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Gravity from Poincaré Gauge Theory of the Fundamental Particles. I

——*General Formulation*——

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We study Poincaré gauge theory with linear and quadratic Lagrangians, where there are the translation and Lorentz gauge fields whose sources are the energy-momentum tensor and the spin tensor, respectively. Ten parameters, a, α, β, γ, a_1, $\cdots$, a_6, are contained in the gravity action. Most general field equations covariant under the Poincaré gauge group are derived, and also their alternative forms are obtained. For geometrical analysis the present space-time is that of Riemann-Cartan, endowed with curvature and torsion. General Relativity is recovered when a spin source vanishes and the parameters satisfy the conditions, $3a_2+2a_5=0=a_5+12a_6$.

§ 1. Introduction

Gravity is usually treated by Einstein's gravity theory called General Relativity, which explains all known experiments in gravity physics. It is based on the geometry of the Riemann space-time manifold which is characterized by the curvature tensor alone and the absence of the torsion tensor. Recently, however, it is shown that New General Relativity is nearly the same as General Relativity on the macroscopic scale and agrees with all gravity experiments:[1] New General Relativity is based on the Weitzenböck space-time manifold which is characterized by the torsion tensor alone and the absence of the curvature tensor. Thus one is not obliged to stay within the conventional theory.

Then what is the principle by which we can go beyond General Relativity and even New General Relativity? It is well known that General Relativity is plagued by nonrenormalizability,[2] which prevents its microscopic version from being a correct quantum field theory. So microscopic gravity theory may have to be a renormalizable field theory.

Everything is made of the fundamental particles of intrinsic spin 1/2, which are called quarks and leptons, whereas the fields of spin 1, such as the electromagnetic field, $W^{\pm}$ mesons, Z meson, gluons, etc., are regarded as the gauge particles. This gauge picture is provided by particle physics with the success of the Weinberg-Salam theory for the electroweak interactions[3] and with the invention of quantum chromodynamics of the strong interaction. We shall also use the gauge principle in describing gravity at the fundamental level of the microscopic scale. The gauge group this time is the Poincaré gauge group, which is the extension of the Poincaré group. In this case the gravitational variable is the tetrad field, which is more fundamental than the metric tensor.

Here are several problems in applying the Poincaré gauge approach to gravity. In this approach there are two sets of the gauge fields. One is associated with the translation gauge group and called the *translation gauge field*, and the other is connected with the internal Lorentz gauge group and called the *Lorentz gauge field*. As for the internal group, Yang and Mills[4] were the first to study the $SU(2)$ isospin gauge group. Then Utiyama investigated the possibility of treating the Poincaré group,[5] which was later inherited by Kibble.[6] However, Poincaré gauge theory is much more complicated than that of the internal group: According to Yang-Mills gauge approach there must be a gauge-invariant Lagrangian which is quadratic in the gauge field strength. The well-known example is the electromagnetic Lagrangian which is quadratic in the electromagnetic field strength. But Einstein's Lagrangian of General Relativity is linear in the Lorentz gauge field strength. This is the first difficulty in treating gravity by the Yang-Mills gauge approach. In 1968 Hayashi[7] tried to resolve this dilemma by extending to Lagrangians quadratic in the Lorentz gauge field strength, and Hayashi and Bregman[8] further pursued this problem. There is a huge amount of the literature since then.[9]

As for quadratic Lagrangians, there are two methods in handling the Lorentz gauge field whose source is an intrinsic spin tensor. One is *conventional* in which the Lorentz gauge field $\boldsymbol{A}$ is decomposed into the Ricci rotation coefficients $\boldsymbol{\Delta}$, which is given by first derivatives of the tetrad field, and contorsion $\boldsymbol{K}$,

$$\boldsymbol{A}=\boldsymbol{\Delta}+\boldsymbol{K}\,. \tag{1·1}$$

When only the linear Lagrangian is used, this conventional method yields Einstein-like equations,[6),7)] but the contorsion field does not propagate in vacuum but is frozen at the place of matter. Quantum field theory for such a frozen field is almost impossible and not appealing. When quadratic Lagrangians come in, field equations, which are derived by taking the translation and Lorentz gauge fields as independent variables, contain third derivatives of the translation gauge field when (1·1) is inserted in field equations.[7),8)] Thus we would be led to higher-derivative gravity theory. However, in fact field equations in the weak field approximation will be of second order differential equations and higher derivatives will be shifted to source terms, as will be shown in this series. Then the contorsion field $\boldsymbol{K}$ is capable of propagating in vacuum when its kinetic energy is supplied by quadratic Lagrangians.

The other is *unconventional* in which the Lorentz gauge field has its own degree of freedom, not decomposable into any other quantities. This case is very much alike the Yang-Mills gauge approach and makes possible for the contorsion field to propagate in vacuum.

We make a few remarks on quadratic Lagrangians. The present approach will differ from the "Quadratic Lagrangian Method in General Relativity",[10] whose quantum theory was investigated by Stelle.[11] Neville studied quadratic Lagrangians of Poincaré gauge theory, however, neglecting the translation gauge field altogether.[12] This is obviously incomplete and differs from our results shown in this series.

In this series we shall explore the problem of what is essential in gravity and various problems concerning gravity of the fundamental particles in order; here we deal with general formulation of the Poincaré gauge invariant gravity theory. Various limits of the theory and the weak field approximation on the field equations will be discussed in the forthcoming papers.

This paper is arranged as follows: In § 2 our arguments are based on the fundamental particles of spin 1/2 by introducing the invariance under the Poincaré gauge group. In § 3 we contemplate two sets of the gauge field strengths, called the translation and Lorentz gauge field strengths, and then build up *three* pieces quadratic in the translation gauge field strength, *six* pieces quadratic in the Lorentz gauge field strength, and finally *one* invariant linear in the Lorentz gauge field strength. (There is no other invariant linear and quadratic in the gauge field strengths.) Then we derive most general gravitational field equations, which consist of two sets of field equations, associated with two variations of the translation and Lorentz gauge fields. In § 4 space-time manifold is analyzed, and the most general one is the Riemann-Cartan space-time with the curvature tensor and the torsion tensor. In § 5 we follow the *conventional* method mentioned above and supply alternative forms of most general gravitational field equations, thus comparing with other approaches. Final section will be devoted to conclusion.

§ 2. Fundamental particles and Poincaré gauge group

We describe the fundamental particles of intrinsic spin 1/2, which are called quarks and leptons; color and flavor indices of quarks are here suppressed. The action integral in special relativity is given by

$$A=\int d^4x L_M \tag{2·1a}$$

with

$$L_M=(i/2)\,(\bar{q}\gamma^k\partial_k q-\partial_k\bar{q}\gamma^k q)-m\bar{q}q\,, \tag{2·1b}$$

where q denotes collectively the quarks and leptons ($\bar{q}=q^*\gamma^0$). Here the electromagnetic field, $W^{\pm}$ mesons, Z meson, and gluons can easily be incorporated in by the Yang-Mills gauge procedure: See, for example, the Weinberg-Salam theory.[3] So they are skipped here.

We now demand that the action should satisfy Poincaré gauge invariance.

To meet this requirement, two steps are needed. First,[13] for arbitrary coordinate transformations, that is, extended translations, the spinor field $q(x)$ should behave like scalar,

$$q'(x') = q(x) \tag{2·2}$$

for $x' = f(x)$, where f is an arbitrary differentiable function. Then the derivative of the spinor field should change as follows:

$$\partial_k q(x) \to (\delta_k^{\ \mu} + c_k^{\ \mu}(x))\,\partial_\mu q(x)\,, \tag{2·3}$$

where Greek index runs from 0 to 3, referring to arbitrary coordinate patch. Here $c_k^{\ \mu}(x)$ is called the *translation gauge field*, because it transforms inhomogeneously under the extended translation,

$$c_k'^{\ \mu}(x') = (\partial x'^\mu/\partial x^\nu)\,c_k^{\ \nu}(x) + (\partial x'^\mu/\partial x^\nu)\,\delta_k^{\ \nu} - \delta_k^{\ \mu} \tag{2·4}$$

for the invariant form of the new derivative,

$$(\delta_k^{\ \mu} + c_k'^{\ \mu}(x'))\,\partial_\mu' q'(x') = (\delta_k^{\ \mu} + c_k^{\ \mu}(x))\,\partial_\mu q(x)\,. \tag{2·5}$$

In fact it can be transformed away for an infinitesimal coordinate patch by appropriate transformation.

Second,[6] for the coordinate x to be invariant, the spinor field changes according to the spin transformation,

$$q'(x) = L(x)\,q(x)\,, \tag{2·6}$$

where L is a 4×4 matrix depending on the coordinate x. The following procedure is quite typical for the case of internal symmetry. The previous derivative (2·3) should be modified as

$$(\delta_k^{\ \mu} + c_k^{\ \mu}(x))\,D_\mu q(x) = (\delta_k^{\ \mu} + c_k^{\ \mu}(x))\,(\partial_\mu + (i/2)\,A_{ij\mu}(x)\,S^{ij})\,q(x)\,. \tag{2·7}$$

Here S^{ij} is the infinitesimal generator of the internal Lorentz group,

$$S^{ij} = (i/4)\,[\gamma^i, \gamma^j]\,, \tag{2·8}$$

and $A_{ij\mu}(x) = -A_{ji\mu}(x)$ is called the *Lorentz gauge field*, which transforms under (2·6) as

$$A'_{ij\mu}(x) = \Lambda_i^{\cdot m}(x)\,\Lambda_j^{\cdot n}(x)\,A_{mn\mu}(x) + (\partial\Lambda_i^{\cdot m}(x)/\partial x^\mu)\,\eta_{mn}\Lambda_j^{\cdot n}(x) \tag{2·9}$$

with η_{mn} the Minkowski metric tensor, $(\eta_{mn}) = \mathrm{diag}(-1, +1, +1, +1)$, for the vector form of the modified derivative,

$$(\delta_k^{\ \mu} + c'_k{}^{\mu}(x))\,D'_\mu q'(x) = L(x)\,\Lambda_k^{\cdot j}(x)\,(\delta_j^{\ \mu} + c_j^{\ \mu}(x))\,D_\mu q(x) \tag{2·10}$$

with $\Lambda_k^{\cdot j}(x)$ a Lorentz transformation depending on x.

Now it is very convenient to introduce $e_k{}^\mu(x)$ as

$$e_k{}^\mu(x) = \delta_k{}^\mu + c_k{}^\mu(x), \tag{2·11}$$

where $e_k{}^\mu$ is called the *tetrad field* which is *not* the gauge field, because it is the Lorentz vector with respect to Latin index and the world vector with respect to Greek index,

$$e'_k{}^\mu(x') = (\partial x'^\mu/\partial x^\nu)\, \Lambda_k{}^{\cdot j}(x)\, e_j{}^\nu(x). \tag{2·12}$$

Thus the inverse field $e^k{}_\mu$ is defined by

$$e_k{}^\mu e^k{}_\nu = \delta^\mu{}_\nu, \qquad e_i{}^\mu e^j{}_\mu = \delta_i^j, \tag{2·13}$$

through which the gauge field $a^k{}_\mu$ is introduced by

$$e^k{}_\mu = \delta^k{}_\mu + a^k{}_\mu. \tag{2·14}$$

Then we are able to define the determinant in terms of this inverse field,

$$\det\,(e^k{}_\mu) = e. \tag{2·15}$$

In summary the modified derivative now takes the form,

$$e_k{}^\mu(x)\, D_\mu q(x) = e_k{}^\mu(x)\, (\partial_\mu + (i/2)\, A_{ij\mu}(x)\, S^{ij})\, q(x), \tag{2·16}$$

where the translation gauge field is contained in $e_k{}^\mu$.

To investigate the gauge field structure in the new derivative (2·16), the following points are essential: The Poincaré group is the semi-direct product of the translation group T and the Lorentz group L, $T \otimes L$, where L consists of the orbital part L_{orb} and of the intrinsic spin part L_{int}. The gauge fields *partially* reflect this group structure, and indeed they are incorporated as if the group structure were the direct product of the translation and internal Lorentz groups, $T \otimes L_{\text{int}}$.

Now we consider an infinitesimal form of the translation and internal Lorentz transformations. First, for arbitrary change of the coordinate x,

$$\delta x^\mu = \varepsilon^\mu(x), \tag{2·17}$$

where $\varepsilon^\mu(x)$ is an arbitrary infinitesimal, differentiable function, the translation gauge field transforms as (2·4) dictates; here $c'_k{}^\mu(x') = c_k{}^\mu(x) + \delta c_k{}^\mu(x)$ and $a'^k{}_\mu(x') = a^k{}_\mu(x) + \delta a^k{}_\mu(x)$ with

$$\left.\begin{aligned} \delta c_k{}^\mu(x) &= \varepsilon^\mu{}_{,\nu} c_k{}^\nu(x) + \varepsilon^\mu{}_{,\nu}\delta_k{}^\nu, \\ \delta a^k{}_\mu(x) &= -\varepsilon^\nu{}_{,\mu} a^k{}_\nu(x) - \varepsilon^\nu{}_{,\mu}\delta^k{}_\nu. \end{aligned}\right\} \tag{2·18}$$

Hence the tetrad field changes as follows:

$$\left.\begin{aligned} \delta e_k{}^\mu(x) &= \varepsilon^\mu{}_{,\nu} e_k{}^\nu(x), \\ \delta e^k{}_\mu(x) &= -\varepsilon^\nu{}_{,\mu} e^k{}_\nu(x). \end{aligned}\right\} \tag{2·19}$$

Here the partial derivative with respect to the coordinate is abbreviated by the comma. This time the spinor field does not change at all,

$$\delta q(x) = 0 = \delta \bar{q}(x). \qquad (2\cdot 20)$$

Second, the spin transformation of the spinor field, denoted by (2·6), now takes the form

$$\delta q(x) = (i/2)\,\omega_{ij}(x)\, S^{ij} q(x), \qquad (2\cdot 21)$$

and the modified derivative changes as follows:

$$\delta e_k{}^\mu(x)\, D_\mu q(x) = (i/2)\,\omega_{ij}(x)\, S^{ij} e_k{}^\mu(x)\, D_\mu q(x) + \omega_k{}^{\cdot j}(x)\, e_j{}^\mu(x)\, D_\mu q(x), \qquad (2\cdot 22)$$

where $\omega_{ij}(x) = -\omega_{ji}(x)$ is a set of six arbitrary infinitesimal, differentiable functions. Then the Lorentz gauge field changes as

$$\delta A_{ij\mu}(x) = \omega_i{}^{\cdot m}(x)\, A_{mj\mu}(x) + \omega_j{}^{\cdot m}(x)\, A_{im\mu}(x) + \omega_{ij}(x)_{,\mu}, \qquad (2\cdot 23)$$

and the tetrad field transforms as

$$\delta e_k{}^\mu(x) = \omega_k{}^{\cdot j}(x)\; e_j{}^\mu(x). \qquad (2\cdot 24)$$

Finally, the action integral for the fundamental particles of intrinsic spin 1/2, which satisfies Poincaré gauge invariance, reads

$$A = \int d^4x e L_M, \qquad (2\cdot 25)$$

where

$$L_M = (i/2)\, e_k{}^\mu (\bar{q}\gamma^k D_\mu q - D_\mu \bar{q}\gamma^k q) - m\bar{q}q \qquad (2\cdot 26)$$

with

$$D_\mu \bar{q} = \bar{q}\overleftarrow{D}_\mu = \bar{q}(\overleftarrow{\partial}_\mu - (i/2)\, A_{ij\mu}(x)\, S^{ij}). \qquad (2\cdot 27)$$

The modified Dirac equation is derived from this action integral as follows:

$$[i\gamma^\mu(D_\mu + (1/2)\, v_\mu) - m]\, q = 0, \qquad (2\cdot 28)$$

$$\bar{q}[i(\overleftarrow{D}_\mu + (1/2)\, v_\mu)\gamma^\mu + m] = 0 \qquad (2\cdot 29)$$

or more concisely,

$$[i\gamma^\mu \partial_\mu + (i/2e)\;(\partial_\mu e\gamma^\mu) - (3/2)\, A_\mu \gamma^5 \gamma^\mu - m]\, q = 0, \qquad (2\cdot 30)$$

$$\bar{q}[i\overleftarrow{\partial}_\mu \gamma^\mu + (i/2e)\;(\partial_\mu e\gamma^\mu) + (3/2)\, A_\mu \gamma^5 \gamma^\mu + m] = 0, \qquad (2\cdot 31)$$

where

$$\gamma^\mu(x) = e_k{}^\mu(x)\, \gamma^k, \qquad (2\cdot 32)$$

$$A_\mu = e^k{}_\mu A_k = (1/6)\, e^k{}_\mu \varepsilon_{kijm} A^{ij\nu} e^m{}_\nu, \qquad (2\cdot 33)$$

$$v_\mu = e^k{}_\mu (e e_k{}^\nu)_{,\nu}/e + e_{k\mu} e_{i\lambda} A^{ki\lambda}. \tag{2·34}$$

Here ε_{kijm} is the usual totally antisymmetric tensor in the Minkowski space-time with $\varepsilon^{0123} = +1 = -\varepsilon_{0123}$.

§ 3. Most general gravitational field equations

We shall derive the most general gravitational action which is invariant under the Poincaré gauge group. It is given by

$$A = \int d^4x e (L_M + L_G), \tag{3·1}$$

where L_G denotes a gravitational Lagrangian density and L_M stands for the matter Lagrangian density of the fundamental particles of spin 1/2, given by (2·26). There are two classes of the gauge field strengths; one is the *translation gauge field strength*, defined through the derivative of the translation gauge field,

$$T_{ijk} = A_{ijk} - A_{ikj} - C_{ijk}, \tag{3·2}$$

where C_{ijk} and A_{ijk} are of the form

$$\left.\begin{aligned} C_{ijk} &= e_j{}^\mu e_k{}^\nu (\partial_\mu e_{i\nu} - \partial_\nu e_{i\mu}), \\ A_{ijk} &= e_k{}^\mu A_{ij\mu}. \end{aligned}\right\} \tag{3·3}$$

The other is the *Lorentz gauge field strength* which is defined through the derivative of the Lorentz gauge field,

$$F_{ijmn} = e_m{}^\mu e_n{}^\nu (\partial_\mu A_{ij\nu} - \partial_\nu A_{ij\mu} + A_{ik\mu} A^k{}_{\cdot j\nu} - A_{ik\nu} A^k{}_{\cdot j\mu}). \tag{3·4}$$

As yet we do not know the behavior of the translation gauge field $c_k{}^\mu$ and the Lorentz gauge field $A_{ij\mu}$. So the translation and Lorentz gauge fields are treated independently of each other until field equations for them are to be solved. Notice again that $e_k{}^\mu$ is not the gauge field, because it transforms like vectors, but we shall use it in place of $c_k{}^\mu$ since it is very convenient to write various formulae.

It is necessary that the gravitational Lagrangian density is of a linear term in the Lorentz gauge field strength and of quadratic terms in the translation and Lorentz gauge field strengths. To derive them all, the irreducible decomposition is best fitted for this purpose. First, let us apply the Young table method[14] to the translation gauge field strength. The first is the tensor, obtained from the Young table [21] minus all possible traces,

$$t_{ijk} = (1/2)(T_{ijk} + T_{jik}) + (1/6)(\eta_{ki} v_j + \eta_{kj} v_i) - (1/3)\eta_{ij} v_k. \tag{3·5}$$

The second is the vector, obtained from trace,

$$v_i = T^k{}_{\cdot ki}, \tag{3·6}$$

and the last is the axial vector, constructed from the Young table [111],

$$a_i = (1/6)\,\varepsilon_{ijmn} T^{jmn}. \tag{3·7}$$

Using the symbol (j_1, j_2) for the irreducible non-unitary representation of the proper orthochronous Lorentz group $L_+^\uparrow$, we find that the tensor transforms according to $(3/2, 1/2) \oplus (1/2, 3/2)$ of 16 dimensions, the vector according to $(1/2, 1/2)$, and the axial vector according to $(1/2, 1/2)$. Furthermore, the tensor satisfies the cyclic identity,

$$t_{ijk} + t_{jki} + t_{kij} = 0\,. \tag{3·8}$$

The original translation gauge field strength is represented from these irreducible building blocks as follows:

$$T_{ijk} = (2/3)\,(t_{ijk} - t_{ikj}) + (1/3)\,(\eta_{ij} v_k - \eta_{ik} v_j) + \varepsilon_{ijkm} a^m. \tag{3·9}$$

Next, we carry out the irreducible decomposition of the Lorentz gauge field strength by means of the Young table method:

(i) The tensor (in fact, the scalar) corresponding to the Young table [1111],

$$A_{ijmn} = (1/6)\,(F_{ijmn} + F_{imnj} + F_{injm} + F_{jmin} + F_{jnmi} + F_{mnij})\,. \tag{3·10}$$

(ii) The tensor corresponding to the Young table [22] minus all possible traces,

$$B_{ijmn} = (1/4)\,(W_{ijmn} + W_{mnij} - W_{injm} - W_{jmin})\,, \tag{3·11}$$

where the tensor W_{ijmn} is the same tensor as F_{ijmn} except that it is traceless in all subscripts, its being called the *Weyl tensor* of F_{ijmn},

$$W_{ijmn} = F_{ijmn} - (1/2)\,(\eta_{im} F_{jn} + \eta_{jn} F_{im} - \eta_{in} F_{jm} - \eta_{jm} F_{in}) + (1/6)\,(\eta_{im}\eta_{jn} - \eta_{in}\eta_{jm})\,F. \tag{3·12}$$

(iii) The tensor corresponding to the Young table [211] minus all possible traces,

$$C'_{ijmn} = (1/4)\,(W_{ijmn} - W_{mnij} + W_{injm} - W_{jmin})\,, \tag{3·13}$$

but 'half' of C'_{ijmn} is indeed irreducible (upon investigation),

$$C_{ijmn} = (1/2)\,(W_{ijmn} - W_{mnij})\,. \tag{3·14}$$

(iv) The tensor corresponding to the Young table [2] minus trace,

$$I_{ij} = (1/2)\,(F_{ij} + F_{ji}) - (1/4)\,\eta_{ij} F. \tag{3·15}$$

(v) The tensor corresponding to the Young table [11],

$$E_{ij} = (1/2)\,(F_{ij} - F_{ji})\,. \tag{3·16}$$

(vi) The scalar,

$$F=\eta^{im}\eta^{jn}F_{ijmn}\,. \tag{3·17}$$

Here we have used the abbreviation

$$F_{ij}=\eta^{mn}F_{imjn}\,. \tag{3·18}$$

With respect to the proper orthochronous Lorentz group $L_+^\uparrow$, the tensor F_{ijmn} of 36 components splits into the following irreducible tensors; the first is $(2,0)\oplus(0,2)$ for the tensor B_{ijmn} of 10 components, the second is $(1,1)$ for the tensor C_{ijmn} of 9 components, the third is $(0,0)$ for the scalar A_{ijmn} of one component, the fourth is $(1,1)$ for the tensor I_{ij} of 9 components, the fifth is $(1,0)\oplus(0,1)$ for the tensor E_{ij} of 6 components, and the last is $(0,0)$ for the scalar F of one component. Finally, the tensor F_{ijmn} is represented in terms of the irreducible building blocks as follows:

$$F_{ijmn}=W_{ijmn}+(1/2)\,\{\eta_{im}(E_{jn}+I_{jn})+\eta_{jn}(E_{im}+I_{im}) \\ -\eta_{jm}(E_{in}+I_{in})-\eta_{in}(E_{jm}+I_{jm})\}-(1/12)\,(\eta_{im}\eta_{jn}-\eta_{in}\eta_{jm})\,F\,, \tag{3·19}$$

where

$$W_{ijmn}=A_{ijmn}+(2/3)\,(2B_{ijmn}+B_{imjn})+C_{ijmn}\,. \tag{3·20}$$

Now we demand that the gravitational Lagrangian density be linear in the Lorentz gauge field strength and quadratic in the translation and Lorentz gauge field strengths, and that it be invariant under the operation of space inversion. The required Lagrangian consists of three parts,

$$L_G=aF+L_T+L_F\,, \tag{3·21}$$

where F is the linear invariant with an arbitrary parameter a, and L_T and L_F are quadratic in the translation gauge field strength and the Lorentz gauge field strength, respectively,

$$L_T=\alpha\,(t_{ijk}t^{ijk})+\beta\,(v_iv^i)+\gamma\,(a_ia^i)\,, \tag{3·22}$$

$$L_F=a_1(A_{ijmn}A^{ijmn})+a_2(B_{ijmn}B^{ijmn})+a_3(C_{ijmn}C^{ijmn}) \\ +a_4(E_{ij}E^{ij})+a_5(I_{ij}I^{ij})+a_6F^2, \tag{3·23}$$

where α, β, γ and a_1, $\cdots$, a_6 are arbitrary parameters. In the past the last term L_F has not been considered.[6),8)] In this paper we shall take account of all possible quadratic terms in the translation and Lorentz gauge field strengths.

Now we shall derive the most general gravitational field equations:[*)] As for

*) The most general gravitational field equations with respect to the Pauli metric were first derived by Hayashi in Ref. 7), but they are rederived here with respect to the Minkowski metric $(\eta_{ij})=\text{diag}\,(-1,+1,+1,+1)$.

the translation gauge field, we get the field equation*)

$$2aF_{ji}+2F^{kmn}_{\cdot\cdot\cdot i}J_{[km][nj]}+2D^kF_{ijk}+2v^kF_{ijk}+2H_{ij}-\eta_{ij}L_G=T_{ij}\,, \tag{3·24}$$

where

$$D^kF_{ijk}=e^{k\mu}D_\mu F_{ijk}=e^{k\mu}(\partial_\mu F_{ijk}-A^m_{\cdot i\mu}F_{mjk}-A^m_{\cdot j\mu}F_{imk}-A^m_{\cdot k\mu}F_{ijm})\,, \tag{3·25}$$

$$F_{ijk}=\alpha(t_{ijk}-t_{ikj})+\beta(\eta_{ij}v_k-\eta_{ik}v_j)-(\gamma/3)\,\varepsilon_{ijkm}a^m, \tag{3·26}$$

$$H_{ij}=T_{mni}F^{mn}_{\cdot\cdot j}-(1/2)\,T_{jmn}F_i^{\cdot mn}, \tag{3·27}$$

$$J_{ijmn}=2a_6F\eta_{im}\eta_{jn}+2\eta_{im}(a_4E_{jn}+a_5I_{jn})$$
$$+2(a_1A_{ijmn}+a_2B_{ijmn}+a_3C_{ijmn})\,. \tag{3·28}$$

Here H_{ij} is symmetric upon introducing the irreducible decomposition of the translation gauge field. The energy-momentum tensor is as usual defined by

$$T_{ij}=e_i^{\ \mu}e_j^{\ \nu}T_{\mu\nu}=e^{-1}e_{j\nu}(\delta eL_M/\delta e^i_{\ \nu})\,. \tag{3·29}$$

Next, for the Lorentz gauge field we obtain the field equation

$$2D_mJ^{[ij][km]}-(T^k_{\cdot mn}-2\delta^k_{\ m}v_n)\,J^{[ij][mn]}+H^{ijk}=+(1/2)\,S^{ijk}, \tag{3·30}$$

where

$$D_mJ^{[ij][km]}=e_m^{\ \mu}D_\mu J^{[ij][km]}=e_m^{\ \mu}(\partial_\mu J^{[ij][km]}$$
$$+A^i_{\cdot n\mu}J^{[nj][km]}+A^j_{\cdot n\mu}J^{[in][km]}+A^k_{\cdot n\mu}J^{[ij][nm]}+A^m_{\cdot n\mu}J^{[ij][kn]})\,, \tag{3·31}$$

$$H_{ijk}=-(\alpha+(2a/3))\,(t_{kij}-t_{kji})-(\beta-(2a/3))$$
$$\times(\eta_{ki}v_j-\eta_{kj}v_i)-(2/3)\,(\gamma+(3a/2))\,\varepsilon_{kijm}a^m \tag{3·32}$$

and the spin tensor is defined by

$$S^{ij\nu}=e_k^{\ \nu}S^{ijk}=-2e^{-1}(\delta eL_M/\delta A_{ij\nu})\,. \tag{3·33}$$

§ 4. Space-time manifold with curvature and torsion

We have so far not considered the geometry of space-time manifold but only paid attention mainly to the Poincaré gauge theory with the translation and Lorentz gauge fields and corresponding strengths. Here we shall discuss the geometrical structure of space-time manifold, thus giving a connection between the Poincaré gauge approach and the purely geometrical approach.

The Riemann-Cartan space-time U_4 is a paracompact, Hausdorff, connected C^∞ four-dimensional manifold with a locally Lorentzian metric $\boldsymbol{g}$ and a linear affine connection Γ which obeys the metric condition,

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*) We denote symmetrization and antisymmetrization of tensor indices by a round bracket ( ) and a square bracket [ ], respectively: For example, $J_{[km][nj]}=(1/4)\,(J_{kmnj}-J_{mknj}-J_{kmjn}+J_{mkjn})$.

$$D_\lambda g_{\mu\nu} = \partial_\lambda g_{\mu\nu} - \Gamma^\rho_{\mu\lambda} g_{\rho\nu} - \Gamma^\rho_{\nu\lambda} g_{\mu\rho} = 0. \tag{4·1}$$

From this equation we get

$$\Gamma^\lambda_{\mu\nu} = \left\{ \begin{matrix} \lambda \\ \mu\nu \end{matrix} \right\} + K^\lambda_{\cdot\mu\nu}, \tag{4·2}$$

where the first term denotes the Levi-Civita connection,

$$\left\{ \begin{matrix} \lambda \\ \mu\nu \end{matrix} \right\} = (1/2)\, g^{\lambda\rho} (\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu}), \tag{4·3}$$

and the second stands for the contorsion tensor,

$$K^\lambda_{\cdot\mu\nu} = (1/2)\, (T^\lambda_{\cdot\mu\nu} - T_{\mu\cdot\nu}^{\cdot\lambda} - T_{\nu\cdot\mu}^{\cdot\lambda}) \tag{4·4}$$

with the torsion tensor

$$T^\lambda_{\cdot\mu\nu} = \Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu}. \tag{4·5}$$

The curvature tensor is defined in terms of the affine connection,

$$R^\rho_{\cdot\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\sigma\nu} - \partial_\nu \Gamma^\rho_{\sigma\mu} + \Gamma^\rho_{\lambda\mu} \Gamma^\lambda_{\sigma\nu} - \Gamma^\rho_{\lambda\nu} \Gamma^\lambda_{\sigma\mu}. \tag{4·6}$$

We are concerned with the fundamental particles of spin 1/2, which are the sum of the two-component spinors transforming under the two-valued representation of the connected, proper, orthochronous Lorentz group. So we must demand that the Riemann-Cartan space-time should permit the *spinor structure*,[15] which means that the spinor can be defined all over the manifold, or mathematically speaking, the tetrad field is smoothly spanned all over the manifold. The metric tensor is expressed in terms of the tetrad field as

$$g_{\mu\nu} = e^i{}_\mu \eta_{ij} e^j{}_\nu. \tag{4·7}$$

The spinor structure thus yields the spinor connection for the two-component spinor, from which the *Lorentz connection* is obtained for the sum of the two-component spinor. Therefore, it is this Lorentz connection that the Poincaré gauge approach has introduced, by demanding that the theory remains invariant under the operation of the internal Lorentz group.

The Lorentz connection $\boldsymbol{A}$, derived from the spinor connection, has the following expression:[16]

$$A_{ij\mu} = \Delta_{ij\mu} + K_{ij\mu} \tag{4·8}$$

with $\boldsymbol{\Delta}$ the Ricci rotation coefficients,

$$\Delta_{ijk} = e_k{}^\mu \Delta_{ij\mu} = (1/2)\, (C_{ijk} - C_{jik} - C_{kij}), \tag{4·9}$$

where C_{ijk} is defined in (3·3); it is constructed from the derivatives of the tetrad field. The second term $\boldsymbol{K}$ is called the contorsion (4·4). Greek or Latin index is converted to Latin or Greek index by the help of the tetrad field, for example,

$$K_{\lambda\mu\nu} = e^i_{\ \lambda} e^j_{\ \mu} K_{ij\nu}\,. \tag{4·10}$$

It follows from the spinor analysis on the Riemann-Cartan space-time that the affine connections with respect to the coordinate basis, $\Gamma^{\lambda}_{\mu\nu}$, and with respect to the Lorentz basis, $A_{ij\mu}$, are related with each other by[16]

$$D_\lambda e^i_{\ \nu} = e^i_{\ \nu,\lambda} + A^{i}_{\cdot j\lambda} e^j_{\ \nu} - \Gamma^{\mu}_{\nu\lambda} e^i_{\ \mu} = 0. \tag{4·11}$$

Therefore, the Lorentz gauge field $\boldsymbol{A}$ is directly connected with the affine connection with respect to the coordinate basis, $\boldsymbol{\Gamma}$,

$$A^{i}_{\cdot j\mu} = e^i_{\ \lambda} e_j^{\ \nu} \Gamma^{\lambda}_{\nu\mu} - e_j^{\ \nu} e^i_{\ \nu,\mu}\,. \tag{4·12}$$

Finally, the connection between the Poincaré gauge approach and the geometrical approach becomes very clear. The translation gauge field strength T_{ijk} of (3·2) is nothing but the torsion tensor $T_{\mu\nu\lambda}$ of (4·5),

$$T_{ijk} = e_i^{\ \mu} e_j^{\ \nu} e_k^{\ \lambda} T_{\mu\nu\lambda} \tag{4·13}$$

or

$$T_{\mu\nu\lambda} = e^i_{\ \mu} e^j_{\ \nu} e^k_{\ \lambda} T_{ijk}\,, \tag{4·14}$$

and the Lorentz gauge field strength F_{ijmn} of (3·4) is the curvature tensor $R^{\rho}_{\cdot\sigma\mu\nu}$ of (4·6),

$$F_{ijmn} = e_{i\rho} e_j^{\ \sigma} e_m^{\ \mu} e_n^{\ \nu} R^{\rho}_{\cdot\sigma\mu\nu} \tag{4·15}$$

or

$$R^{\rho}_{\cdot\sigma\mu\nu} = e^{i\rho} e^j_{\ \sigma} e^m_{\ \mu} e^n_{\ \nu} F_{ijmn}\,. \tag{4·16}$$

§ 5. Alternative form of the gravitational field equations

Following the *conventional* method mentioned in the Introduction, we shall rewrite the gravitational field equations, denoted by (3·24) and (3·30), by using the following expression for the Lorentz gauge field $A_{ij\mu}$:

$$A_{ij\mu} = \Delta_{ij\mu} + K_{ij\mu}\,. \tag{5·1}$$

(See Eq. (4·8).) Using (5·1) in (3·4), we find that the Lorentz gauge field strength $F_{ij\mu\nu}$ is decomposed into two parts,

$$F_{ij\mu\nu} = R_{ij\mu\nu}(\{\ \}) + F_{ij\mu\nu}(K) \tag{5·2}$$

with

$$R_{ij\mu\nu}(\{\ \}) = \partial_\mu \Delta_{ij\nu} - \partial_\nu \Delta_{ij\mu} + \Delta_{ik\mu}\Delta^{k}_{\cdot j\nu} - \Delta_{ik\nu}\Delta^{k}_{\cdot j\mu}\,, \tag{5·3}$$

$$F_{ij\mu\nu}(K) = \nabla_\mu K_{ij\nu} - \nabla_\nu K_{ij\mu} + K_{im\mu} K^{m}_{\cdot j\nu} - K_{im\nu} K^{m}_{\cdot j\mu}\,, \tag{5·4}$$

where $\nabla_\mu K_{ij\nu}$ is the covariant derivative with respect to the Ricci rotation coefficients when the index is Latin, and with respect to the Christoffel symbol when the index is Greek. As is well known, $R_{ij\mu\nu}(\{\ \})$ is related to the Riemann-Christoffel curvature tensor by

$$R_{ij\mu\nu}(\{\ \})=e_{i\kappa}e_j{}^\lambda R^\kappa{}_{\cdot\lambda\mu\nu}(\{\ \})$$

$$=e_{i\kappa}e_j{}^\lambda\left[\partial_\mu\begin{Bmatrix}\kappa\\ \lambda\nu\end{Bmatrix}-\partial_\nu\begin{Bmatrix}\kappa\\ \lambda\mu\end{Bmatrix}+\begin{Bmatrix}\kappa\\ \rho\mu\end{Bmatrix}\begin{Bmatrix}\rho\\ \lambda\nu\end{Bmatrix}-\begin{Bmatrix}\kappa\\ \rho\nu\end{Bmatrix}\begin{Bmatrix}\rho\\ \lambda\mu\end{Bmatrix}\right]. \tag{5·5}$$

Using (5·2) in the formulae (3·10) ~ (3·18), we find that each irreducible part of F_{ijmn} is split into two parts: In particular, J_{ijmn} of (3·28) can be expressed as

$$J_{ijmn}=J_{ijmn}(\{\ \})+J_{ijmn}(K), \tag{5·6}$$

where $J_{ijmn}(\{\ \})$ and $J_{ijmn}(K)$ are formed of the irreducible parts of $R_{ijmn}(\{\ \})$ and $F_{ijmn}(K)$, respectively: In particular, the antisymmetric part of $J_{ijmn}(\{\ \})$ is given by

$$J_{[ij][mn]}(\{\ \})=\frac{3}{2}a_2R_{ijmn}(\{\ \})-\frac{1}{4}(3a_2-2a_5)\,[\eta_{im}R_{jn}(\{\ \})$$
$$+\eta_{jn}R_{im}(\{\ \})-\eta_{in}R_{jm}(\{\ \})-\eta_{jm}R_{in}(\{\ \})]$$
$$+\frac{1}{4}(a_2-a_5+4a_6)\,(\eta_{im}\eta_{jn}-\eta_{in}\eta_{jm})\,R(\{\ \}), \tag{5·7}$$

where $R_{ij}(\{\ \})$ and $R(\{\ \})$ are the Ricci tensor and the Riemann-Christoffel scalar curvature, respectively,

$$R_{ij}(\{\ \})=e_i{}^\mu e_j{}^\nu R^\lambda{}_{\cdot\mu\lambda\nu}(\{\ \}),\qquad R(\{\ \})=\eta^{ij}R_{ij}(\{\ \}). \tag{5·8}$$

The gravitational Lagrangian density L_G of (3·21) with (3·22) and (3·23) can then be rewritten as

$$L_G=aR(\{\ \})+\frac{3}{4}a_2R_{ijmn}(\{\ \})R^{ijmn}(\{\ \})-\frac{1}{2}(3a_2-2a_5)\,R_{ij}(\{\ \})R^{ij}(\{\ \})$$
$$+\frac{1}{4}(a_2-a_5+4a_6)\,R(\{\ \})^2+L'_T+L'_F+2a\partial_\mu(ev^\mu)/e\,, \tag{5·9}$$

where

$$L'_T=\left(\alpha+\frac{2a}{3}\right)t_{ijk}t^{ijk}+\left(\beta-\frac{2a}{3}\right)v_kv^k+\left(\gamma+\frac{3a}{2}\right)a_ka^k, \tag{5·10}$$

and L'_F is obtained from L_F by subtracting all the quadratic terms of the Riemann-Christoffel curvature tensor,

$$L'_F = L_F - \Big[\frac{3}{4}a_2 R_{ijmn}(\{\ \})R^{ijmn}(\{\ \})$$
$$-\frac{1}{2}(3a_2-2a_5)R_{ij}(\{\ \})R^{ij}(\{\ \})+\frac{1}{4}(a_2-a_5+4a_6)R(\{\ \})^2\Big]. \tag{5·11}$$

We are now ready to rewrite the gravitational field equations. Using the above formulae in (3·24), we get the alternative form of the field equation for the translation gauge field,

$$2aG_{ij}(\{\ \})+(3a_2+2a_5)\Big[R_{im}(\{\ \})R_j{}^{;m}(\{\ \})+R^{mn}(\{\ \})\Big(R_{imjn}(\{\ \})$$
$$-\frac{1}{2}\eta_{ij}R_{mn}(\{\ \})\Big)\Big]-(2a_2+a_5-4a_6)R(\{\ \})\Big(R_{ij}(\{\ \})-\frac{1}{4}\eta_{ij}R(\{\ \})\Big)$$
$$+2R^{kmn}{}_{\cdots i}(\{\ \})J_{[km][nj]}(K)+2F^{kmn}{}_{\cdots i}(K)\,(J_{[km][nj]}(\{\ \})+J_{[km][nj]}(K))$$
$$+2D^kF'_{ijk}+2v^kF'_{ijk}+2H'_{ij}-\eta_{ij}(L'_T+L'_F)$$
$$=T_{ij}\,, \tag{5·12}$$

where $G_{ij}(\{\ \})$ is the Einstein tensor,

$$G_{ij}(\{\ \})=R_{ij}(\{\ \})-\frac{1}{2}\eta_{ij}R(\{\ \}), \tag{5·13}$$

and we have used the identity,[17]

$$R^{kmn}{}_{\cdots i}(\{\ \})R_{kmnj}(\{\ \})$$
$$=2R_{imjn}(\{\ \})R^{mn}(\{\ \})+2R_{im}(\{\ \})R_j{}^{;m}(\{\ \})-R(\{\ \})R_{ij}(\{\ \})$$
$$+\frac{1}{4}\eta_{ij}[R_{klmn}(\{\ \})R^{klmn}(\{\ \})-4R_{mn}(\{\ \})R^{mn}(\{\ \})+R(\{\ \})^2]. \tag{5·14}$$

Here F'_{ijk} and H'_{ij} are given by

$$F'_{ijk}=\Big(\alpha+\frac{2a}{3}\Big)(t_{ijk}-t_{ikj})$$
$$+\Big(\beta-\frac{2a}{3}\Big)(\eta_{ij}v_k-\eta_{ik}v_j)-\frac{1}{3}\Big(\gamma+\frac{3a}{2}\Big)\varepsilon_{ijkm}a^m, \tag{5·15}$$

$$H'_{ij}=T_{mni}F'^{mn}{}_{\cdot\cdot j}-\frac{1}{2}T_{jmn}F'_i{}^{;mn}. \tag{5·16}$$

Comparing (5·15) with (3·32), we find that

$$F'_{ijk} = \frac{1}{2}(H_{ijk} - H_{ikj} - H_{jki}), \tag{5·17a}$$

or conversely

$$H_{ijk} = F'_{ijk} - F'_{jik}. \tag{5·17b}$$

From (3·30) we obtain the field equation for the Lorentz gauge field,

$$\begin{aligned}(3a_2+2a_5)\nabla_{[i}G_{j]k}(\{\ \}) &+ (a_2+a_5+4a_6)\eta_{k[i}\partial_{j]}G(\{\ \})\\ &-2(D^m-\nabla^m)J_{[ij][km]}(\{\ \}) - 2D^m J_{[ij][km]}(K)\\ &+(T_k{}^{;mn}-2\delta_k{}^m v^n)(J_{[ij][mn]}(\{\ \})+J_{[ij][mn]}(K)) - H_{ijk}\\ &= -\frac{1}{2}S_{ijk}, \end{aligned} \tag{5·18}$$

where $G(\{\ \}) = \eta^{ij}G_{ij}(\{\ \})$, and we have used the identity,[18)]

$$\nabla^m R_{ijkm}(\{\ \}) = \nabla_j R_{ik}(\{\ \}) - \nabla_i R_{jk}(\{\ \}). \tag{5·19}$$

These alternative forms of the gravitational field equations, (5·12) and (5·18), lead to the following results:

(i) Vacuum solutions of the Einstein equation also satisfy the gravitational field equations in vacuum. Namely, if the metric tensor, $g_{\mu\nu} = e^k{}_\mu e_{k\nu}$, satisfies the Einstein equation in vacuum,

$$R_{ij}(\{\ \}) = 0, \tag{5·20}$$

then the translation gauge field $c_k{}^\mu$ and the Lorentz gauge field $A_{ij\mu} = \Delta_{ij\mu}$ (i. e., torsion $=0$) satisfy the gravitational field equations without sources, $T_{ij} = 0 = S_{ijk}$.

(ii) *Equation* (5·18) *contains third derivatives of the metric tensor*, unless the parameters satisfy the conditions,*)

$$3a_2 + 2a_5 = 0 = a_5 + 12a_6. \tag{5·21}$$

(iii) Suppose that the condition (5·21) is satisfied, then the left-hand side of the gravitational field equations, denoted by (5·12) and (5·18), are considerably simplified. In Eq. (5·12), the quadratic terms of the Riemann-Christoffel curvature tensor are all vanishing. In (5·18), the first two terms disappear, and all the remaining terms are linear or quadratic in the torsion tensor. Accordingly, if the intrinsic spin of the source can be ignored, i.e., if $S_{ijk} = 0$, then Eq. (5·18) is trivially satisfied by the vanishing torsion, and therefore, Eq. (5·12) reduces to the Einstein equation,

$$G_{ij}(\{\ \}) = \kappa T_{ij} \quad \text{with} \quad a = 1/2\kappa, \tag{5·22}$$

*) Equation (5·21) is equivalent to $3a_2 + 2a_5 = 0 = a_2 + a_5 + 4a_6$.

where T_{ij} is symmetric because of $S_{ijk}=0$ by the assumption. Here κ denotes Einstein's gravitational constant, $\kappa \equiv 8\pi G$, with G Newton's gravitational constant. Thus *the case of* $3a_2+2a_5=0=a_5+12a_6$ *reduces to General Relativity, if the intrinsic spin tensor of the source is vanishing.*

§ 6. Conclusion

We have started general formulation of Poincaré gauge theory with linear and quadratic Lagrangians, applied to the fundamental particles of spin 1/2, that is, quarks and leptons. According to the group structure of the Poincaré gauge group, there were two sets of the gauge field, namely, the translation and Lorentz gauge fields, from which we constructed the translation and Lorentz gauge field strengths. We fixed the gravity Lagrangian which consists of the three pieces quadratic in the translation gauge field strength, the six pieces quadratic in the Lorentz gauge field strength, and one invariant linear in the Lorentz gauge field strength, with ten free parameters, a, α, β, γ, a_1, $\cdots$, a_6. Our main conclusion here is the gravitational field equations invariant under the Poincaré gauge group; see (3·24) and (3·30) for the independent variations of the translation and Lorentz gauge fields, respectively. Further, using the *conventional* method mentioned in the Introduction, we rewrite the gravitational field equations more transparent in the familiar language; see (5·12) and (5·18). These are the fundamental equations we shall follow in forthcoming articles in this series. From these alternative forms we have obtained the following results: (i) Vacuum solutions of the Einstein equation also satisfy the gravitational field equations in vacuum. (ii) Equation (5·18) contains third derivatives of the metric tensor, unless the parameters obey the conditions, $3a_2+2a_5=0=a_5+12a_6$. (iii) The case of $3a_2+2a_5=0=a_5+12a_6$ reduces to General Relativity if the intrinsic spin tensor of the source is vanishing.

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