac{5}{2} h^{\mu\nu} T_{\mu\nu}$$

↑ propagator --

$$\text{Now } C_n = \left[\bar{D}_\nu h_\mu^\nu - \frac{1}{2} \bar{D}_\mu h_{\lambda}^\lambda \right] \leftarrow \text{harmonic gauge}$$

add

$$\Delta(h) \delta(C_n - F_n)$$

$$\begin{aligned} & \text{exponentials } S dF \delta(C_n - F_n(x)) e^{i \int d^4x \sqrt{-g} \frac{1}{2} F_\mu F^\mu} \\ &= \ell^{i \int d^4x \sqrt{-g} \mathcal{L}_{gf.}} \end{aligned}$$

$$\mathcal{L}_{gf} = \frac{1}{2} \left[\bar{D}_\mu h_\nu^\nu - \bar{D}^\nu h_{\nu}^\lambda \right]^2 = \frac{1}{2} C_m C^m$$

$$\text{Now } \boxed{\mathcal{L}_g + \mathcal{L}_{gf} = \frac{1}{2} \bar{D}_\alpha h_{\mu\nu} \bar{D}^\alpha h^{\mu\nu} - \frac{1}{4} \bar{D}_\alpha h_{\lambda}^\lambda \bar{D}^\alpha h_{\sigma}^\sigma + R \text{ terms}}$$

Ghosts

$$C_m = \bar{D}_v h_v^\nu - \frac{1}{2} \bar{D}_m h^\lambda_\lambda$$

$$h_{\mu\nu} \rightarrow h_{\mu\nu} + \bar{D}_v F_m + \bar{D}_m \zeta_v$$

$$\begin{aligned}\delta C_m &= \bar{D}_v (\bar{D}_v \zeta_m + D_m \zeta_v) - \frac{g}{2} \bar{D}_m D_\alpha \zeta^\alpha \\ &= \bar{D}_v \bar{D}^\nu \zeta_\nu - \underbrace{[\bar{D}_m, D_v]}_{R_{\mu\nu\alpha}^\nu} \zeta^\alpha \\ &\quad R_{\mu\nu\alpha}^\nu \zeta^\alpha = -R_{\mu\alpha} \zeta^\alpha \\ &= (g_{\mu\alpha} \bar{D}^2 + R_{\mu\alpha}) \zeta^\alpha\end{aligned}$$

$$\Delta = \det \left(\frac{\partial \mathcal{L}_M}{\partial \dot{x}_\alpha} \right) = (g_{\mu\nu} \bar{D}^2 + R_{\mu\nu})$$

F.P. ghost fields η^μ

$$Z_{\text{ghost}} = \eta^\mu (g_{\mu\nu} \bar{D}^2 + R_{\mu\nu}) \eta^\nu$$

Note Feynman rule

$$k_1^\nu \downarrow \begin{cases} k_1^\mu \\ q_1^\alpha \beta \end{cases} = -i \left[k_\alpha k_\beta g_{\mu\nu} + q \cdot h g_{\mu\nu} g_{\alpha\beta} + g_{\mu\alpha} h_\beta f^{\alpha\beta} - g_\nu h_\beta g_{\mu\alpha} + g^2 g_{\mu\alpha} g_{\nu\beta} \right] \quad \boxed{\text{Exercise}}$$

Summary $\leftarrow -\frac{2}{k^2} \bar{R}$

$$S = \bar{S} + \Gamma$$

$$e^{i\Gamma} = S[dh_{\mu\nu}][d\eta][d\bar{\eta}] \leftarrow i S d^4x \left[\mathcal{L}_g + \mathcal{L}_{gf} + \mathcal{L}_{\eta\bar{\eta}} \right] \quad \times \times \times$$

$$\mathcal{L}_g + \mathcal{L}_{gf} = \frac{1}{2} \bar{D}_\alpha h_{\mu\nu} \bar{D}^\alpha h^{\mu\nu} - \quad \left. \right\} \text{like ordinary matter loops}$$

$$\mathcal{L}_{\eta\bar{\eta}} = \eta^\mu (\bar{g}_{\mu\nu} \bar{D}^\nu + \bar{R}_{\mu\nu}) \eta^\nu$$

Or add $S d\phi e^{i S d^4x - \frac{1}{2} \phi (\bar{D}^2 + m^2 - \xi \bar{R}) \phi}$ $\leftarrow \text{scalar matter}$

remain. We do not feel that this is the last word on this subject, because the situation as described in section 7 is so complicated that we feel less than sure that there is no way out. A certain exhaustion however prevents us from further investigation, for the time being.