General relativity with spin and torsion: Foundations and prospects*

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A generalization of Einstein's gravitational theory is discussed in which the spin of matter as well as its mass plays a dynamical role. The spin of matter couples to a non-Riemannian structure in space-time, Cartan's torsion tensor. The theory which emerges from taking this coupling into account, the U_4 theory of gravitation, predicts, in addition to the usual infinite-range gravitational interaction mediated by the metric field, a new, very weak, spin contact interaction of gravitational origin. We summarize here all the available theoretical evidence that argues for admitting spin and torsion into a relativistic gravitational theory. Not least among this evidence is the demonstration that the U_4 theory arises as a local gauge theory for the Poincaré group in space-time. The deviations of the U_4 theory from standard general relativity are estimated, and the prospects for further theoretical development are assessed.

CONTENTS

I. INTRODUCTION

A. Motivation for our study

Of the fundamental interactions known to modern physics, the gravitational interaction is the weakest and, at a microscopic level, the least well understood. All available evidence from experiments in macrophysics attests to the validity of Einstein's general theory of relativity as a description of this interaction. Why, then, is it worthwhile to propose alternative or more general gravitational theories? An empiricist might argue that it is better to wait for new experimental results and then, only under the challenge posed by possible contradictions to general relativity, to revise or

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‖Sections of this article which, taken together, comprise a short course in U_4 theory.

modify Einstein's theory or propose an alternative.

There is, however, a dichotomy in theoretical physics at present. Strong, electromagnetic, and weak interactions find their most successful description within the framework of relativistic quantum field theory in flat Minkowski space-time. These quantum fields reside in space-time but are separate from it. Gravitation, according to Einstein, deforms Minkowski space and inheres in the dynamic Riemannian geometry of space-time. One branch of fundamental physics is highly successful in a flat and rigid space-time, but gravitation requires a nonflat and dynamic space-time. This state of affairs seems, at least from an epistemological point of view, to be unsatisfactory.

It is not our purpose here to try to express one conception in the framework of another, as is attempted in the flat space-time approach to gravitation (everything expressed as fields in a flat space-time background) or in the so-called unified field theories of the general relativistic type (everything expressed as non-Euclidean geometry). Rather, we would like *to extend the concepts of general relativity to the microphysical realm* in order to facilitate a comparison and possibly a link to the theories of the other interactions. General relativity was originally formulated as a theory valid for mass distributions on a macroscopic, as opposed to atomic scale and for classical electromagnetic fields. The dichotomy in geometrical frameworks alluded to above might be due to the different domains of applicability of the respective theories. Accordingly, forms of gravitational theories for microphysical processes, theories which should go over to general relativity in some macrophysical limit, should be studied as possible roads towards the unified space-time picture we desire.

The gravitational interaction is extremely weak, and there is little hope at present for direct measurements of gravitational effects between elementary particles. Therefore, to some degree, we ought to argue heuristically and even to speculate, and leave the final word to future experiments.

B. Spin and gravitation

Spin angular momentum of matter, occurring in nature in units of $\hbar/2$ (where $\hbar$ is Planck's reduced constant), is the physical notion which seems pertinent and necessary for a successful extension of general relativity to microphysics. The constituents of macroscopic matter are elementary particles which obey, at least locally, quantum mechanics and special relativity theory. As a consequence, all elementary particles can be classified by means of irreducible unitary representations of the Poincaré group and can be labeled by mass m and spin s. Mass is connected with the translational part of the Poincaré group and spin with the rotational part. Mass and spin are *elementary* notions, each with an analogous standing not reducible to that of the other. Distributing mass-energy and spin over space-time leads us to the field-theoretical notions of an energy–momentum tensor Σ_{ij} and a spin angular momentum tensor $\tau_{ij}^{\cdot\cdot k}$ of matter.

In the macrophysical limit, mass (or energy-momentum) adds up because of its monopole character, whereas spin, being of dipole character, usually averages out. (The so-called spin of planets or of billiard balls is of an orbital origin and has no direct relation to the intrinsic spin we discuss here.) Because spin averages out in the large, the dynamical characterization of a continuous distribution of macroscopic matter can be successfully achieved by energy-momentum alone. Einstein's general relativity has taught us that the energy-momentum tensor of matter (the "momentum current") is the source of the gravitational field. This current is coupled to the metric tensor g_{ij} of a Riemannian space-time continuum.

When we venture forth into the microphysical realm of matter, we find that spin angular momentum (the "spin current") also comes into play and characterizes matter dynamically. The hypothesis is near at hand that spin angular momentum is the source of a field, too, in fact the source of a gravitational field. By a gravitational field we mean here a field inseparably coupled to the geometry of space-time. We expect that, in analogy to the coupling of energy-momentum to the metric, spin is coupled to a geometrical "state quantity" of space-time, a quantity which should relate to rotational degrees of freedom in space-time. In this way we are led not to the Riemannian space-time of general relativity but to a slightly more general space-time, the 4-dimensional Riemann–Cartan space-time U_4. The non-Riemannian part of the affine connection characterizing a U_4 is the contortion tensor $K_{ij}^{\cdot\cdot k}$ defined in Eq. (2.11) below, which sould be coupled to the spin. The spin current finds a dynamical basis in this extended framework, the so-called U_4 theory of gravitation. Sometimes U_4 theory is also called the Einstein–Cartan–(Sciama–Kibble) theory. For a guide to the relevant literature see Sec. I.D.

C. Organization of the article

In Sec. II we shall recount in a fashion as elementary as possible the development of the Riemann–Cartan geometry of space-time. There we shall mainly follow the guidelines laid down so beautifully by Schrödinger (1960). We use the mathematical formalism and the conventions of Schouten (1954). We start from an affinely connected space L_4 with an asymmetric connection Γ_{ij}^{k} and impose a (symmetric) metric tensor field g_{ij} on it. In order to guarantee *a local Minkowski structure*, we will postulate $\overset{\Gamma}{\nabla}_k g_{ij}=0$ and deduce the mathematical consequences of this postulate.

In Sec. III we shall develop a field theory of gravitation which is in many ways reminiscent of Einstein's general relativity. The field of spinning matter will be minimally coupled to the U_4 space-time. Then we shall couple the non-Riemannian part of the connection, the contortion tensor $K_{ij}^{\cdot\cdot k}$, to spin, and the metric g_{ij} to energy-momentum of matter. We shall set up the total action function of the matter field interacting with gravitation, first generally, and then with a specific gravitational Lagrangian. By Hamilton's principle, we will derive the gravitational field equations by means of independent variations with respect to metric and torsion. These field equations will be expressed in several alternative forms.

Torsion in U_4 theory supplies an additional contact interaction of spinning matter which leads to deviations from general relativity only in extreme situations (big bang, gravitational collapse and microphysics). For the time being, one is forced to judge the relative merits of U_4 theory according to the available theoretical evidence. In Sec. IV we prove that U_4 theory is the local gauge theory of the Poincaré group in space-time. Belief in the validity of special relativity with respect to *local* inertial systems as represented by tetrad frames, leads to a Riemann–Cartan space-time U_4; the more restricted Riemannian space-time doesn't fit as naturally into this gauge scheme. This derivation, though more demanding of the reader's time, may appear more satisfactory than the one in Sec. III; it supports what is in some ways a more pleasing theory than Einstein's version of general relativity. For Sec. IV, we claim some originality as to the proper interpretation of the local Poincaré transformation, in particular its translational part, and as to the rigor and completeness of the arguments. We regard the question of identifying the Poincaré gauge theory with the U_4 theory as having been settled here.

In Sec. V, we discuss the very slight deviations of U_4 theory from conventional general relativity. In particular we recognize that torsion is tied to matter and leads to new effects only in extreme situations such as the big bang in cosmology. The specific couplings of some matter fields (scalar, Maxwell, Yang–Mills, Proca, Dirac, neutrino) to the U_4 are studied. Some numbers in a typical cosmology with spinning matter are estimated.

In Sec. VI, we first compare the geometrical approach of Secs. II and III with the gauge approach of Sec. IV. Then, within a U_4 space-time, possibilities for a more general dynamics for the gravitational field are explored. Next we sketch the new idea of hypermomentum which requires a spacetime (L_4, g) equipped with an affine connection and an independent metric. Finally, we put forward some speculations.

When not stated otherwise, the mathematical conventions are those of Schouten (1954), the physical conventions those of Landau–Lifshitz (1962). The holonomic indices $i, j, k \ldots$ and the anholonomic indices $\alpha, \beta, \gamma \ldots$ run from 0 to 3 and summation over repeated indices is implied. Symmetrization and antisymmetrization are denoted by ( ) and [ ], respectively. δ_i^j is the Kronecker symbol and $\epsilon^{0123} = -1$ is one component of the totally antisymmetric Levi–Civita symbol.

A fair understanding of general relativity, such as presented in Einstein (1955), is all that is required for the "short course in U_4 theory" (those sections of the article denoted by "||" in the Contents). For the rest of the article, we would recommend as extremely helpful Schrödinger (1960) and Sec. 19 of Corson (1953).

D. Guide to the literature

We have tried to make this article self-contained and to present all material needed for understanding the foundations and the theoretical framework of U_4 theory. We will select here some references in order to give an idea of the history of this theory and to suggest further readings. These references, together with the works cited in them, should embody all work relevant to U_4 theory.

The notion of an asymmetric affine connection was casually mentioned by Eddington (1921) in discussing possible extensions of general relativity (see also Eddington, 1924, Secs. 91 and 98). He was aware that infinitesimal parallelograms were broken in a space-time with an asymmetric connection. He pointed out that applications in microphysics are conceivable, but he did not develop his idea.

Torsion as the antisymmetric part of an asymmetric affine connection was introduced by Élie Cartan (1922, 1923, 1924, 1925), also in the context of a study of general relativity. He recognized the tensor character of torsion and developed a differential geometric formalism. He had some idea that the torsion of space-time might be connected with the intrinsic angular momentum of matter and that it should vanish in matter-free regions. Cartan provided only the rudiments of a theory of intrinsic angular momentum and torsion in general relativity.

The possible link between torsion and intrinsic angular momentum was forgotten later on, probably because the modern concept of spin had its breakthrough only with the discovery of the spin of the electron (1925–26), and because Cartan's work seems not to have been widely read by relativists.[1] The concept of an asymmetric affine connection reappeared in the context of different attempts of a unified field theory of the general relativistic type. This is of no direct interest to us since U_4 theory is a dualistic theory in which matter and geometry are kept separate. For the general framework of these theories, see for instance Einstein (1955), App. 2, Schrödinger (1960), or Tonnelat (1955, 1965).

In the forties it became increasingly clear from the work of Costa de Beauregard (1942, 1943, 1964), Weyssenhoff and Raabe (1947), and Papapetrou (1949), that the energy-momentum tensor of massive spinning fields, say the Dirac field, must be asymmetric.[2] This result, together with Einstein's field equation, can be taken to imply the insufficiency of standard general relativity for fields with spin.

At about the same time or later there appeared several articles with appealing ideas that suggested some sort of a general relativistic space-time with torsion in a dualistic framework: Stueckelberg (1948), Weyl (1950), Finkelstein (1960, 1961), Rodichev (1961), Ivanenko (1962), and Pellegrini and Plebanski (1963).

Continuum physicists also appreciated the notion of Cartan's torsion. Mainly from the work of Kondo

[1] In May 1929, É. Cartan wrote a letter to Einstein. Cartan pointed out that his studies on torsion might be of physical relevance to general relativity. In particular he argued that Einstein's teleparallelism theory is but a special case of a theory with torsion. According to his answer, it seems that Einstein suspected only a claim for priority by Cartan in regard to the discovery of torsion. He did not enter into a physical discussion of Cartan's papers.

[2] For a recent article uniting the arguments for an asymmetric energy-momentum tensor, see Hehl (1976).

(1952), Bilby, Bullough, and Smith (1955), and Kröner (1960), it became clear that torsion plays a central rôle in the continuum theory of crystal dislocations. If we describe by differential geometric methods the deformation of a continuum where dislocations are allowed, Cartan's torsion can be shown to be equivalent to the dislocation density (compare also Kondo, 1955, 1958, 1962, 1968; Bilby, 1960; Kröner, 1964). The relation of continua with dislocations to the polar continua of modern continuum mechanics and their generalized stress states, is discussed in Kröner (1968) and Truesdell and Noll (1965).

U_4 theory proper begins with Sciama (1962, 1964) and Kibble (1961). Their gauge approaches to gravitation will be examined in detail in Sec. IV.C.3. Both Sciama[3] and Kibble arrived at the same set of field equations (3.21, 3.22) and laid down the basic structure of U_4 theory.

Later, the geometrical framework of U_4 theory was set up, the affine connection was written down explicitly, and the formalism presented here in Secs. II and III was developed (Hehl and Kröner, 1965; Hehl, 1966; Hehl, 1970[4]). The canonical energy-momentum tensor was split in the U_4 framework and the combined field equation (3.23) which facilitates comparison of U_4 theory with general relativity was derived. For further developments see Hehl and von der Heyde (1973), Hehl, von der Heyde, and Kerlick (1974), and von der Heyde and Hehl (1975).

In the meantime Trautman (1972a, b, c; 1973a, b, d; 1975) worked out a beautiful mathematical analysis of U_4 theory based on modern differential geometric techniques and the theory of tensor-valued differential forms. He inquired in particular into the compatibility of the affine and metric structures of space-time. His variational principle was discussed in Kopczyński (1973b, 1975) and Trautman (1975). Trautman (1973b, c) also proposed that the singularity behavior of U_4 theory could differ from that of general relativity.

We believe that the foundations and the theoretical framework of U_4 theory have by now been well established (compare Kerlick, 1975a, and Salam, 1975).

Trautman's conjecture about singularities was taken up by several authors. Kopczyński (1972, 1973a) first demonstrated the bouncing of certain simple cosmological models with torsion (see also Tafel, 1973, and Kuchowicz, 1973). For reviews on this development see Kerlick (1976), Kuchowicz (1975a, b, c), Tafel (1975), and Sec. V.3.

Kibble's (1961) paper and an earlier paper by Utiyama (1956) were the starting points for various gauge approaches to gravitation. As we will outline in Sec. IV, we believe that the Poincaré group actively interpreted leads to the most plausible classical field theory of gravitation. This is more or less in accord with Hayashi and Bregman (1973). Translational gauge theories were developed by Hayashi and Nakano (1968), Utiyama and Fukuyama (1971), Cho (1976), and others; rotational (Lorentz) gauge theories by Carmeli (1974), Lord (1971), and others. For some highly interesting discussions on gravitation and gauge theories see Kaempffer (1965), Dürr (1971, 1973), and Yang (1974). Earlier papers advocating the use of Poincaré gauge invariance include those of Brodski, Ivanenko, and Sokolik (1962) and Ivanenko (1962, 1972).

[3]The tetrad formalism used by Sciama was taken from Weyl (1929).

[4]Part of this article appeared translated and in a slightly revised version in Hehl (1973, 1974).

II. THE RIEMANN–CARTAN SPACE-TIME U_4

A. Differential manifold

We will assume that space-time has the properties of a continuum, that it is a four-dimensional differential manifold X_4. Any point of it can be labeled by real coordinates x^k with $k = 0, 1, 2, 3$, where 0 refers to the time coordinate, and 1, 2, 3 to the space coordinates. A renaming of the coordinates (coordinate transformation in the passive sense) does not change the continuum under consideration. On an X_4, using the pattern laid down by the transformation properties of coordinate differentials and gradients of scalar fields, we can define contra- and covariant vectors, and subsequently general tensors and tensor densities.

B. Connection

In order to do physics in such a space-time, we should have additional structures on the X_4. In an X_4 it does not make sense to say "a (nonzero) vector field is constant." To give such a statement meaning, one must introduce the notion of parallel transfer of vectors. Parallelly displaced from x^k to $x^k + dx^k$, a vector C^k changes according to the prescription

$$dC^k = -\Gamma^k_{ij}(x) C^j dx^i \, . \tag{2.1}$$

Here, in the sense that "in the infinitesimal everything is linear," dC^k is assumed to be bilinear in C^j and dx^i, the set of the 64 coefficients Γ^k_{ij} is the affine connection. An X_4 equipped with a Γ is called a linearly connected space or L_4.

The antisymmetric part of the affine connection

$$S_{ij}{}^{k} := \tfrac{1}{2}(\Gamma^k_{ij} - \Gamma^k_{ji}) \equiv \Gamma^k_{[ij]}, \tag{2.2}$$

in contrast to the symmetric part, transforms as a tensor. It is *Cartan's torsion* tensor, a purely affine quantity. If one builds up infinitesimal parallelograms in an L_4, it turns out that they do not close in general, the closure failure being proportional to the torsion tensor: torsion breaks infinitesimal parallelograms.

In an L_4, the torsion with its 24 independent components can be covariantly split into a traceless part and a trace. Therefore it will not be surprising in later applications to find that both of its parts enter with different coefficients. The modified torsion tensor

$$T_{ij}{}^{k} := S_{ij}{}^{k} + 2\delta^k_{[i} S_{j]l}{}^{l} \tag{2.3}$$

which differs from the torsion $S_{ij}{}^{k}$ in its trace, will be particularly useful.

The parallel transport law (2.1) can be extended to higher rank tensor fields and densities, and it is possible to define their covariant differentiation $\nabla \equiv \overset{\Gamma}{\nabla}$ with respect to Γ. Now it makes sense to state that "a field is constant over spacetime." In this case its covariant derivative has to vanish.

Parallel transfer is a path-dependent concept. If we parallelly transfer a vector around an infinitesimal area back to its starting point, we find that its components change. This change turns out to be proportional to the Riemann curvature tensor

$$R_{ijk}^{\cdot\cdot\cdot l} := 2\partial_{[i}\Gamma^l_{j]k} + 2\,\Gamma^l_{[i|m}\Gamma^m_{|j]k}\,. \tag{2.4}$$

The Riemann tensor in an L_4 obeys a differential identity (Bianchi's identity) and four more algebraic identities, which are somewhat more complicated than in a Riemannian space (see Schouten, 1954, pp. 144 ff.).

C. Metric

Now we introduce an assumption motivated by special relativity: In the L_4 at any point there exists an independent metric tensor field $g_{ij} = g_{ji}(x)$ in order to allow for local measurements of distances and angles. Such a space is called an (L_4, g). The square of the infinitesimal interval ds between x^k and $x^k + dx^k$ is then determined by

$$ds^2 = -g_{ij}(x)\,dx^i dx^j\,. \tag{2.5}$$

A corresponding invariant for a covariant vector field can be built up by means of the contravariant metric field g^{jk} which is defined by $g_{ij}g^{jk} = \delta^k_i$. We assume that g_{ij} has the signature $+2$. Then locally in suitable coordinates, g_{ij} can always be expressed in the Minkowski form

$$g_{ij} \overset{*}{=} \mathrm{diag}(-1, +1, +1, +1)\,. \tag{2.6}$$

In an (L_4, g), by means of the tensor of nonmetricity

$$Q_{ijk} := -\overset{\Gamma}{\nabla}_i g_{jk} \tag{2.7}$$

and the torsion tensor (2.2), we can establish the identity

$$\Gamma^k_{ij} \equiv g^{kl}\,\Delta^{abc}_{jil}\left(\tfrac{1}{2}\partial_a g_{bc} - g_{cd}S_{ab}^{\cdot\cdot d} + \tfrac{1}{2}Q_{abc}\right), \tag{2.8}$$

where we define the permutation tensor by

$$\Delta^{abc}_{jil} := \delta^a_j\delta^b_i\delta^c_l + \delta^a_i\delta^b_l\delta^c_j - \delta^a_l\delta^b_j\delta^c_i\,. \tag{2.9}$$

D. Local Minkowski structure

The interval defined by Eq. (2.5) becomes invariant under parallel transfer and therefore a useful concept in ordinary space-time physics only when we adopt an additional idea from special relativity: We require the metric field g_{ij} to be covariantly *constant*, thereby constraining the connection in a certain way

$$Q_{ijk} = 0 \quad (\text{nonmetricity} = 0)\,. \tag{2.10}$$

The metric postulate (2.10) guarantees that lengths, in particular the unit length, and angles are preserved under parallel displacement. This local Euclidean, or rather Minkowskian structure of space-time is established by all experiments supporting special relativity. The metric postulate is an *a posteriori* constraint which reflects in a precise manner the results of numerous experiments. A space-time where Q_{ijk} vanishes can be pictured as a set of Minkowskian "grains" glued together by means of the affine connection, obtained from constraining the general connection (2.8) by the metric postulate (2.10),

$$\Gamma^k_{ij} = \left\{{k \atop ij}\right\} - K_{ij}^{\cdot\cdot k};\quad K_{ij}^{\cdot\cdot k} := -S_{ij}^{\cdot\cdot k} + S_{j\cdot i}^{\cdot k} - S^k_{\cdot ij} = -K_{i\cdot j}^{\cdot k}\,. \tag{2.11a,b}$$

The quantity $\left\{{k \atop ij}\right\}$, the Christoffel symbol computed from the metric g_{ij}, is familiar and established by Einstein's general relativity. The contortion tensor[5] $K_{ij}^{\cdot\cdot k}$ (the non-Riemannian part) has 24 independent components and depends on metric and torsion. Tensor indices are raised and lowered by means of the metric. An L_4 with the most general unit-preserving affine connection (2.11) is a Riemann–Cartan space-time U_4. If torsion vanishes, we recover the Riemannian space-time V_4 of general relativity and, if the curvature additionally vanishes, the Minkowski space-time R_4 of special relativity,

$$(L_4, g) \xrightarrow{Q=0} U_4 \xrightarrow{S=0} V_4 \xrightarrow{R=0} R_4\,. \tag{2.12}$$

In a U_4, as well as in a V_4, the curvature tensor is antisymmetric not only in the first two indices, but also in the last two indices. Hence in a U_4, the Ricci tensor $R_{ij} := R_{kij}^{\cdot\cdot\cdot k}$ remains the only essential contraction of the curvature tensor. The Ricci tensor R_{ij} is asymmetric in general, as is the Einstein tensor of a U_4 which is defined according to the usual prescription

$$G_{ij} := R_{ij} - \tfrac{1}{2}g_{ij}R_k^{\cdot k}\,. \tag{2.13}$$

Its antisymmetric part, by means of the second algebraic identity of the curvature tensor, can be represented by the modified divergence ($\overset{*}{\nabla}_k := \nabla_k + 2S_{kl}^{\cdot\cdot l}$)

$$G_{[ij]} = \overset{*}{\nabla}_k T_{ij}^{\cdot\cdot k}\,. \tag{2.14}$$

Notice that for a contravariant vector density $\mathfrak{v}^i$ we have $\overset{*}{\nabla}_i\mathfrak{v}^i = \partial_i\mathfrak{v}^i$.

E. Autoparallels and extremals

When we discuss the preferred curves in a Riemann–Cartan space-time U_4, we must distinguish between two classes of curves, both of which reduce to the "geodesics" of the corresponding Riemannian space V_4 when we set torsion equal to zero.

Autoparallel curves (or straightest lines) are those curves over which a vector is transported parallel to itself according to the connection Γ^k_{ij} of the manifold. We start from Eq. (2.1) which defines the parallelism, choose a suitable affine parameter s and obtain the differential equation of the autoparallels

$$\frac{d^2x^k}{ds^2} + \Gamma^k_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds} = 0\,. \tag{2.15}$$

Observe that only the symmetric (but torsion dependent) part $\Gamma^k_{(ij)} = \left\{{k \atop ij}\right\} + 2S^{\cdot k}_{(ij)}$ of the connection enters (2.15).

Extremal curves (shortest or longest lines) are those curves which are of extremal length with respect to the metric of the manifold. According to (2.5), the length between two given points depends only on the metric

[5] In German "Verdrehungstensor."

field (and not on the torsion). Therefore the differential equation for the extremals can be derived from $\delta\int ds = \delta\int(-g_{ij}\,dx^i\,dx^j)^{1/2}=0$ exactly as in the corresponding Riemannian space and we are led to

$$\frac{d^2x^k}{ds^2}+\left\{{k \atop ij}\right\}\frac{dx^i}{ds}\frac{dx^j}{ds}=0. \qquad (2.16)$$

In a U_4 the autoparallels and the extremals coincide if and only if the torsion is totally antisymmetric $S_{ijk}=S_{[ijk]}$, that is, the dual of an axial vector. This can be proved by taking the symmetric part of Γ_{ij}^k in (2.11).

Let us anticipate a result of Sec. V.D.3: the trajectories of test particles in U_4 theory are neither autoparallels nor extremals in general. Neither notion is central to U_4 theory. Nor is the deviation equation for neighboring autoparallels, the analog of the geodesic deviation equation in a V_4, endowed with any direct operational significance.

F. Orthonormal tetrads as anholonomic coordinates

In a Riemann–Cartan space-time U_4, just as in the Riemannian V_4 of general relativity, we may introduce a (pseudo-)*orthonormal* basis of vectors $\mathbf{e}_\alpha = e^i{}_\alpha\partial_i$ (a "tetrad"—Greek indices $\alpha,\beta,\gamma\ldots=0,1,2,3$ number the vectors) and the dual basis $\theta^\alpha=e_i{}^\alpha dx^i$ of one-forms ("co-frames") at each point of space-time as anholonomic coordinates. It should be understood that the tool of anholonomic coordinates has the same standing in a U_4 as in a V_4: it is necessary when one wants to introduce *spinors* in a U_4 (or a V_4), and it is convenient when one looks for the operational interpretation of quantities appearing in the formalism.

The components $e^i{}_\alpha$ and their reciprocals $e_i{}^\alpha$ satisfy

$$e^i{}_\alpha e_j{}^\alpha=\delta^i_j;\quad e^i{}_\alpha e_i{}^\beta=\delta^\beta_\alpha;\quad g_{ij}=e_i{}^\alpha e_j{}^\beta g_{\alpha\beta}, \qquad (2.17)$$

where $g_{\alpha\beta}$ is the Minkowski metric $(-1,1,1,1)$. All quantities may be referred to these anholonomic coordinates. The object of anholonomity

$$\Omega_{\alpha\beta}{}^{\gamma}:=e^i{}_\alpha e^j{}_\beta\partial_{[i}e_{j]}{}^\gamma;\quad \Omega_{\alpha\beta\gamma}:=\Omega_{\alpha\beta}{}^{\delta}g_{\delta\gamma} \qquad (2.18)$$

measures the noncommutativity of the tetrad basis. The U_4 connection expressed in these anholonomic coordinates is

$$\Gamma_{\alpha\beta\gamma}:=\Gamma^\delta_{\alpha\beta}g_{\delta\gamma}=-\Omega_{\alpha\beta\gamma}+\Omega_{\beta\gamma\alpha}-\Omega_{\gamma\alpha\beta}-K_{\alpha\beta\gamma}=-\Gamma_{\alpha\gamma\beta}. \qquad (2.19)$$

Observe that the Christoffel part has vanished here because $g_{\alpha\beta}$ is constant and that $\Gamma_{\alpha\beta\gamma}$ has 24 independent components. The parallel transport of an orthonormal tetrad basis $\mathbf{e}_\alpha$ is given by

$$d\mathbf{e}_\beta=(-\Gamma^\gamma_{\alpha\beta}e_i{}^\alpha dx^i)\,\mathbf{e}_\gamma=(-\Gamma^\gamma_{\alpha\beta}\theta^\alpha)\,\mathbf{e}_\gamma=:-\omega^\gamma{}_\beta\,\mathbf{e}_\gamma. \qquad (2.20)$$

The connection one-form $\omega^\gamma{}_\beta$ $(\omega_{\gamma\beta}=\omega_{[\gamma\beta]})$ describes the rotation of the parallelly transported tetrad relative to the given tetrad system. This rotation consists of two pieces, the Ricci rotation $\tilde\omega^\gamma{}_\beta$ due to the Riemannian metric and depending on $\Omega_{\alpha\beta\gamma}$ (see Eq. 2.19), and an independent "added twist" $K^\gamma{}_\beta:=K_{\alpha\beta}{}^{\gamma}\theta^\alpha$ proportional to the contortion. The tetrad vectors in a U_4 thus have new degrees of freedom—independent rotations not specified by the metric structure. For more information on this see Schouten (1954) and Misner, Thorne, and Wheeler "MTW" (1973).

Trautman (1973b) bases his study of the U_4 theory on the calculus of tensor valued differential forms, in which the local basis of forms θ^α at each point, the exterior product $\wedge$, the exterior derivative, and the connection one-form $\omega^\gamma{}_\beta=\tilde\omega^\gamma{}_\beta-K^\gamma{}_\beta$ play a fundamental role. His formalism is equivalent to the formalism used here. The torsion two-form Θ^δ and the curvature two-form $\Omega^\delta{}_\gamma$ in Trautman's work are given in terms of the quantities used by us by

$$\Theta^\delta=-S_{\alpha\beta}{}^{\delta}\theta^\alpha\wedge\theta^\beta;\quad \Omega^\delta{}_\gamma=+\tfrac{1}{2}R_{\alpha\beta\gamma}{}^{\delta}\theta^\alpha\wedge\theta^\beta. \qquad (2.21)$$

We have seen that the presence of contortion in a U_4 supplies space-time with new rotational degrees of freedom. We know that matter possesses spin angular momentum in general, and, in the spirit of general relativity, we would like space-time to reflect the properties of matter. In the next section we will achieve this result by coupling the contortion of space-time to the spin of matter. Henceforth, we will consider space-time to be a U_4 whose affine connection is given by Eq. (2.11).

III. THE FIELD EQUATIONS OF U_4 THEORY

A. Matter action function and minimal coupling to gravitation

We start with the flat Minkowski space-time R_4 of special relativity. The group of motions is the Poincaré group (four-dimensional rotations and translations). Let us imagine a classical matter field $\psi(x^k)$ embedded in the R_4. It is supposed to transform as a tensor with respect to the Poincaré group.[6] The special relativistic Lagrangian density of matter in Cartesian coordinates $\mathfrak{L}(\psi,\partial\psi;\eta)$ is assumed to depend on the constant Minkowski metric η, the matter field, and the gradient of the matter field.

Now imagine that gravitation is "switched on." Then the special relativistic Lagrangian has to be coupled to gravity ("geometry") in the sense of the equivalence principle. This requires us to substitute the U_4 metric $g(x)$ for the Minkowski metric η and to couple minimally the U_4 connection Γ to the matter field ψ. Thus we arrive at

$$\eta\to g;\quad \partial\psi\to\overset{\Gamma}{\nabla}\psi \qquad (3.1a,b)$$

leading from the realm of Minkowski space-time to the

[6] The holonomic formalism to be developed in this section can be extended without difficulty to include spinor fields by introducing orthonormal tetrads as anholonomic coordinates according to the procedure described in Sec. II.F (see Hehl, 1973, 1974). Spinor fields have no special relationship to torsion. This is because energy-momentum and spin angular momentum are tensors for material spinor fields as well as for tensor fields and because the effect of matter on space-time is mediated by energy-momentum and spin. In the gauge formalism of Sec. IV, which we shall show to be equivalent to the formalism of this section, spinor fields will be included right from the beginning.

U_4 with the connection (2.11).[7] The minimal substitution (3.1b) will be applied only to matter fields, but not to gauge fields of internal symmetry groups, such as Maxwell's field (see Sec. V.B.2).

After the minimal coupling procedure, the action function of the matter field ψ interacting with gravitation reads

$$W_m = (1/c)\int d^4x\,\mathfrak{L}(\psi, \overset{\Gamma}{\nabla}\psi, g) = (1/c)\int d^4x\,\mathfrak{L}(\psi, \partial\psi, g, \partial g, S). \quad (3.2)$$

Here c is the velocity of light and d^4x the coordinate four-volume element. The action function depends on the independent variables

$$\psi,\ g_{ij}\ (10\text{ components}),\quad S_{ij}^{\cdot\cdot k}\ (24\text{ components}). \quad (3.3)$$

According to (2.11), we could take, instead of the torsion S, the contortion $K=K(g,S)$ as an independent variable. The choice (3.3) is the more fundamental one, however, since S (unlike K) is *a priori* independent of the metric.

B. Dynamical definitions of energy-momentum and spin

When we vary metric and torsion independently, we can define the *metric* energy-momentum tensor σ and a tensor μ which has the meaning of a spin energy potential (see Eq. 3.8 below) as follows[8]:

$$e\sigma^{ij} := 2\delta\mathfrak{L}/\delta g_{ij};\quad e\mu_k^{\cdot ji} := \delta\mathfrak{L}/\delta S_{ij}^{\cdot\cdot k}. \quad (3.4)\ (3.5)$$

Observe that Eq. (3.4) must be evaluated for $S_{ij}^{\cdot\cdot k}$ held constant.

According to our considerations in Sec. II.F, spin should couple to contortion rather than to torsion. We need only use Eq. (2.11b) and simple algebra to show that the spin angular momentum tensor

$$e\tau_k^{\cdot ji} := \delta\mathfrak{L}/\delta K_{ij}^{\cdot\cdot k} \quad (3.6)$$

can be expressed in terms of the spin energy potential

$$\tau^{ijk} = \mu^{[ji]k};\quad \mu^{ijk} = -\tau^{ijk} + \tau^{jki} - \tau^{kij}. \quad (3.7)$$

We need still one more definition before we can link the geometry up to more familiar quantities. The variation of the torsion as well as that of the metric contributes to the total energy of the matter field. We introduce the asymmetric total energy-momentum tensor

$$\Sigma^{ij} := \sigma^{ij} - \overset{*}{\nabla}_k\mu^{ijk} = \sigma^{ij} + \overset{*}{\nabla}_k(\tau^{ijk} - \tau^{jki} + \tau^{kij}) \quad (3.8)$$

which will be justified in the next section.

[7]Should the matter Lagrangian depend on second or higher derivatives of the matter field or should a nonminimal coupling to gravitation be allowed, we would still be able to arrive at a consistent theory within the framework of a U_4 space-time. However, the dualistic character of the theory, that is, the strict separation of matter and geometry, would be lost. This supports our belief that, in a dualistic framework, elementary matter fields have first-order Lagrangians. Thus the equivalence principle, and hence minimal coupling according to Eq. (3.1), applies to these matter fields. For a detailed discussion of the equivalence principle in U_4 theory and the coupling process see von der Heyde (1975b) and the end of Sec. IV.B.4.

[8]$e := [-\det(g_{ij})]^{1/2}$; $\delta\mathfrak{L}(Q,\partial Q)/\delta Q := \partial\mathfrak{L}/\partial Q - \partial_k[\partial\mathfrak{L}/\partial(\partial_k Q)]$.

C. Canonical tensors and conservation laws

The action function (3.2) has to be a scalar. By means of Noether's theorem, this property leads to identities for the Lagrangian density. If the equation

$$\delta\mathfrak{L}/\delta\psi = 0\quad (\text{matter equation}) \quad (3.9)$$

is fulfilled, these identities supply the identifications

$$e\Sigma_i^{\cdot j} = \mathfrak{L}\delta_i^j - \frac{\partial\mathfrak{L}}{\partial(\partial_j\psi)}\nabla_i\psi; \quad (3.10)$$

$$e\tau^{ijk} = \frac{\partial\mathfrak{L}}{\partial(\partial_k\psi)}h^{[ji]}\psi \quad (3.11)$$

and the conservation laws of energy-momentum and angular momentum

$$\overset{*}{\nabla}_j\Sigma_i^{\cdot j} = 2\Sigma_k^{\cdot j}S_{ij}^{\cdot\cdot k} + \tau_{jk}^{\cdot\cdot l}R_{il}^{\cdot\cdot jk}; \quad (3.12)$$

$$\overset{*}{\nabla}_k\tau_{ij}^{\cdot\cdot k} - \Sigma_{[ij]} = 0. \quad (3.13)$$

In Eq. (3.11) the quantities h^{ij} are the representation matrices of an infinitesimal coordinate transformation appropriate to ψ. Equations (3.10) and (3.11) reveal $\Sigma_i^{\cdot j}$ and τ^{ijk} as the canonical energy-momentum and spin angular momentum tensors, respectively, which are well known from special relativistic Lagrangian field theory (see also Sec. IV.A.4). Accordingly, the definitions (3.4), (3.5), (3.6), and (3.8) are reasonable in the light of our experience with special relativity. In particular, the definition (3.8) allows us to decompose the (asymmetric) canonical energy-momentum tensor into the metric one, which is symmetric by definition, and into a term supplied by spin. The interpretation of $\Sigma_i^{\cdot j}$ as the total energy-momentum tensor leads to an understanding of the usual symmetrization procedure of the canonical energy-momentum tensor.[9]

D. Total action function

Comparison with general relativity suggests that we equate the action function of the field, that is the action function of the space-time continuum with the connection (2.11), to an integral over an effective field Lagrangian density $\mathfrak{V}/2k$:

$$W_f = (1/c)\int d^4x\,\mathfrak{V}(g, \partial g, S, \partial S)/2k \quad (3.14)$$

($k := 8\pi c^{-4}G$; $G =$ Newton's gravitational constant). As in general relativity (see Landau-Lifshitz, 1962), $\mathfrak{V}$ has the dimension (length)$^{-2}$. No new coupling constant is necessary in order to accommodate torsion in this action function.

Adding up matter and field contributions (3.2) and (3.14) leads to the total action function of the interacting system

$$W = (1/c)\int d^4x[\mathfrak{L}(\psi, \partial\psi, g, \partial g, S) + (1/2k)\mathfrak{V}(g, \partial g, S, \partial S)]. \quad (3.15)$$

E. Field equations

We vary the total action function (3.15) with respect to the independent variables (3.3). First we get the matter

[9]. . . and to a solution of the localization problem of energy-momentum for massive matter (compare Hehl, 1976).

equation (3.9) and then the 10 plus 24 independent field equations

$$-\delta\mathcal{V}/\delta g_{ij}=ke\sigma^{ij}\,, \qquad (3.16)$$

$$-\delta\mathcal{V}/\delta S_{ij}{}^{k}=2ke\,\mu_{k}{}^{ji}\,. \qquad (3.17)$$

If we introduce as sources the canonical tensors as defined in Eqs. (3.8) and (3.6), we have

$$-(\delta\mathcal{V}/\delta g_{ij})-(g^{li}/2)\overset{*}{\nabla}_k(\delta\mathcal{V}/\delta S_{jk}{}^{l})=ke\Sigma^{ij}\,, \qquad (3.18)$$

$$-(g^{l[i}/2)(\delta\mathcal{V}/\delta S_{j]k}{}^{l})=ke\tau^{ijk}\,. \qquad (3.19)$$

The antisymmetric part of (3.18) is satisfied identically (see Eq. 3.13).

Now we have to specify $\mathcal{V}$. Because the Ricci tensor is the only essential contraction of the Riemann tensor in a U_4 (as in general relativity), the density of the curvature scalar $\mathfrak{R}:=eg^{ij}R_{ij}$ suggests itself as the most natural choice. A divergence can be split off

$$\mathfrak{R}=\tilde{\mathfrak{R}}+\partial_i(2eg^{k[j}\Gamma^{i]}_{jk})\,; \qquad (3.20)$$

$$\tilde{\mathfrak{R}}=\tilde{\mathfrak{R}}(g,\partial g,S):=2eg^{ij}(\Gamma^{k}_{l[i}\Gamma^{l}_{k]j}-\Gamma^{l}_{li}S_{jk}{}^{k})\,.$$

Just as in general relativity, $\tilde{\mathfrak{R}}$ can be taken as an effective field Lagrangian: $\delta\mathcal{V}=\delta\tilde{\mathfrak{R}}=\delta\mathfrak{R}$. The computations of the variation of $\mathfrak{R}$ with respect to g and S are straightforward but involved (see the Appendix). If we substitute the results into (3.18) and (3.19), we have

$$G^{ij}=k\Sigma^{ij} \quad \text{(1st field equation)}\,, \qquad (3.21)$$

$$T^{ijk}=k\tau^{ijk} \quad \text{(2nd field equation)}\,, \qquad (3.22)$$

or, in words,

Einstein tensor $= k\times$ energy-momentum ,

modified torsion $= k\times$ spin angular momentum .

Equation (3.21) is a generalized Einstein equation, Eq. (3.22) an algebraic relation linking spin and Cartan's torsion.

Because the second field equation is algebraic, one is able to substitute everywhere spin for torsion and cast out effectively torsion from the formalism. For this purpose we split the Einstein tensor G^{ij} of the U_4 into its Riemannian part $G^{ij}(\{\})$ and its non-Riemannian part. The torsion terms in the latter part are substituted by means of Eq. (3.22). This yields

$$G^{ij}(\{\})=k\tilde{\sigma}^{ij} \quad \text{(combined field equation)} \qquad (3.23)$$

with the combined energy-momentum tensor

$$\tilde{\sigma}^{ij}:=\sigma^{ij}+k[-4\tau^{ik}{}_{[l}\tau^{jl}{}_{k]}-2\tau^{ikl}\tau^{j}{}_{kl}+\tau^{kli}\tau_{kl}{}^{j}$$
$$+\tfrac{1}{2}g^{ij}(4\tau_{m}{}^{k}{}_{[l}\tau^{ml}{}_{k]}+\tau^{mkl}\tau_{mkl})]\,, \qquad (3.24)$$

which is symmetric by definition and which obeys the conservation law $\overset{\{\}}{\nabla}_j\tilde{\sigma}^{ij}=0$. We have formally eliminated torsion, but only by sacrificing the interpretation of Eq. (3.1b) as a minimal coupling. Now Eq. (3.1b) has to be applied with

$$\Gamma^{k}_{ij}=\left\{{k \atop ij}\right\}+k(\tau_{ij}{}^{k}-\tau_{j}{}^{k}{}_{i}+\tau^{k}{}_{ij}+\delta^{k}_{i}\tau_{jl}{}^{l}-g_{ij}\tau^{kl}{}_{l})\,. \qquad (3.25)$$

With respect to the Riemannian connection $\{\}$, the substitution (3.1b) looks nonminimal.

The fundamental framework of U_4 theory has been established. What is left to do is to understand U_4 theory better and to consider the physical consequences. Section IV gives an alternative derivation of the field equations in terms of the principle of local gauge invariance with respect to the Poincaré group. Readers whose primary interest lies in the consequences of the theory may wish to skip to Sec. V.

IV. LOCAL GAUGE THEORY FOR THE POINCARÉ GROUP

We have already alluded in Sec. I to the fundamental role of the Poincaré group in the characterization of elementary particles and its close connection with the U_4 theory. We shall show in this section that to require the invariance of a special relativistic theory of matter under local space-time rotations and translations (independent action of the Poincaré group at every point) leads inexorably to the introduction of torsion and curvature. In view of the fundamental significance of the Poincaré group for physics, and of the local gauge approach in general, we take such a derivation as the strongest evidence that the geometry of the physical world is indeed a Riemann–Cartan geometry.

For physical reasons, as well as reasons of presentation, we shall not follow too closely the original expositions of Sciama (1962) and Kibble (1961), although we shall discuss these works in due course. We wish to emphasize here two crucial points which are usually overlooked in discussions of the Poincaré transformations.

We first wish to clarify the distinction between *coordinate systems* and *systems of orthonormal reference frames* (tetrads) in space-time. We employ (holonomic) coordinate systems x^i in the usual manner, merely as a means for labeling events. A frame of reference, comprising an orthonormal basis (tetrad, vierbein) of vectors $\mathbf{e}_\alpha$ will be given a more important, operational meaning: it will represent in principle our standard apparatus for measurements in space and time.

Our second point for emphasis is that, in analogy to the usual procedure in local gauge theories of internal symmetry groups (Yang and Mills, 1954; Utiyama, 1956), a Poincaré transformation will be interpreted as an *active* transformation of the matter fields. That is, coordinates and frames will be regarded as fixed once and for all, while the matter fields [here given the collective designation $\psi(x^i)$] are replaced by fields $[\Pi\psi](x^i)$ which have been rotated and translated with respect to $\psi(x^i)$. It is, of course, to be expected that a complementary, *passive* interpretation of Π (the same matter field viewed from transformed frames of reference) will always be possible, and we shall discuss this at the end of Sec. IV.B.3. In this section we put $c=1$.

A. Special relativistic kinematics

1. Coordinate transformation

Suppose that we start with the usual interpretation (e.g., Kibble, 1961) of an infinitesimal global Poincaré transformation as a transformation of the global Cartesian coordinates in Minkowski space

$$x^i \to {}'x^i = x^i + \mathring{\omega}_j{}^i x^j + \mathring{\epsilon}^i\,, \tag{4.1}$$

where $\mathring{\omega}_{ji} = \mathring{\omega}_{[ji]}$ and $\mathring{\epsilon}^i$ are both infinitesimal constant parameters. The components of the matter field $\psi(x^i)$ with respect to the coordinate basis $\mathbf{e}_i$, orthonormal in Minkowski space, transform under (4.1) as

$$\psi(x) \to {}'\psi('x) = (1 + \mathring{\omega}^{ji} f_{ij})\psi(x)\,. \tag{4.2}$$

Here, the constant quantities $f_{ij} = f_{[ij]}$ are the representations of the generators of the Lorentz group [Eq. (4.8), below] associated with the fields ψ.

Further, suppose that we now require, as is usual, that the action function $W = \int L(\psi, \partial\psi)\, d^4x$, which is invariant under transformations (4.1) and (4.2), be required to exhibit invariance under a *local* gauge transformation, where the constant parameters $\mathring{\omega}$ and $\mathring{\epsilon}$ are replaced by the arbitrary functions $\mathring{\omega}(x)$ and $\mathring{\epsilon}(x)$. But then the rotational part of (4.1) loses its status as an independent part of the gauge group, since it can now be absorbed into a redefined function $\epsilon^i(x)$. What was once a ten-parameter global transformation has been reduced to the four-parameter group of general coordinate transformations

$$x^i \to {}'x^i = x^i + \epsilon^i(x)\,. \tag{4.3}$$

What happens above is plausible from a mathematical point of view, but our physical experience tells us that it is inadequate. We know that there exist in nature, corresponding to the rotations and translations in Eq. (4.1), independent and irreducible currents of spin $\tau_{ij}{}^{\cdot\cdot k}$ and of momentum $\Sigma_i{}^{\cdot j}$. The spin current τ cannot be reduced to the form of an orbital angular momentum. Neither, then, should we expect local rotations to be reduced to local translations. A local gauge theory based on Eq. (4.1) can yield no independent concept of spin, and can only make sense for spinless matter, or for matter approximated as spinless. Fermions, in particular, are described by spinor representations of the Lorentz group, but general coordinate transformations have no spinor representation.

2. Frames of reference

In order to maintain the independence of the rotational gauge transformations, we must first recall the operational meaning of a Poincaré transformation in special relativity. The global Cartesian coordinate system in which the transformation (4.1) is defined represents, at least in principle, a global network of standard clocks and standard measuring rods (see, for example, Taylor and Wheeler, 1966, Ch. 1). A Poincaré transformation, in this context, is no mere renaming of points, but a relation between measurements. A transformation (4.1) presupposes the existence of standard lengths (and angles) which are independent of position and direction.

To implement this physical aspect of a Poincaré transformation, we choose at each point of the matter distribution a set of orthonormal vectors (tetrad basis) with respect to which all measurements of the "physical components" of the matter field are made. This tetrad basis will be denoted by $\mathbf{e}_\alpha$, where the Greek letters label the vectors of the basis $(0,1,2,3)$. Thus,

$$\mathbf{e}_\alpha \cdot \mathbf{e}_\beta = g_{\alpha\beta} = \mathrm{diag}(-1,+1,+1,+1)\,. \tag{4.4}$$

The tetrads θ^α dual to $\mathbf{e}_\alpha$ ("co-frames") are given in a differential manifold with metric by $\theta^\beta \cdot \mathbf{e}_\alpha = \delta_\alpha^\beta$. In a Minkowski space R_4, the tetrad bases $\mathbf{e}_\alpha$ and the Cartesian coordinate bases $\mathbf{e}_i$ coincide according to

$$\mathbf{e}_\alpha \overset{R_4}{=} \delta_\alpha^i \mathbf{e}_i\,; \quad \theta^\alpha \overset{R_4}{=} \delta_i^\alpha \mathbf{e}^i\,. \tag{4.5}$$

Henceforth we shall regard the matter fields ψ (except gauge potentials, see below) as having components with respect to such an orthonormal frame.

3. Global Poincaré transformation

In order to avoid the problems associated with Eq. (4.3) and to make explicit the active interpretation of a Poincaré transformation, we start now from the global gauge transformation

$$\psi(x) \to [\Pi\psi](x) := (1 + \omega^{\alpha\beta} f_{\beta\alpha} - \epsilon^\gamma \partial_\gamma)\psi(x)\,; \tag{4.6}$$

$$\epsilon^\gamma := \mathring{\epsilon}^\gamma + \mathring{\omega}_\alpha{}^{\cdot\gamma} \delta_i^\alpha x^i\,; \quad \omega^{\alpha\beta} := \mathring{\omega}^{\alpha\beta}\,. \tag{4.7a,b}$$

Here, $\mathring{\epsilon}$ and $\mathring{\omega}$ are the same parameters as in (4.1), but referred now to an orthonormal basis, and claim the interpretation of measured lengths and angles. The representations of the generators of rotations $f_{\alpha\beta}$ and of translations $\partial_\gamma = \delta_\gamma^i \partial_i$ satisfy in an R_4 the well known commutation relations of the Poincaré group,

$$[f_{\alpha\beta}, f_{\gamma\delta}] = g_{\gamma[\alpha} f_{\beta]\delta} - g_{\delta[\alpha} f_{\beta]\gamma}\,, \tag{4.8}$$

$$[f_{\alpha\beta}, \partial_\gamma] = g_{\gamma[\alpha}\partial_{\beta]}\,, \tag{4.9}$$

$$[\partial_\alpha, \partial_\beta] = 0\,. \tag{4.10}$$

It may seem at first glance that nothing has been gained, for $[\Pi\psi](x)$ in (4.6) is via Eqs. (4.7), (4.5), (4.1) identical with $'\psi(x)$ in Eq. (4.2). The essential difference is that, even when the parameters $\mathring{\epsilon}$ and $\mathring{\omega}$ are generalized to arbitrary functions, Π in Eq. (4.6) still makes physical sense, and the rotational part $\mathring{\omega}$ remains independent of the translational part.

This formulation also offers the advantage that we can now regard the transformation (4.6) as an active gauge transformation of all the matter fields ψ at a fixed coordinate x^i measured with respect to a fixed tetrad $\mathbf{e}_\alpha$. The process prescribed by Eq. (4.6) is this: Replace the fields ψ at a point x^i by fields which have first been rotated by an amount $-\omega$, that is, $\psi(x) \to [\Lambda\psi](x) = (1 + \omega f)\psi(x)$, and then have been translated by an amount $+\epsilon = \mathring{\epsilon} + \mathring{\omega} \cdot x$, that is, $[\Lambda\psi](x) \to [\Pi\psi](x) = [\Lambda\psi](x - \epsilon)$. Then, as experience shows, matter distributions $\Pi\psi$ and ψ are physically equivalent. Apart from their relative orientation and position, there are no measurable properties by which the distributions can be distinguished.

The translation generators in Π do not cause a rotation of ψ because the orthonormal frames $\mathbf{e}_\alpha$ are parallel throughout Minkowski space. One must notice, however, that the magnitude of the translation, i.e., the parameter ϵ, is not completely independent, inasmuch as it contains, according to Eq. (4.7a), a rotation-induced part $\mathring{\omega} \cdot x$ as well as the independent part $\mathring{\epsilon}$. This fact is closely related to the existence of orbital angular momentum.

4. Matter Lagrangian and conservation theorems

Let the material system be specified by a Lagrangian of the form

$$L(\psi, \partial_i\psi) = L(\psi, \delta^i_\alpha \partial_i \psi; g_{\alpha\beta}, \gamma^\alpha \ldots). \tag{4.11}$$

Except for the matter field ψ, all other quantities (the Kronecker delta δ^i_α, the Dirac matrices γ^α, the Minkowski metric $g_{\alpha\beta}$, etc.) are invariant under Π.

Equivalence of physical measurements as defined above demands the invariance of the action function $W = \int L\, d^4x$ under Π

$$\delta W := \Pi W - W = \int_{\Pi\Omega} L(\Pi\psi, \partial_i \Pi\psi)\, d^4x - \int_\Omega L(\psi, \partial_i\psi)\, d^4x = 0\,, \tag{4.12}$$

where $\Pi\Omega$ is the volume Ω translated by an amount $+\epsilon(x)$. We calculate

$$\delta W = \int_\Omega \left\{ \left(\frac{\partial L}{\partial \psi} - \partial_i \frac{\partial L}{\partial(\partial_i \psi)} \right) \delta\psi + \partial_i \left(\frac{\partial L}{\partial(\partial_i\psi)} \delta\psi + \epsilon^\alpha \delta^i_\alpha L \right) \right\} d^4x\,. \tag{4.13}$$

The first term in parentheses vanishes by reason of the Euler–Lagrange equation satisfied by the matter field ψ. Since Eq. (4.12) must hold for arbitrary volumes Ω, the divergence term in (4.13) must also vanish. Taking into account the definition of $\delta\psi$ from Eq. (4.6) and the relation $\partial_i \epsilon^\gamma = \overset{\circ}{\omega}{}_\alpha{}^{\cdot\gamma} \delta^\alpha_i$ obtained by differentiating Eq. (4.7a), we find as coefficients of the independent constants $\overset{\circ}{\omega}$ and $\overset{\circ}{\epsilon}$ the conservation laws of angular momentum and energy-momentum

$$\partial_i \tau_{\alpha\beta}^{\cdot\cdot\,i} - \Sigma_{[\alpha\beta]} = 0\,, \tag{4.14a}$$

$$\partial_i \Sigma_\gamma^{\cdot\,i} = 0\,. \tag{4.14b}$$

Here, the canonical currents of spin angular momentum $\tau_{\alpha\beta}^{\cdot\cdot\,i}$ and of energy-momentum $\Sigma_\gamma^{\cdot\,i}$ have been defined by

$$\tau_{\alpha\beta}^{\cdot\cdot\,i} := -\frac{\partial L}{\partial(\partial_i\psi)} f_{\alpha\beta}\psi\,, \tag{4.15a}$$

$$\Sigma_\gamma^{\cdot\,i} := \delta^i_\gamma L - \frac{\partial L}{\partial(\partial_i\psi)} \partial_\gamma \psi\,. \tag{4.15b}$$

In Eq. (4.14a), the term $-\Sigma_{[\alpha\beta]} = \partial_i(x_j \delta^j_{[\alpha} \Sigma_{\beta]}^{\cdot\,i})$, the divergence of the orbital angular momentum, results from the rotation-dependent part $\overset{\circ}{\omega}\cdot x$ of the total translation ϵ.

Having finished this detailed exposition, we are now prepared to proceed in the generalization of this globally Poincaré invariant theory to a theory invariant under local Poincaré transformations.

B. General relativistic kinematics

1. Currents and potentials

Suppose that we now define a local Poincaré gauge transformation by allowing the hitherto constant parameters $\overset{\circ}{\omega}$ and $\overset{\circ}{\epsilon}$ to vary freely over space-time. Within the framework of special relativity, to each $\psi(x)$ is assigned a matter distribution $[\Pi\psi](x)$ which is no longer measurably equivalent, in the sense of Sec. IV.A. Applying the conservation laws (4.14) leads to the following nonvanishing term in the variation (4.13) of the action under $\Pi(x)$

$$\delta W = \int_\Omega \{(-\partial_i \overset{\circ}{\omega}{}^{\alpha\beta})(\tau_{\beta\alpha}^{\cdot\cdot\,i} + x_\beta \Sigma_\alpha^{\cdot\,i}) + (\partial_i \overset{\circ}{\epsilon}{}^\gamma) \Sigma_\gamma^{\cdot\,i}\} d^4x\,, \tag{4.16a}$$

or, using Eq. (4.7),

$$\delta W = \int_\Omega \{(-\partial_i \omega^{\alpha\beta}) \tau_{\beta\alpha}^{\cdot\cdot\,i} + (\partial_i \epsilon^\gamma - \omega_\beta^{\cdot\gamma} \delta^\beta_i) \Sigma_\gamma^{\cdot\,i}\} d^4x\,. \tag{4.16b}$$

In Eq. (4.16a), the gradients of the independent parameters $\overset{\circ}{\omega}$ and $\overset{\circ}{\epsilon}$ act upon total angular momentum $J := \tau + x \circ \Sigma$ and energy-momentum Σ, respectively. Correspondingly, the currents J and Σ ought to be the sources of their respective gauge fields. But J cannot couple to a gauge potential as a dynamical current in a Poincaré invariant manner for it is not a translation-invariant property of matter. We must rather identify the independent tensors τ and Σ of (4.16b) as the dynamical currents of matter.

In order to obtain Poincaré gauge invariance under local transformations $\Pi(x)$, we couple six rotational gauge potentials $\Gamma_i^{\ \alpha\beta} = \Gamma_i^{\ [\alpha\beta]}(x)$ and four translational gauge potentials $e_i^{\ \alpha}(x)$ to the matter fields, that is,

$$L(\psi, \partial_i \psi) \to \mathfrak{L}(\psi, \partial_i\psi, \Gamma_i^{\ \alpha\beta}, e_i^{\ \alpha})\,. \tag{4.17}$$

Then the variations $\delta\Gamma$, δe of the gauge potentials under $\Pi(x)$ contribute to the change in the action function. These variations compensate for the terms in (4.16) in the limit of weak fields provided

$$\partial\mathfrak{L}/\partial\Gamma_i^{\ \alpha\beta} \simeq \tau_{\alpha\beta}^{\cdot\cdot\,i}\,, \tag{4.18a}$$

$$\delta\Gamma_i^{\ \alpha\beta} \simeq -\partial_i \omega^{\alpha\beta}\,, \tag{4.18b}$$

$$\partial\mathfrak{L}/\partial e_i^{\ \alpha} \simeq \Sigma_\alpha^{\cdot\,i}\,, \tag{4.19a}$$

$$\delta e_i^{\ \alpha} \simeq \omega_\beta^{\cdot\alpha} \delta_i^{\ \beta} - \partial_i \epsilon^\alpha\,. \tag{4.19b}$$

That these relations [(4.18), (4.19)] can only hold in the weak field limit can be readily seen from the definitions (4.15) of the canonical tensors τ and Σ: *A* coupling which exactly produces (4.18a) and (4.19a) necessarily alters the canonical tensors and their conservation laws (4.14), whose validity is presupposed in the derivation of (4.16). Thus we are led to expect a highly nonlinear coupling as the final result.

In order to proceed further without *ad hoc* assumptions, we will try to obtain a more exact picture of the structure of this coupling by clearing up the physical meaning of the gauge potentials and of the compensation process. For this purpose we will deduce a condition from the Poincaré transformation which makes quantitative the notion of "equivalent matter distributions" without reference to an action principle.

2. Rigidity condition and coupling

The global gauge transformation (4.6), (4.7) may be completely characterized by two important properties: it changes neither the distance between events nor the relative orientation of neighboring matter fields.

We may state this more exactly as follows: let $\xi^i(x) = \delta^i_\alpha \xi^\alpha(x)$ be an infinitesimal vector field which we imagine to be bound with the matter field. Then a simple calculation shows that the Poincaré transformation (4.6), (4.7) with *constant* parameters $\overset{\circ}{\omega}, \overset{\circ}{\epsilon}$ leads to

$$\Pi(\xi^i \partial_i \psi) = (\Pi \xi^\alpha) \delta^i_\alpha \partial_i (\Pi\psi)\,. \tag{4.20}$$

The meaning of this relation is clear: It makes no difference whether (a) we compare field amplitudes ψ at nearby points before the transformation Π and transform the result, or whether (b) we compare the transformed amplitudes $\Pi\psi$ at the corresponding points shifted with the material continuum. Were this not so, we could distinguish experimentally between $\Pi\psi$ and ψ. Because the matter field behaves like a rigid body under such a global transformation, we call Eq. (4.20) a *rigidity condition.*

For a local transformation with position-dependent parameters $\mathring{\omega}(x)$ and $\mathring{\epsilon}(x)$, we obtain instead of Eq. (4.20) the relation

$$\Pi(\xi^i\partial_i\psi)=(\Pi\xi^\alpha)\delta^i_\alpha\partial_i(\Pi\psi)+\xi^i[(-\partial_i\omega^{\alpha\beta})f_{\beta\alpha}+(\partial_i\epsilon^\gamma-\omega_\delta^{\cdot\gamma}\delta^\delta_i)\partial_\gamma]\psi\,, \tag{4.21}$$

or, by comparison with (4.18b, 4.19b),

$$\Pi(\xi^i\partial_i\psi)\simeq(\Pi\xi^\alpha)(\delta^i_\alpha+\delta e^i{}_\alpha)(\partial_i+\delta\Gamma_i{}^{\beta\gamma}f_{\gamma\beta})(\Pi\psi)\,. \tag{4.22}$$

Here, the reciprocal quantities $e^i{}_\alpha(x)$ satisfy

$$e^i{}_\alpha:=\frac{1}{e}\frac{\partial e}{\partial e_i{}^\alpha}\;;\;e:=\det(e_i{}^\alpha)\;\Rightarrow\;e^i{}_\alpha e_j{}^\alpha=\delta^i_j\;;\;e^i{}_\alpha e_i{}^\beta=\delta^\beta_\alpha\,. \tag{4.23}$$

From the viewpoint of special relativity, Eq. (4.21) would have to be interpreted as an irregular deformation of the matter field; the distributions $\Pi\psi$ and ψ would be measurably different. Special relativity is no longer an adequate framework as soon as we demand, in the sense of a gauge theory, that local Poincaré transformations should lead to measurably equivalent matter distributions. Just how to leave special relativity is strongly suggested by Eq. (4.22).

Interpreted as a rigidity condition, Eq. (4.22) tells us that two changes are required, namely:

(a) an adjustable connection between the orthonormal frames $\mathbf{e}_\alpha$ [which still satisfy Eq. (4.4), however] and the coordinate bases $\mathbf{e}_i$,

$$\delta^\alpha_i\to e_i{}^\alpha(x)\,;\quad\delta^i_\alpha\to e^i{}_\alpha(x)\,, \tag{4.24}$$

and

(b) an adjustable relative rotation $dx^i\Gamma_{i\beta}{}^\alpha(x)\mathbf{e}_\alpha$ of the tetrads $\mathbf{e}_\beta$ at neighboring points x^i and x^i+dx^i which induces a change in the derivative operator,

$$\partial_i\to D_i:=\partial_i+\Gamma_i{}^{\alpha\beta}(x)f_{\beta\alpha}\,. \tag{4.25}$$

The operator D_i is defined in such a way that the connection coefficients Γ operate only on Greek (anholonomic) indices. The connection Γ does not couple to gauge potentials of internal symmetries such as the electromagnetic potential A_i. Were this not so, we would have to admit the possibility of breaking of internal symmetries by the action of the gravitational field.

3. Local Poincaré transformation

According to its definition (4.25), the operator D generates rotation-free *parallel* translations, where "parallel" is locally defined in the same way as in Minkowski space and is verifiable in principle by the same local measuring device as before. The operator D assumes the operational significance formerly possessed by the operator ∂ in Minkowski space. The Poincaré transformation (4.6), (4.7) ought therefore to be viewed as the special relativistic limit of the local gauge transformation

$$\psi(x)\to[\tilde{\Pi}\psi](x):=(1+\omega^{\alpha\beta}f_{\beta\alpha}-\epsilon^\gamma D_\gamma)\psi(x)\,, \tag{4.26}$$

where $D_\gamma:=e^i{}_\gamma D_i$. The symbol $\tilde{\Pi}$ is used in order to stress that the *group of rotations and parallel translations* in Eq. (4.26) has, in the mathematical sense, a more general group structure than the original Poincaré group. The commutation relations of this group are given by Eq. (4.8) and, instead of Eqs. (4.9), (4.10), by

$$[f_{\alpha\beta},D_\gamma]=g_{\gamma[\alpha}D_{\beta]}\,, \tag{4.27}$$

$$[D_\alpha,D_\beta]=e^i{}_\alpha e^j{}_\beta(F_{ij}^{\cdot\cdot\gamma\delta}f_{\delta\gamma}-F_{ij}^{\cdot\cdot\gamma}D_\gamma)\,, \tag{4.28}$$

where the quantities

$$F_{ij}^{\cdot\cdot\gamma\delta}:=2(\partial_{[i}\Gamma_{j]}{}^{\gamma\delta}+\Gamma_{[i}{}^{\alpha\gamma}\Gamma_{j]}{}^{\beta\delta}g_{\alpha\beta})=-F_{ij}^{\cdot\cdot\delta\gamma}\,, \tag{4.29}$$

$$F_{ij}^{\cdot\cdot\gamma}:=2(\partial_{[i}e_{j]}{}^\gamma+\Gamma_{[i}{}^{\alpha\gamma}e_{j]}{}^\beta g_{\alpha\beta})=2D_{[i}e_{j]}{}^\gamma\,, \tag{4.30}$$

are the rotational and translational gauge fields, respectively (compare Trautman, 1975). Equation (4.28) shows us that the parallel translations no longer form an Abelian group as soon as the gauge fields appear. Such nonlocal behavior distinguishes parallel translations from other symmetry operations.

Equation (4.22) is the limiting case of the rigidity condition

$$\tilde{\Pi}(\xi^iD_i\psi)=(\tilde{\Pi}\xi^\alpha)(e^i{}_\alpha+\delta e^i{}_\alpha)(D_i+\delta\Gamma_i{}^{\beta\gamma}f_{\gamma\beta})(\tilde{\Pi}\psi)\,. \tag{4.31}$$

Together with Eqs. (4.23), (4.26), (4.29), and (4.30), Eq. (4.31) yields the transformation properties of the gauge potentials under $\tilde{\Pi}$

$$\delta\Gamma_i{}^{\alpha\beta}=-D_i\omega^{\alpha\beta}-\epsilon^\gamma F_{\gamma i}^{\cdot\cdot\alpha\beta}\,, \tag{4.32}$$

$$\delta e_i{}^\alpha=\omega_\gamma^{\cdot\alpha}e_i{}^\gamma-D_i\epsilon^\alpha-\epsilon^\gamma F_{\gamma i}^{\cdot\cdot\alpha}\,, \tag{4.33}$$

which should be compared to (4.18) and (4.19). It is understood that Latin and Greek indices can be interchanged by transvection with $e^j{}_\gamma$ or $e_j{}^\gamma$, e.g., $F_{\gamma i}^{\cdot\cdot\alpha}=F_{ji}^{\cdot\cdot\alpha}e^j{}_\gamma$.

Finally, we point out how a passive interpretation of these gauge transformations can be given. With help from Eqs. (4.29) and (4.30), the transformations for the potentials take the following form:

$$\delta\Gamma_i{}^{\alpha\beta}=\delta_{\rm hom}\Gamma_i{}^{\alpha\beta}-\partial_i(\omega^{\alpha\beta}-\epsilon^j\Gamma_j{}^{\alpha\beta})-(\partial_i\epsilon^j)\Gamma_j{}^{\alpha\beta}\,, \tag{4.34}$$

$$\delta e_i{}^\alpha=\delta_{\rm hom}e_i{}^\alpha-(\partial_i\epsilon^j)e_j{}^\alpha\,. \tag{4.35}$$

The homogeneous variation $\delta_{\rm hom}:=(\omega\cdot f-\epsilon\cdot D)$ yields tensorial relations under $\tilde{\Pi}$ according to Eq. (4.26). The additional terms are the real compensating inhomogeneities. Equations (4.35), (4.34), and (4.26) may now be interpreted as a transformation of the tetrad system and coordinates during which the matter field is held fixed. Equation (4.35) is then interpreted as a parallel transport of the tetrads by an amount $-\epsilon(x)$, followed by a rotation $+\omega(x)$ as well as a transformation of the coordinates from x^i to $'x^i=x^i+\epsilon^i$.

4. Conservation theorems revisited

So far we have made no use of the definition (4.7a). This is in fact not necessary since Eq. (4.33) now guarantees invariance under transformations with $\omega \neq 0$, where ϵ vanishes, that is, under local rotations without induced translations. This was not permitted in special relativity, for according to Eq. (4.16b), a separate conservation for orbital angular momentum would have been implied. From now on, $\omega(x)$ and $\epsilon(x)$ may be regarded as completely independent.

In analogy with Eqs. (4.12), we now require the invariance of the action $W = \int \mathfrak{L} d^4x$ under the local gauge transformation $\tilde{\Pi}(x)$. Then Eqs. (4.26), (4.32), and (4.33) lead to the following identities as the coefficients of the independent quantities $D_i\omega^{\alpha\beta}$, $D_i\epsilon^\alpha$, $\omega^{\alpha\beta}$, and ϵ^α

$$e\tau_{\alpha\beta}^{\cdot\cdot i} := \frac{\partial\mathfrak{L}}{\partial\Gamma_i{}^{\alpha\beta}} \equiv -\frac{\partial\mathfrak{L}}{\partial(\partial_i\psi)} f_{\alpha\beta}\psi\,, \tag{4.36}$$

$$e\Sigma_\alpha^{\cdot i} := \frac{\partial\mathfrak{L}}{\partial e_i{}^\alpha} \equiv e^i{}_\alpha\mathfrak{L} - \frac{\partial\mathfrak{L}}{\partial(\partial_i\psi)} D_\alpha\psi\,, \tag{4.37}$$

$$D_i(e\tau_{\alpha\beta}^{\cdot\cdot i}) - \Sigma_{[\alpha\beta]} \equiv 0\,, \tag{4.38}$$

$$D_i(e\Sigma_\alpha^{\cdot i}) \equiv F_{\alpha i}^{\cdot\cdot\beta\gamma} e\tau_{\beta\gamma}^{\cdot\cdot i} + F_{\alpha i}^{\cdot\cdot\beta} e\Sigma_\beta^{\cdot i}\,. \tag{4.39}$$

As we had hoped, the dynamically defined currents τ and Σ agree with the special relativistic quantities as coupled according to Eqs. (4.24), (4.25). The angular momentum conservation law (4.38) is essentially the same as occurs in special relativity (4.14a), in that the gauge fields exert no torques on the matter distribution. The conservation law of energy-momentum (4.14b) is altered, for (4.39) implies that both gauge fields act upon the corresponding sources.

From Eqs. (4.23), (4.24), (4.25), (4.36), and (4.37) we obtain the explicit form of the material Lagrangian density

$$L(\psi, \delta^i_\alpha\partial_i\psi; g_{\alpha\beta}, \gamma^\alpha) \to \mathfrak{L}(\psi, \partial\psi, \Gamma, e) = eL(\psi, e^i{}_\alpha D_i\psi; g_{\alpha\beta}, \gamma^\alpha)\,. \tag{4.40}$$

It can be shown that by means of finite gauge transformations the potentials at a fixed point X can be *locally* transformed to the form $\Gamma_i{}^{\alpha\beta}(X) \triangleq 0$ and $e_i{}^\alpha(X) \triangleq \delta^\alpha_i$, and that the Lagrangian density $\mathfrak{L}(X)$ and the currents $\tau(X)$, $\Sigma(X)$ then reduce to their special relativistic form. This property may be recognized as the realization of the equivalence principle: the properties of matter in a gravitational field cannot be distinguished locally from the properties of special relativistic matter. We would like to point out that the nonvanishing force density $(D\Sigma)(X)$ in Eq. (4.39), absent in special relativity, does not contradict the validity of the equivalence principle. From its definition, this principle can be applied only to local (first-order) quantities. Because the energy-momentum tensor Σ contains derivatives of the fields, however, the special relativistic energy momentum conservation law (4.14b) is of second order, and thus a nonlocal concept (von der Heyde, 1975b).

5. U_4 geometry recognized

The discussion of the rigidity condition in Sec. IV.B.2 suggests that the gauge potentials describe geometrical properties of space-time. A nongeometrical interpretation as special relativistic fields is problematic, since the global Minkowski geometry can be verified by measurement only when the gauge fields vanish. We can now demonstrate that the geometry which results is the Riemann–Cartan geometry described in Sec. II.

According to Eq. (4.25), $dx^i\Gamma_{i\alpha}{}^\beta \mathbf{e}_\beta$ is the relative rotation encountered by a tetrad $\mathbf{e}_\alpha$ in going from x to $x + dx$. From this we can calculate that the relative rotation of the respective coordinate bases $\mathbf{e}_j = e_j{}^\alpha\mathbf{e}_\alpha$ is $dx^i(\partial_i e_j{}^\alpha + \Gamma_{i\beta}{}^\alpha e_j{}^\beta)\mathbf{e}_\alpha = dx^i(D_i e_j{}^\alpha)e^k{}_\alpha\mathbf{e}_k$. In a holonomic coordinate system, the parallel transport is thus given by

$$\nabla_i := \partial_i + \Gamma^k_{ij}h_k^{\cdot j}\,;\quad \Gamma^k_{ij} := e^k{}_\alpha D_i e_j{}^\alpha\,, \tag{4.41}$$

where h represents the generator of coordinate transformations for tensor fields. The components of the covariant derivative of a tensor A in a coordinate basis are given with respect to its tetrad components by

$$\nabla_i A_{j\cdots}{}^{k\cdots} = e_j{}^\alpha \cdots e^k{}_\beta \cdots D_i A_{\alpha\cdots}{}^{\beta\cdots}\,. \tag{4.42}$$

The concept of parallelism with respect to the coordinate basis as defined in Eq. (4.41) is by construction locally identical with Euclidean parallelism, as is measured in a local tetrad. In a similar way, the local Euclidean angle and length measurements define the metric in a coordinate basis

$$g_{ij}(x) := e_i{}^\alpha(x)e_j{}^\beta(x)g_{\alpha\beta}\,. \tag{4.43}$$

From the antisymmetry $\Gamma_i{}^{\alpha\beta} = \Gamma_i{}^{[\alpha\beta]}$ in its definition (4.25) and from (4.42) results

$$D_i g_{\alpha\beta} = 0 \;\Rightarrow\; \nabla_i g_{jk} = 0\,, \tag{4.44}$$

and the metric postulate (2.10) which restricts the geometry (L_4, g) to a U_4 is fulfilled. Moreover, Eqs. (4.41), (4.29), and (4.30) identify the rotational gauge field as curvature and the translational gauge field as torsion in a Riemann–Cartan space, as defined in (2.4), (2.2),

$$R_{ijk}^{\cdot\cdot\cdot l} = e_k{}^\alpha e^l{}_\beta F_{ij\alpha}^{\cdot\cdot\cdot\beta}\,;\quad S_{ij}^{\cdot\cdot k} = \tfrac{1}{2}e^k{}_\alpha F_{ij}^{\cdot\cdot\alpha}\,. \tag{4.45a,b}$$

The gauge theory of the Poincaré group leads in this way to U_4 geometry.

C. General relativistic dynamics

1. Field equations

In order to complete the gauge theory, we must find a gauge-invariant Lagrangian density $\mathfrak{L}_f = \mathfrak{L}_f(\Gamma, \partial\Gamma, e, \partial e)$ for the fields Γ, e which is as simple as possible, and which in its macroscopic limit does not contradict observation. In gauge theories of internal symmetries (at least for semisimple groups), the simplest such Lagrangian is quadratic in the gauge fields F. In the case of the space-time symmetries considered here, however, it is possible to find a nontrivial *linear* Lagrangian. Indeed this is a choice which leads to Einstein's theory in the absence of spin, and so we follow Sciama and Kibble in setting

$$\mathfrak{L}_f(\mathbf{\Gamma}, \partial\mathbf{\Gamma}, e) = (e/2k)e^i{}_\alpha e^j{}_\beta F_{ji}^{\cdot\cdot\alpha\beta}\,. \tag{4.46}$$

By varying the total action $W = \int(\mathfrak{L} + \mathfrak{L}_f)d^4x$ with respect to the independent potentials Γ and e, and keeping in mind the definitions (4.36), (4.37), we obtain the field equations

$$(1/e)D_j(ee^i_{[\alpha}e^j{}_{\beta]}) = k\tau_{\alpha\beta}^{\cdot\cdot i}\,, \tag{4.47}$$

$$F_{\beta\alpha}^{\cdot\cdot i\beta} - \tfrac{1}{2}e^{i}{}_{\alpha}F_{\beta\gamma}^{\cdot\cdot\gamma\beta} = k\Sigma_{\alpha}^{\cdot i}. \tag{4.48}$$

These dynamical equations are supplemented by the identities

$$D_{[i}F_{jk]}^{\cdot\cdot\alpha\beta} \equiv 0, \tag{4.49a}$$

$$D_{[i}F_{jk]}^{\cdot\cdot\alpha} \equiv e_{[i}{}^{\beta}F_{jk]\beta}^{\cdot\cdot\alpha}. \tag{4.49b}$$

By means of the relations (4.45), (4.30) the field equations (4.47), (4.48) can be shown to coincide with the formulation (3.22), (3.21). From a purely gauge-theoretical point of view, the choice (4.46) for $\mathfrak{L}_f$ and the equations (4.47), (4.48) are somewhat problematic. We shall return to this point in Sec. VI.B.

2. Relation to the Riemannian structure

A difference between alternative formulations of the U_4 theory persists. It is characteristic of a gauge approach that it leads directly to U_4 geometry without visible reference to a Riemannian structure. For purposes of comparison of U_4 theory with Einstein's theory, keeping in mind the field equation (3.22), the torsion $S_{ij}^{\cdot\cdot k}$ suggests itself as a more convenient independent variable than $\Gamma_i{}^{\alpha\beta}$. Equation (4.30) with (4.45b) may be solved for the Riemann–Cartan connection

$$\Gamma_{i\alpha}^{\cdot\beta} = (-\Omega_{i\alpha}^{\cdot\cdot\beta} + \Omega_{\alpha\cdot i}^{\cdot\beta} - \Omega^{\beta}_{\cdot i\alpha}) - K_{ij}^{\cdot\cdot k}e_k{}^{\beta}e^{j}{}_{\alpha}, \tag{4.50}$$

where $\Omega_{ij}^{\cdot\cdot\beta} := \partial_{[i}e_{j]}{}^{\beta}$ is the *object of anholonomity* (see Sec. II.F) and $K_{ij}^{\cdot\cdot k}(g,S)$, which depends on torsion and metric, is defined in Eq. (2.11).

The connection (4.50) may then be interpreted to consist of two independent parts: the Ricci rotation coefficients $\Gamma(\Omega)$ [in parentheses in Eq. (4.50)], and the contortion tensor K. The Ricci rotation coefficients depend on the tetrad components alone and in holonomic coordinates equal the Christoffel symbol. The contortion tensor describes an additional non-Riemannian tetrad rotation. Substitution of (4.50) in the preceeding equations of this section allows a separation of Riemannian and non-Riemannian contributions.

Clearly, a variation of $\Gamma(\Omega,K)$ in Eq. (4.50), with the tetrads $e_i{}^{\alpha}$ held constant, is equivalent to a variation with respect to $K_{ij}^{\cdot\cdot k}$. The dynamically defined spin (4.36) is thus identical with the spin defined in Eq. (3.6). In a variation with respect to $e_i{}^{\alpha}$, keeping torsion fixed, the connection $\Gamma_i{}^{\alpha\beta}$ now contributes to the tetrad variation. The corresponding dynamical current is the metric energy-momentum tensor σ^{ij} defined in Eq. (3.4), where the difference $\sigma - \Sigma$ in (3.8) is generated from the variation of $\Gamma(\Omega)$.

3. The Sciama and Kibble approaches reconsidered

Finally, we wish to comment briefly upon the assumptions made by Sciama (1962) and Kibble (1961). Sciama's approach is that of a gauge theory of the homogeneous Lorentz group starting from a Riemannian background, that is from Einstein's theory formulated in terms of tetrads, with an affine connection $\Gamma(\Omega)$ given by (4.50) for the case $K=0$. Since $\Gamma(\Omega)$ carries the total inhomogeneity of $\Gamma(\Omega,K)$ with respect to local tetrad rotations, Einstein's theory is manifestly gauge invariant under the local Lorentz group. Notwithstanding, the Riemannian connection $\Gamma(\Omega)$ cannot in itself be identified as an independent gauge potential, because it is completely determined by the tetrad components.

Led by analogy between spin and electric charge, Sciama argued that the independent current τ corresponds to an independent gauge potential Γ. Replacing $\Gamma(\Omega)$ by Γ in the total Lagrangian density for general relativity then leads to the field equations (4.47), (4.48) of the U_4 theory.

Kibble starts from Minkowski space and a passive interpretation of the Poincaré transformation. Going from a global to a local transformation, the total translation ϵ is interpreted as in (4.3) as a general coordinate transformation, but an independent active rotation ω of the matter field is defined in addition

$$\psi(x) \to {}'\psi('x) := (1+\omega^{\alpha\beta}(x)f_{\beta\alpha})\psi(x); \quad x^i \to {}'x^i := x^i + \epsilon^i(x). \tag{4.51}$$

Formally, the system (4.51) agrees with the active transformation (4.6), (4.7). Still Kibble's theory seems to us to be a gauge theory of the homogeneous Lorentz group which is required to be invariant under general coordinate transformations rather than a gauge theory of the Poincaré group *per se*. The splitting into independent total translation and rotation seems well motivated only after introducing the gauge potentials $e_i{}^{\alpha}(x)$. For example, special relativity is not invariant under a transformation (4.51) with $\epsilon = 0, \omega = \text{const} \neq 0$. Moreover, the transformation (4.51), interpreted actively,

$${}'\psi(x) = (1+\omega^{\alpha\beta}f_{\beta\alpha} - \epsilon^i\partial_i)\psi(x), \tag{4.52}$$

does not describe a rotation-free translation of the matter field in curved space-time, because the rotation of the field ψ is composed of ω and the tetrad rotation $\epsilon\cdot\Gamma$. This should be compared with Eq. (4.26), where the independence of rotational and translational deformations of the material continuum, exhibited by a global Poincaré transformation, is guaranteed also for local transformations.

Formally, Kibble's transformations can be derived from those discussed here by addition of the local tetrad rotation, $\omega^{\alpha\beta} \to \omega^{\alpha\beta} + \epsilon^i\Gamma_i{}^{\alpha\beta}$, and going over to a passive interpretation of translations. For instance, Eq. (4.35) then implies ${}'e_i{}^{\alpha}('x) = e_i{}^{\alpha}(x) + \omega_{\beta}^{\cdot\alpha}e_i{}^{\beta} - (\partial_i\epsilon^j)e_j{}^{\alpha}$ which shows that in Kibble's formulation the interpretation of $e_i{}^{\alpha}$ as a gauge potential is not obvious, because $e_i{}^{\alpha}$ transforms homogeneously under (4.51). Since the theory which results is invariant under local rotations, both formulations lead to the same geometry. That is, the invariance of the U_4 theory under (4.26), and thus under local rotations and parallel translations, implies invariance under general coordinate transformations.

V. CONSEQUENCES OF U_4 THEORY

We assume throughout this section that matter fields with first-order Lagrangians are minimally coupled to the torsion, and that the field equations (3.21), (3.22), which result from choosing the scalar curvature for the field Lagrangian, hold. Theories with more general field Lagrangians will be dealt with in Sec. VI.B.

A. A new spin contact interaction

1. Torsion does not propagate

The second field equation (3.22) is an *algebraic* rather than a differential relation between torsion and spin. The consequence of this fact is clear: In the U_4 theory, *there can be no torsion of space-time outside the spinning matter distribution itself*. Torsion is inextricably bound to matter and cannot propagate through the vacuum as a torsion wave or via any interaction of nonvanishing range.

Outside the matter distribution, spin makes itself felt only by means of its influence on the metric tensor. The effect of spin on the metric is, however, of higher order in the gravitational coupling constant than the effect of mass on the metric. That both spin and mass play a role in the generation of the metric is particularly clear from the combined field equation (3.23). On the left-hand side of this equation, the same second-order differential operator as in general relativity acts on the metric tensor; only the sources on the right-hand side are redefined. This redefined source, the combined energy-momentum tensor $\tilde{\sigma}_{ij}$, contains both explicit and implicit spin contributions (Sec. V.A.2, below). Thus a change in the spin distribution τ^{ijk} implies a change in $\tilde{\sigma}^{ii}$ which via the field equations causes a change in the metric tensor that propagates through space and time. It is possible, for example, to envision the generation of ordinary gravitational waves from a time-varying spin distribution.[10]

The effect of spin on metric geometry has been studied in the linear approximation by Arkuszewski, Kopczyński, and Ponomariev (1974) [compare also von der Heyde and Hehl (1975), Sec. 9]. Using this simplest approximation (linear in both metric and torsion so that the spin contact terms are neglected) one finds that distant observers, who measure only the metric field, cannot distinguish between a (ferromagnetically) polarized source of spinning matter (which causes torsion locally) and a rotating distribution of matter with the same total angular momentum (which nowhere causes torsion) [compare also Adamowicz (1975)[11]].

2. Short-range behavior

We have already shown that the only spin-spin interaction mediated by torsion is a contact interaction. A detailed examination of this interaction begins with a splitting of the action (3.14) for the U_4 theory into Riemannian and non-Riemannian parts. We choose as independent geometrical variables the metric tensor g_{ij} and the contortion $K_{ij}^{\cdot\cdot k}$ (because spin couples to contortion).

The Lagrangian density for matter can then be split as follows:

$$\mathfrak{L}(\psi, \overset{\Gamma}{\nabla}\psi, g) = \mathfrak{L}(\psi, \overset{\{\}}{\nabla}\psi, g) + e\tau_{k}^{\cdot ji}(\psi, \overset{\Gamma}{\nabla}\psi, g)K_{ij}^{\cdot\cdot k} + eZ(\psi, \overset{\{\}}{\nabla}\psi, g, K). \tag{5.1}$$

Varying with respect to $K_{ij}^{\cdot\cdot k}$ gives us a differential equation for Z. The minimal coupling assumption, along with the field equation (3.22), here guarantees that Z, when not identically zero, is of higher order in the gravitational coupling constant than is the term $e\tau_{k}^{\cdot ji}K_{ij}^{\cdot\cdot k}$.

The curvature scalar density $\mathfrak{R}$ can also be decomposed into Riemannian and non-Riemannian parts and a divergence. Hence the total action $W(\Gamma)$ may be decomposed,

$$W(\Gamma) = W(\{\,\}) + (1/c)\int d^4x\, e[(\tau_{k}^{\circ ji} - T_{k}^{\cdot ji}/2k)K_{ij}^{\cdot\cdot k} + Z]. \tag{5.2}$$

In the lowest order in k, there appears a spin-contortion and a (modified) torsion-contortion coupling. The explicit spin square terms in the combined energy-momentum tensor $\tilde{\sigma}^{ij}$ of Eq. (3.24) originate in the torsion-contortion term (from the field part of the non-Riemannian contribution to the action). The U_4 spin corrections implicit in σ^{ij} (and thus also in $\tilde{\sigma}^{ij}$) derive from the matter part $(\tau_{k}^{\cdot ji}K_{ij}^{\;\;k} + Z)$ of the non-Riemannian contribution.

Applying the second field equation (3.22) to Eq. (5.2) for vanishing Z leads, in first order in k, to the form

$$\tau_{k}^{\cdot ji}K_{ij}^{\cdot\cdot k}/2 = k(\tau_{ij}^{\cdot\cdot k}\tau_{\cdot\cdot k}^{ji}/2 + \tau_{ij}^{\cdot\cdot k}\tau_{k}^{\cdot ij} + \tau_{ik}^{\cdot\cdot k}\tau_{\cdot\cdot l}^{il}) \tag{5.3}$$

for the non-Riemannian contribution to the Lagrangian. Thus the U_4 theory predicts, without further assumption, that in addition to all the gravitational interactions between particles that occur in general relativity[12] and which are derivable from $W(\{\,\})$ there is a new, very weak, universal spin contact interaction of gravitational origin.

From the above analysis, we can conclude that although the behavior of the U_4 theory matches that of general relativity at large distances, the behavior at short range is distinctly different. The magnitude of this difference is the subject of our next discussion.

3. Critical density and matter creation

Before analyzing the details of the spin contact interaction as it applies to matter fields, we can estimate its strength and predict the physical regimes in which we should expect significant deviations from the predictions of Einstein's theory.

First, we re-emphasize that spin, in the context of the U_4 theory, means intrinsic spin, that is the irreducible spin of elementary particles which occurs in nature in units of $\hbar/2$, and not the rotational motion of planets or

[10] We hasten to point out that such processes are only of theoretical interest. Typical values for spin associated with astronomical objects are very much smaller than typical values of orbital angular momentum. For example, a completely polarized neutron star of one solar mass with a rotational period of one second has approximately 10^{16} times as much orbital angular momentum as spin (see also Kerlick, 1973). The strength of gravitational waves produced by processes involving spin are thus of no significance for astronomical observation.

[11] We disagree with the conclusion of Adamowicz that "torsion appears as a by-product of the process of averaging of General Relativity." In our article the spin is meant to be the spin of elementary particles, therefore torsion exists already on an elementary level (see Sec. I.B).

[12] These include the spin contact interactions of conventional general relativity (at least in a quantum version thereof) reported by O'Connell (1973).

of galaxies which is sometimes called "spin" but is really reducible to an orbital motion.

Matter in the U_4 theory is treated as a continuum. The masses and spins of the elementary particles which constitute matter are smeared out into continuous functions over space-time, so that the concepts of an energy-momentum tensor and a spin angular momentum tensor are meaningful ones[13].

Consider such a continuum made up of elementary particles so that its number density n is a continuous function. The mass density is then $\rho = mn$, where m is the particle mass, and the spin density is $s = \hbar n/2$. (We assume that all the spins within a volume element are polarized; however, the argument which follows can also be extended to the case of randomly oriented spins by an averaging procedure— Sec. V.B.7.)

The combined field equation (3.23) tells us that the mass density receives corrections due to the spin contact interaction which are of order ks^2, so that we can expect spin effects to be of equal importance with mass terms whenever the number density $n = m/k\hbar^2$ or the *critical mass density*

$$\bar{\rho} = \frac{m^2}{k\hbar^2} = \frac{m}{\lambda\!\!\!^{-} l^2} \approx \begin{cases} 10^{47}\ \mathrm{g\,cm^{-3}} \text{ for electrons} \\ 10^{54}\ \mathrm{g\,cm^{-3}} \text{ for neutrons} \end{cases} \tag{5.4}$$

is achieved (Hehl and von der Heyde, 1973). Here the reduced Compton wavelength is $\lambda\!\!\!^{-} := \hbar/mc$ and the Planck length[14] is $l := (\hbar ck)^{1/2}$. The huge mass densities (5.4) may well be unphysical, especially since we have ignored nongravitational interactions. Still, even higher densities are encountered in cosmological models and in discussions of quantum gravity.

The differences between U_4 theory and Einstein's theory come about because the Planck length l has entered the dynamics of the theory in a consistent way as a sort of "cutoff parameter" for short distances. Correspondingly, the critical density $\bar{\rho}$ acts as a cutoff for densities, beyond which the behavior of the theory is markedly different.

Certainly, the size of $\bar{\rho}$ indicates that U_4 corrections are totally negligible even at nuclear densities. Spin corrections need only be considered in connection with the ultrahigh densities near the final singularity in gravitational collapse, near the big bang in cosmological models, and in the study of quantum gravitational processes.

Quantum processes like particle pair creation are characterized by the Planck length l and the critical density $\bar{\rho}$, as can be seen from the form (5.14) of the Dirac equation in a U_4. A nonlinear term of axial vector type corrects in first approximation the mass term by

$$\Delta m \approx m\lambda\!\!\!^{-} l^2 \bar{\psi}\psi\,, \tag{5.5}$$

and signals the onset of particle creation whenever Δm exceeds $2m$. Not surprisingly, this occurs when the mass density reaches the critical density $\bar{\rho}$ (Kerlick, 1975b).

B. Matter fields in a U_4

We shall now examine the consequences of minimal coupling of the torsion in a U_4 to matter fields.

1. Scalar field

Scalar matter has no spin and should neither feel nor produce torsion. That this is indeed the case is obvious from the minimal coupling prescription, for in any affine space the covariant derivative of a scalar field is identical with its partial derivative.

2. Maxwell and Yang–Mills fields

If we would try to perform the minimal coupling procedure for Maxwell's field in a U_4, we would obtain as a result the spin angular momentum tensor $\tau^{ijk} = A^{[i}F^{j]k}$ which is not $U(1)$ gauge invariant. Preserving this gauge invariance forbids us to apply the minimal coupling procedure in this case. We observe that Maxwell's equations are geometrically rather special: they can be expressed in terms of exterior derivatives which are already covariant objects on any differential manifold X_4. Of course, Maxwell's equations,

$$\partial_{[i}F_{jk]} = 0; \quad \partial_k \mathfrak{F}^{ik} = (4\pi/c)\mathfrak{J}^i; \quad [\partial_k \mathfrak{J}^k = 0]\,, \tag{5.6}$$

where $\mathfrak{F}^{ik} = e\,g^{il}g^{km}F_{lm}$, may be rewritten in terms of U_4 covariant derivatives as (compare Prasanna, 1975c)

$$\nabla_{[i}F_{jk]} - 2S_{[ij}{}^{l}F_{k]l} = 0;$$
$$\overset{*}{\nabla}_k \mathfrak{F}^{ik} + S_{kl}{}^{i}\mathfrak{F}^{kl} = (4\pi/c)\mathfrak{J}^i; \quad [\overset{*}{\nabla}_k \mathfrak{J}^k = 0], \tag{5.7}$$

but doing so offers no fundamental new insight.

Because the Maxwell Lagrangian is not minimally coupled to geometry, photons in the U_4 theory are unaffected by the presence of torsion. The causal structure of a U_4, based as it is on light signals, is determined completely by the (conformal) metric structure of that space-time.

Gauge fields which arise from local invariance with respect to a non-Abelian symmetry group (Yang–Mills fields) share with Maxwell's field this exemption from minimal coupling. They can be minimally coupled to torsion only at the cost of breaking the gauge symmetry (see also the end of Sec. IV.B.2).

3. Proca field

The Proca field (massive Maxwell field) has spin one (see Corson, 1953). Since it has a mass, the problem of gauge non-invariance of the spin that we encountered in the case of electrodynamics does not appear here. The minimal coupling procedure for this field yields the Lagrangian density[15]

$$\mathfrak{L} = -(\hbar c/2)e[\nabla_{[i}U_{j]}\nabla^{[i}U^{j]} - (mc/\hbar)^2 U_i U^i]. \tag{5.8}$$

Splitting this formula into Riemannian and non-Riemannian parts allows us to compute the canonical spin angu-

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[13]First quantized fields which can be represented by continuous wave functions are not excluded by this treatment.

[14]Our definition of l differs from the definition of MTW (1973) by the factor $(8\pi)^{1/2}$.

[15]Compare also Rochev (1974).

lar momentum tensor

$$\begin{aligned}\tau_{kji} &= (\hbar c/2)[U_k \nabla_{[j} U_{i]} - U_j \nabla_{[k} U_{i]}] \\ &= \tau_{kji}(U, \overset{\{\}}{\nabla} U) + \hbar c S_{i\,[j}^{\cdot\ \ \cdot l} U_{k]} U_l \,. \end{aligned} \tag{5.9}$$

The important new feature here is that spin depends on torsion. Thus when this tensor is substituted into the second field equation (3.22), one finds a relation (omitting indices)

$$[\partial T/\partial S - l^2 U^2] S = l^2 U \,\mathrm{curl}\, U \,. \tag{5.10}$$

The term in the brackets is an algebraic operator on the 24 components of the torsion tensor. This operator might become singular when $U^2 \approx l^{-2}$, that is at the number density $U \partial U \approx (\lambda l^2)^{-1}$ corresponding to the critical density $\bar{\rho}$. The nature of such a "torsion singularity" has not yet been investigated.[16]

4. Dirac field

The introduction of Dirac spinors into a U_4 geometry is in every way as straightforward as their introduction into general relativity, requiring only the introduction of orthonormal tetrads as anholonomic coordinates (see Sec. II.F). The minimally coupled Dirac Lagrangian in a U_4 is (see Lenoir, 1971; Datta 1971a, b; Hehl and Datta, 1971)

$$\begin{aligned}\mathfrak{L}(\Gamma) &= (\hbar c/2) e [(\nabla_\alpha \bar{\psi})\gamma^\alpha \psi - \bar{\psi}\gamma^\alpha \nabla_\alpha \psi - 2(mc/\hbar)\bar{\psi}\psi] \\ &= \mathfrak{L}(\{\}) + e\tau^{\alpha\beta\gamma} K_{\gamma\beta\alpha} \,,\end{aligned} \tag{5.11}$$

where the conventions used are those of Jauch and Röhrlich (1955). The canonical energy-momentum tensor for a Dirac field is

$$\Sigma_{\alpha\beta} = -(\hbar c/2)[(\nabla_\alpha \bar{\psi})\gamma_\beta \psi - \bar{\psi}\gamma_\beta \nabla_\alpha \psi] \,. \tag{5.12}$$

The canonical spin angular momentum tensor for a Dirac field is totally antisymmetric (dual to an axial vector) and given by

$$\tau^{\alpha\beta\gamma} = \tau^{[\alpha\beta\gamma]} = (\hbar c/4)\bar{\psi}\gamma^{[\alpha}\gamma^\beta \gamma^{\gamma]}\psi \,. \tag{5.13}$$

The Dirac equation in a U_4 ($\delta\mathfrak{L}/\delta\bar{\psi} = 0$), after using the second field equation, takes the form

$$\gamma^\alpha \overset{\{\}}{\nabla}_\alpha \psi + \tfrac{3}{8} l^2 (\bar{\psi}\gamma_5 \gamma^\alpha \psi)\gamma_5 \gamma_\alpha \psi + (mc/\hbar)\psi = 0 \,. \tag{5.14}$$

The nonlinear term, which represents a spin contact interaction (or self interaction), repulsive for aligned spin (Kerlick, 1975b), is an axial vector interaction with characteristic length l. The combined energy-momentum tensor for the Dirac field in a U_4 is

$$\tilde{\sigma}_{\alpha\beta} = \Sigma_{(\alpha\beta)}(\{\}) - \tfrac{1}{2} g_{\alpha\beta} k \tau^{\mu\nu\lambda}\tau_{\mu\nu\lambda} \,. \tag{5.15}$$

5. Neutrino field

Neutrinos in the U_4 theory obey the Dirac equation (5.14) in the limit of zero mass. They are invariant under the same duality rotation $\psi \to (1+\gamma_5)\psi$, under which they are invariant in special relativity. Solutions of the neutrino equation in a cosmological context have been studied by Kuchowicz (1974).

The spin of a neutrino (and therefore its torsion) possesses a special feature: it is a dual of a lightlike axial vector field. This means that the spin contact term in Eq. (5.14) vanishes, and that the paths of neutrinos (in the absence of other spinning matter fields) are, like photons, still null extremal curves of the metric.

Letelier (1975) has studied the problem of "ghost neutrinos" in a U_4. He finds that such "ghost" solutions (neutrino fields which generate no contribution to curvature and are therefore problematic in general relativity) do not exist in a U_4. This fact seems to argue the greater plausibility of the U_4 theory.

6. Semiclassical spin fluid

The semiclassical "spinning dust" matter distribution (see Weyssenhoff and Raabe, 1947; Weyssenhoff, 1958; Halbwachs, 1960; and also Maugin, 1974) generalizes the "perfect fluid" of general relativity to the case of nonvanishing spin. As such, it is an attempt to model on a classical level the Dirac electron. Unfortunately, there seems to be no satisfactory Lagrangian for this distribution, and therefore no unambiguous road to a minimally coupled theory. Rather, we must postulate the following *convective forms* for the energy-momentum and spin angular momentum tensors ($c = 1$):

$$\Sigma_i^{\cdot\, j} = p_i u^j + P(u_i u^j + \delta_i^j) \,, \tag{5.16}$$

$$\tau_{ij}{}^k = \tau_{ij} u^k \,; \quad \tau_{ij} = \tau_{[ij]} \,. \tag{5.17}$$

Here, p_i and u^j are the momentum density and velocity of the fluid, P is the hydrostatic pressure, and τ_{ij} is the spin density. In order to insure that the equations of motion for the particles be integrable, it is necessary to further restrict the spin by requiring $\tau_{ij}\xi^j = 0$, where the timelike vector ξ^j is usually taken to be the velocity u^j (see Frenkel, 1926) or the momentum p^j (see Tulczyjew, 1959; Dixon, 1964, 1965).

The combined energy-momentum tensor for this distribution is

$$\begin{aligned}\tilde{\sigma}^{ij} &= (\rho + P - 2ks^2)u^i u^j + (P - ks^2) g^{ij} \\ &\quad + 2(u_m u^k - \delta_m^k)\overset{\{\}}{\nabla}_k(\tau^{m(i}u^{j)}) + \tau^{(i}(2\delta_m^{j)}u^k - u^{j)}\delta_m^k)\overset{\{\}}{\nabla}_k u^m \,.\end{aligned} \tag{5.18}$$

Here, the "polarization current" is given by $\tau_i := \tau_{ik}u^k$, and the square of the spin density $s^2 := \tau_{kl}\tau^{kl}$ is positive for either restriction mentioned above on τ_{ij}.

7. Macroscopic average

For all the spinning matter fields that we have discussed, we expect that in most macroscopic situations spins are not polarized but are randomly oriented, their polarizations undergoing rapid fluctuations with time. When we want to apply the field equations in the macroscopic domain, we are obliged to perform a space-time average of the combined energy-momentum tensor $\tilde{\sigma}^{ij}$. Indeed the spins and the gradients of spin may cancel out when such an average is performed. However, many of the spin corrections embodied in $\tilde{\sigma}^{ij}$ are *quadratic*, so that they do not average out to zero. Thus we expect, even in the macroscopic limit of the U_4 theory, nonvanishing deviations from general relativity.

[16] Preliminary investigations seem to show that such a singularity fails to appear in homogeneous cosmological models of Bianchi Types I and IX.

In the application of the field equations to studies of gravitational collapse or to cosmological models, we would like to know how these quadratic corrections scale with volume. Certainly, the density of spin-squared for a completely polarized fluid scales as the inverse square of the volume. It is consistent with our view of spinning matter as a continuum to assume such a scaling for randomly oriented particles as well. Of course, such an assumption must be justified physically.

We ought to bear in mind that taking any of the matter distributions above to represent matter at high densities can at best be regarded as a very naive approximation to physical reality, since we have neglected nongravitational interactions. Keeping this in mind, it is nevertheless interesting to examine the U_4 theory for its deviations from general relativity in model cosmologies.

C. Cosmological models with spin and torsion

Cosmological models with torsion were first studied in the hope that the singularities so ubiquitous in the solutions of Einstein's equations might be averted. This hope has dimmed by now, since only under rather unrealistic circumstances (practically no shear or electromagnetic radiation, spinning matter at high densities described by a semiclassical spinning dust) can we obtain singularity-free solutions of the U_4 field equations. Nevertheless, these cosmological solutions retain their importance as an important proving ground for the U_4 theory, for here is one of but few places where we can expect matter densities high enough to cause significant deviations from general relativity.

We will not attempt a comprehensive review of all solutions here (see rather Kerlick, 1976; Tafel, 1975; and Kuchowicz 1975a, c), but will focus on several features common to all models. We will discuss singularity aversion in terms of violations of the energy condition of the Hawking–Penrose singularity theorems. Then we will look at the Friedmann-like equation for a typical model to estimate the orders of magnitude of shear, vorticity, and magnetic field effects, and check for possible consequences to observation.

1. Global considerations

For cosmological models endowed with polarized (as opposed to random) spin, the distribution of that spin strongly determines the allowable symmetries of the metric tensor, as witness the following two examples:

(a) Kopczynski (1973) first observed in an anisotropic, spatially homogeneous cosmological model with metric

$$-ds^2 = -dt^2 + \sum_{b,c=1}^{3} a_{bc}dx^b dx^c \,, \tag{5.19}$$

that the direction of spin polarization must be an eigenvector of the spatial metric a_{bc}. For nonvanishing spin in the x^3 direction, the metric reduces to

$$-ds^2 = -dt^2 + a_{11}\big((dx^1)^2 + (dx^2)^2\big) + a_{33}(dx^3)^2 \,. \tag{5.20}$$

Similar restrictions have been encountered in other spinning dust models as well as models with a Dirac field as source of torsion (Kerlick, 1975a).

(b) In a study of Misner's (1969) Mixmaster Universe, Kerlick (1975a) found that the closed topology of that universe precludes any homogeneous polarized spin distribution whatsoever.

Clearly, then, a study of the compatibility between metric and spin that is exacted by the field equations is prerequisite to a systematic study of cosmological models with torsion. Studies of these conditions for nonrotating models with spinning dust, based on the Bianchi classification of homogeneous spatial geometries, have been carried out by Tafel and by Kuchowicz in the papers cited earlier.

2. Modified singularity theorems

Having found a model whose metric and spinning matter distributions are compatible, we now ask whether that model will become singular. The theorems of Hawking, Penrose, and others show that under a wide variety of conditions on a spacelike hypersurface, and subject to a reasonable energy condition on that hypersurface, solutions of Einstein's equations must inevitably develop singularities. (For details about these theorems and their proofs, and their domains of applicability, we refer the reader to the treatise of Hawking and Ellis, 1973.)

It is easy to reutilize these theorems for the U_4 theory. One need only notice two facts: First, spinless (scalar) test particles and photons, which determine causal structure also in a U_4, are insensitive to torsion (Sec. V.B) and follow extremal curves of the metric tensor; it is incompleteness of extremal curves which defines a singularity in these theorems.[17] Second, the U_4 theory may be recast into the quasi-Einsteinian form (3.23) with the combined energy-momentum tensor $\tilde{\sigma}_{ij}$ replacing the energy-momentum tensor of Einstein's theory.

Furthermore, the kinematic variables (shear, vorticity, convergence) used in constructing singularity theorems represent deformations of the cosmological fluid seen by Fermi-propagated observers comoving with the cosmological fluid. Here, too, the Riemannian connection has the appropriate operational meaning (Kerlick, 1976).

The singularity behavior of a cosmological model with torsion can be deduced from the generalization of the Raychaudhuri (1955) equation to a U_4 (Stewart and Hájíček, 1973):

$$\dot{\kappa} := u^i \overset{\{\}}{\nabla}_i \kappa = -\overset{\{\}}{\nabla}_i \dot{u}^i + \kappa^2/3 + 2(\sigma^2 - \omega^2) + k\tilde{E} \,. \tag{5.21}$$

The vector $u^i(u_k u^k = -1)$ is the four-velocity of the cosmological fluid. The acceleration $\dot{u}^i := u^k \overset{\{\}}{\nabla}_k u^i$ of the fluid is due both to gravitational and nongravitational interaction. It vanishes for the simple spatially homogeneous models we will consider here. The shear σ and vorticity ω are defined and explained in Ellis (1971). They describe how a unit sphere of cosmological fluid shears and rotates as it propagates forward in time. The convergence κ (negative of Ellis's volume expansion θ) measures the rate at which worldlines of the cosmological fluid approach each other. Clearly, a positive definite right-hand side of Eq. (5.21) means

[17] Only singularities in the metric are considered here. For possible singularities in torsion compare Sec. V.B.3.

that κ increases without limit, and that a singularity ensues.

The quantity $\tilde{E} := (\tilde{\sigma}_{ij} - g_{ij}\tilde{\sigma}^{k}_{\cdot k}/2)u^i u^j$ in Eq. (5.21) represents the contribution of matter, and occurs in the strong energy condition of the generalized Hawking–Penrose theorem, $\tilde{E} \geq 0$. Subject to the other requirements of these theorems, positivity of $\tilde{E}$ is sufficient to ensure that a singularity will occur.

For a Weyssenhoff fluid, the ansatzes (5.16), (5.17) and the definitions of $\tilde{\sigma}_{ij}$ [Eq. (3.24)] and of $\tilde{E}$ lead to

$$\tilde{E}_{\text{dust}} = (\rho + 3P)/2 - 2ks^2 - 2\tau^{kl}\partial_{[k}u_{l]} - \overset{\{\}}{\nabla}_k \tau^k \ , \tag{5.22}$$

where $\rho := -p_i u^i$ is the mass density, the next to last term is an interaction between spin and vorticity, and the last term is the divergence of the polarization current (which we expect to be small or vanishing). When pressure, vorticity, and polarization currents all vanish, $\tilde{E}$ becomes negative whenever $4ks^2$ exceeds ρ, which is near the critical density $\bar{\rho}$. The prevention of a singularity in spinning dust models with torsion can therefore be explained by the presence of an effective negative combined energy density $\tilde{E}$ which depends on the postulated form of the canonical energy-momentum and spin angular momentum tensors (Hehl, von der Heyde, and Kerlick, 1974). It may well be argued (Kerlick, 1976) that these forms are unreasonable, particularly at high densities. Effects of shear, vorticity, and magnetic fields are examined in greater detail in the next subsection.

If the spinning matter is described by a homogeneous solution $\psi = \psi(x^0)$ of the Dirac equation (5.14), we find that

$$\tilde{E}_{\text{Dirac}} = \tfrac{1}{2} mc^2 \bar{\psi}\psi + \frac{3l^4}{8k}(\bar{\psi}\gamma_5\gamma^\mu\psi)(\bar{\psi}\gamma_5\gamma_\mu\psi) + \tfrac{3}{2}\hbar c \Omega^0 \bar{\psi}\gamma_5\gamma_0\psi - 4\sum_{a=1}^{3}\Omega^a s_a . \tag{5.23}$$

Here $s^\alpha := -\epsilon^{\alpha\beta\gamma\delta}\tau_{\beta\gamma\delta}/3!$, and $\Omega^\alpha := -\epsilon^{\alpha\beta\gamma\delta}\Omega_{\beta\gamma\delta}/3!$ is computed from the anholonomic object (2.18). The third term vanishes in Bianchi Type I and some other types, and the last term is again a spin-rotation term. For the simple case of Bianchi Type I, both remaining terms are positive definite. The spin-squared terms here exacerbate rather than alleviate the singularity problem (Kerlick, 1975b).

3. Orders of magnitude in a typical cosmology

Consider once more the spatially flat anisotropic metric (5.20), and let there be a polarized spinning dust distribution with present mass density ρ_{m0} and spin density s_0, as well as electromagnetic fields and radiation with present density ρ_{r0}, and let the present value of the shear be σ_0 and the present radius of the universe be a_0. Then the equation for the radius parameter $a := (a_{11})^{1/3}(a_{33})^{1/6}$ [compare Eq. (5.20)] has the form of an "effective potential equation"

$$3\left(\frac{\dot{a}}{a}\right)^2 + \frac{k\rho_{m0}a_0^3}{a^3} + \frac{k\rho_{r0}a_0^4}{a^4} + \frac{2\sigma_0^2 a_0^6}{a^6} - \frac{k^2 s_0^2 a_0^6}{a^6} = 0 \,, \tag{5.24}$$

where the numerators in each term are constants.

The simplest model universe in U_4 theory has no shear and no electromagnetic radiation. If we estimate the present radius $a_0 \approx 1.2\times 10^{28}$ cm and take for the mass density (of neutrons) $\rho_{m0} \approx 2\times 10^{-31}$ g cm^{-3}, and for the spin density $s_0 = \rho_{m0}(\hbar/m_N)$, then we find that this effective "potential" has a minimum at $a_{\min} \approx 1$ cm and $\rho_{\max} \approx \bar{\rho} \approx 10^{54}$ g cm^{-3}. Allowing ordinary hydrostatic pressure proportional to the density does not change this picture much.

Allowing the slightest amount of shear makes the model singular, as was pointed out by Steward and Hájíček (1973). Present observed upper limits on shear (Ellis, 1971) are of the order 10^{84} times greater than necessary to kill off spin-torsion effects.

Even for models with vanishing shear, the presence of the 3K blackbody radiation implies a present electromagnetic radiation density of $\rho_{r0} \approx 10^{-33}$ g cm^{-3}. That much radiation brings the minimum radius of the universe in Eq. (5.24) down to the order of 10^{-16} cm, or less than the neutron Compton wavelength. At such dimensions, a study of the nonquantized gravitational field loses all validity (see also Raychaudhuri, 1975).

A simple metric like (5.20) allows no vorticity ω, but we can still estimate the sizes of two additional terms,

$$-\frac{\omega_0^2 a_0^4}{a^4} \pm \frac{k\omega_0 s_0 a_0^5}{a^5} \ , \tag{5.25}$$

to be added to the left-hand side of Eq. (5.24), which would appear in rotating models. Hawking (1969) has estimated a present value $\omega_0 \approx 10^{-14}$ rad yr^{-1}. In the presence of shear or blackbody radiation, neither term can prevent a singularity within the domain of validity of this nonquantized approach.

Alas, it seems that even under the most favorable of circumstances, if torsion causes a "bounce" of the universe at all, it does so at much too early a time to have left any directly observable and easily accessible trace in the cosmic blackbody radiation or in the relative abundances of the elements. The effects of torsion are of significance only during a much earlier regime, namely the era of quantum geometrical effects and pair creation by gravitational forces.

We have mentioned (Sec. V.A.3) the possibility of pair creation via spin–spin interaction for densities greater than 10^{54} g cm^{-3}. It should be noted that this density is much less than the 10^{92} g cm^{-3} or so required for pair production via curvature (Zel'dovich, 1970). At least in the realm of matter creation, the presence of torsion makes a crucial difference.

D. Further consequences

1. Junction conditions

A clear consequence of the discussion of the critical density $\bar{\rho}$ in U_4 theory (Sec. V.A.3) is that spin-torsion corrections are of absolutely no significance for the physics of stars (even neutron stars). Nevertheless, studies of static solutions of the U_4 equations for spin fluid distributions are of theoretical interest for the light that they shed on the structure of the field equations.[18] In this regard, the most interesting consequence

[18] Compare Prasanna (1975a, b).

is the form that the junction conditions have between the Ricci–flat Riemannian geometry outside matter and the Riemann–Cartan geometry within. Arkuszewski, Kopczyński, and Ponomariev (1975) have studied this problem using Lichnerowicz's formalism of tensor-valued distributions. They have derived conditions on the spinning matter which must be satisfied to ensure regular solutions of the vacuum Einstein equations outside the matter.

The singularity theorems discussed in Sec. V.C.2 are of course valid for collapsing stars as well. Indeed, the final singularity for a collapsing sphere of Weyssenhoff fluid is avoided,[19] but only after it has passed through the event horizon. Unsolved questions here are whether a "bounce" or destruction of the event horizon occurs at some future time. Also unsolved is whether a spin fluid may provide an interior solution to match an exterior Kerr solution in general relativity.

2. Initial value problem

The decomposition of the Einstein field equation into space-plus-time form (Arnowitt, Deser, and Misner, 1962), of crucial importance to a canonical quantization of the gravitational field, has been extended to the U_4 theory by Alvarez (1974). An alternate method of generalization would rely on the combined field equation (3.23), and substitute the combined tensor $\tilde{\sigma}_{ij}$ for the energy-momentum tensor of Einstein's theory wherever it occurs. The solution of the initial value problem, at least for nonquantized matter fields, should present no additional problem when so corrected for torsion effects.

3. Equations of motion

We have already indicated that photon and spinless test particles sense no torsion. A test particle in U_4 theory, one which could sense torsion, is a particle with dynamical spin like the electron. Its equation of motion can be obtained by integrating the conservation law of energy-momentum (3.12). In so doing, we obtain directly the Mathisson–Papapetrou type equation[20] for the motion of a spinning test particle (Hehl, 1971; Trautman, 1972c).[21] Adamowicz and Trautman (1975) have studied the precession of such a test particle in a torsion background. All these considerations seem to be of only academic interest, however, since torsion only arises inside matter. There, the very notion of a spinning test particle becomes obscure (H. Gollisch, 1974, unpublished). Only neutrinos, whose spin self interaction vanishes, seem to be possible candidates for U_4 test particles.

VI. DISCUSSION AND PROSPECTS

A. Approaches to U_4 theory compared

We have presented two different but equivalent approaches to U_4 theory: the geometrical approach of Secs. II and III and the gauge approach of Sec. IV.

In Sec. II, starting from an (L_4, g), we constrained space-time to be a U_4 by the postulates (2.6) and (2.10), that is, by assuming a local Minkowski structure. In Sec. III, matter was embedded in the space-time arena U_4, where the independent geometrical variables of the action principle are metric and torsion [see Eq. (3.3)].[22]

The gauge approach of Sec. IV starts with a global Minkowski space-time and maintains it locally. As can be expected, the general relativistic kinematics is once again realized in a U_4 and described by tetrads and a tetrad connection. In any case, the local Minkowski structure of the U_4 is the cornerstone of U_4 theory.

Each approach has its advantages and disadvantages; they are in a certain way complementary. The geometrical approach may be suggestive to general relativists, since it is reminiscent of and patterned after the conventional presentation of Einstein's theory in holonomic coordinates.

The gauge approach may appeal in particular to elementary particle physicists, for it reveals those aspects which gravitation shares with other interactions. The underlying gauge idea is simple, but its application entails a great deal of detailed and refined thought. The conceptual framework required for this approach is considerably more complicated than in the geometrical approach. This can also be inferred from a look at the literature on this subject, where no concensus as to the true Poincaré gauge theory has yet been achieved.

The Riemannian structure within a U_4 is concealed in the gauge approach; but it is readily seen in the geometrical approach. This has a practical value when a comparison with Einstein's theory is desired. If the U_4 theory were indeed the correct one, however, this would be irrelevant.

The gauge approach deals with well-defined special relativistic notions like matter fields, currents, lengths, and angles. In the geometrical approach the corresponding notions have to be identified afterwards in order to link up with special relativistic experience. For example, the definition (3.8) of the asymmetric energy-momentum tensor appears to be *ad hoc*. Only later is it justified in Eq. (3.10) by identification with the canonical tensor from special relativity.

As we can see, each of both approaches has its merits. Since the theory which results is the same, we can always switch from one formalism to the other one when it is convenient.

Trautman (1972 a, b, c; 1973 a, b) has provided an effectively equivalent approach to U_4 theory. His beautiful formalism is developed in an (L_4, g) space-time. He carries out a variational procedure with metric and connection as independent geometrical variables. Afterwards the metric postulate (2.10) is assumed. Trautman then proves that the U_4 field equations result when the matter Lagrangian satisfies certain constraints. If

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[19] Arkuszewski, Kopczyński, and Ponomariev (1975) claim that a sphere of Weyssenhoff fluid collapses to a singularity so long as the effective energy density $\tilde{E}$ remains positive. But in their example, this quantity indeed becomes negative and prevents the singularity.

[20] Mathisson (1937), Papapetrou (1951); see also Bailey and Israel (1975).

[21] See also Liebscher (1973).

[22] Sandberg (1975) has criticized torsion theories because of the metric postulate (2.10). The possible inconsistencies that he suspects are not present in our work he cites or in this article. His alternative theory, based on the (physically unmotivated) requirement of projective invariance for the matter Lagrangian, does not seem convincing.

these constraints are not satisfied, it is still possible to redefine the Lagrangian suitably and to remedy this defect, as shown by Kopczyński (1973b, 1975) and Trautman (1975). Naturally, Trautman must also assume, for physical reasons, the metric postulate (2.10), and hence a U_4 space-time.

B. More general field Lagrangians?

The U_4 theory is a dualistic theory in which the concepts of matter and geometry remain segregated. Correspondingly, in both approaches the construction of the theory is executed in two independent steps.

The first step, based on a simple postulate (the metric postulate in Sec. II, local Poincaré invariance in Sec. IV), leads to the kinematics of the theory, that is, to the embedding of the matter fields in a Riemann–Cartan space-time. Since these postulates are based directly on special relativity, they are valid wherever the local Minkowski structure of space-time is experimentally verifiable. Thus the U_4 kinematics for distances greater than typical nuclear distances must be regarded as an unalterable part of the theory. (A gravitational theory which may be useful at shorter distances is suggested in Sec. VI. C.)

In the second step, the dynamics of the U_4 is established by means of the choice of the scalar curvature $\mathfrak{R}$ for the field Lagrangian. Lovelock (1969) has shown in general relativity that this choice is unique up to the cosmological term in the following sense: the field equation (as Euler–Lagrange equation resulting from a second-order Lagrangian) is required to contain no higher than second derivatives of the metric; see also Rund and Lovelock (1972). In extending this theorem to a U_4, von der Heyde (1975b) has shown that besides the cosmological term, only terms quadratic in torsion could be added. Such terms do not alter the dynamics in an essential way. The extended theorem is based on the assumption that the field Lagrangian density $\mathfrak{V}$ is a polynomial in S and in the derivatives of S and g, and that, except for Einstein's gravitational constant k, no new dimensional coupling constants are admitted. Any change in the dynamics of the theory must violate at least one of the above assumptions.

There are, in fact, a few indications which may suggest a generalization of the dynamics:

(a) If one proposed to quantize the gravitational fields in a U_4 by presently available methods (see van Nieuwenhuizen, 1975, and also Deser and van Nieuwenhuizen, 1974), new problems, besides those familiar from general relativity, appear. The torsion field does not possess a conjugate momentum because the effective Lagrangian $\tilde{\mathfrak{R}}$ in Eq. (3.20) contains no derivatives of the torsion. Correspondingly, the gauge transformation (4.32), (4.33) could not be represented according to the postulates of Schwinger (1970). Furthermore, the spin-spin contact interaction (5.3), (5.14) is of vanishing range, and, in analogy to the conventional coupling in the weak interaction, nonrenormalizable.

(b) From a gauge theory point of view, choice (4.46) for the field Lagrangian seems to favor the rotational gauge field $F_{ij}{}^{\alpha\beta}$ and to shut out the translational gauge field $F_{ij}{}^{\alpha}$ (torsion). [Torsion, remember, is not contained *a priori* in $F_{ij}{}^{\alpha\beta}$ so long as $\Gamma_i{}^{\alpha\beta}$ is considered an independent variable, and has not yet been decomposed according to Eq. (4.50).] Consequently, the field equations (4.47), (4.48) appear to relate the gauge potentials to the wrong currents.

(c) In the limit of special relativity, the variation in Eq. (4.16a) of the rotational gauge potential $\Gamma_i{}^{\alpha\beta}$ (the term $\partial\omega$) acts upon the total angular momentum J. It might therefore be expected that, besides the spin, some remnant of orbital angular momentum (though certainly nothing which depends explicitly on position), say the spin of the translational gauge potential, could enter the field equations. A related remark is already found in Kibble (1961).

(d) The kinematics of the U_4 theory proclaims that spin is an independent concept, on an equal footing with energy-momentum. Nevertheless, the algebraic character of the second field equation (3.22) seems to hinder the complete emancipation of spin. By going to the combined field equation (3.23), the first field equation (3.21) is made to incorporate torsion and the concept of spin is subsumed under the concept of energy-momentum. It is unexpected to find that spin and torsion have no more profound role in physics than to serve as strange-looking source terms in an Einstein-like theory.

All of these indications point to one "solution": One could supplement the field Lagrangian density $\mathfrak{L}_f = \mathfrak{R}/2k$ by some quadratic functions of the gauge fields according to the pattern (indices are omitted)

$$\mathfrak{L}_f' = \hbar c e[\alpha R^2 + \beta S^2/l^2]. \tag{6.1}$$

The dimensionless tensors α and β of eighth and sixth rank, respectively, depend only on the metric. Note that $\hbar$ has now been introduced into $\mathfrak{L}_f'$ in order to maintain dimensional consistency.

In a (quantized) theory derived from an action principle with an extended field Lagrangian density containing $\mathfrak{L}_f'$, one would find, besides the familiar graviton, a new boson which one could call a *tordion*. The spin-spin interaction in this extended theory would result from a tordion exchange. Details of the theory depend, of course, on the choice of α and β in Eq. (6.1), but the tordions would in any case be very massive and tightly bound to matter.

Similar particles were introduced by Ivanenko (1964), but only under the assumption that torsion was the gradient of some other field. Hehl (1966, 1970) considered the addition of a quadratic term $R_{[ij]}R^{[ij]} \approx (\nabla S)^2$ to $\mathfrak{R}$. General quadratic Lagrangians in the U_4 theory were studied by Hayashi (1968) and Hayashi and Bregman (1973), but in order to attain general relativity as a limit, the curvature-squared terms were eventually dropped. Lopez (1975) suggests terms like (6.1) which are formally analogous to the Lagrangian of the Maxwell field.

To our knowledge, there exists no formulation of dynamics in a U_4 which is simple and established from physically transparent assumptions, which agrees with experiment, *and* which is also free of the "problems" *a* through *d*. Such a dynamics for the U_4 theory would be accessible to the quantization and renormalization methods developed by 't Hooft and Veltman (1972), DeWitt (1975), 't Hooft (1975), and others. As the true gauge theory for the Poincaré group, the U_4 theory may pave the way towards a solution of some unsolved problems in quantum gravity.

C. Hypermomentum

In modern continuum mechanics three-dimensional elastic polar continua are studied.[23] These continua allow, in addition to the usual concept of a (force) stress, the concept of "hyperstress"; such a stress characterizes intrinsic double forces with and without moment. Space-time can be interpreted as a four-dimensional elastic continuum. Accordingly, Hehl, Kerlick, and von der Heyde (1976a) proposed that matter, besides being endowed with an energy-momentum tensor $\Sigma_i{}^k$ (analogous to force stress in elastic continua), may also be endowed with a "hypermomentum" tensor Δ^{ijk} (analogous to the hyperstress mentioned above).

The antisymmetric part $\tau^{ijk} := \Delta^{[ij]k}$ of this tensor is already familiar as the spin angular momentum of matter. The trace of this tensor $\Delta^k := \Delta_l^{\cdot\, lk}$ can be identified with the intrinsic part of the *dilation current*, which is important in the high-energy "scaling limit" of elementary particle physics. The traceless proper hypermomentum $\overline{\Delta}^{ijk} := \Delta^{(ij)k} - g^{ij}\Delta^k/4$ is something new.

The space-time description appropriate to a matter field bearing both momentum and hypermomentum is the linearly connected manifold with metric (L_4, g) whose connection is given by Eq. (2.8) (Hehl, Kerlick, and von der Heyde, 1976b). The nonmetricity tensor (2.7) can be split into a trace Q_i and a traceless part $\overline{Q}$,

$$Q_{ijk} = Q_i g_{jk} + \overline{Q}_{ijk}; \quad Q_i := \tfrac{1}{4} Q_{i\,\cdot\, l}^{\cdot\, l}. \tag{6.2}$$

An (L_4, g) with vanishing $\overline{Q}$ is called a Y_4 or a Weyl space with torsion. The connection in a Y_4 preserves the light cone under parallel transport. In such a space-time, we find that the dilatation current Δ^k can be dynamically related to the Weyl vector Q_k. A Y_4 theory ought to be the local gauge theory for the Weyl group (Poincaré group plus dilatation) and should be meaningful in the "scaling limit" mentioned above. Results of this type, some of them for torsionless connections, are scattered throughout the literature.[24]

If additionally a nonvanishing $\overline{Q}_{ijk}$ is admitted, it would couple to (and dynamically define) the traceless proper hypermomentum $\overline{\Delta}^{ijk}$. The currents Δ^k and $\overline{\Delta}^{ijk}$ would then serve as new sources of the gravitational interaction (Hehl, Kerlick, and von der Heyde, 1976c).

D. Further speculations

The geometrical framework of U_4 theory could be also extended in another direction. If space-time turned out to be *locally* anisotropic, we would have to consider for its description a *Finsler* geometry with the line element $ds = f(x, dx)$, homogeneous of first degree in dx. In such a space-time, the concept of length still survives. Most interesting among such attempts is the work of Bogoslowskii (1974).

The U_4 theory comprises both the conventional gravitational interaction and the contact interaction mediated by spin and torsion. Because of the current-current nature of this interaction, we could try to understand it as a prototype of the weak interaction of elementary particle physics.[25]

Only interactions with a high degree of universality like gravitation (which acts between all particles) and the weak interaction (which acts between all fermions directly and between all massive bosons indirectly) have a chance of being "geometrized" in the sense of the general relativity theory of 1915. Furthermore, strong and electromagnetic interaction (when we take $\hbar = c = 1$) both have dimensionless coupling constants, whereas weak interaction and gravitation have coupling constants with the dimension of a length, the weak interaction length[26] $l_F = (G_F/\hbar c)^{1/2} \approx 10^{-16}$ cm, and the Planck length $l = (k\hbar c)^{1/2} \approx 10^{-32}$ cm. Is there a chance to link up both coupling constants after quantizing the matter field in a U_4 framework? This would be a necessary prerequisite for a possible relation between *weak interaction* and *torsion* (and the intermediate boson and the tordion), which should be investigated.

The dual resonance models in elementary particle physics have been quite successful in describing hadronic phenomena. "The dual theory... is the spinning particle theory par excellence" (D. Olive in Jacob, 1974). In certain dual model theories, Yoneya (1974), Scherk and Schwarz (1974a), and others have tried to obtain in a suitable limit massless spin-two quanta interpreted as gravitons. They found Einsteinian structures and later also additional torsion-like contributions (Scherk and Schwarz, 1974b), which, however, only formally resemble the torsion discussed in this work. In a dual theory, one introduces Poincaré invariance at the beginning. If a dual theory is a correct model of nature, then the gravitational theory that comes out should also carry a torsion structure.

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[23]See, for example, Eringen (1962), Jaunzemis (1967), Kröner (1964, 1968), Truesdell and Noll (1965), and Truesdell and Toupin (1960). See also Maugin and Eringen (1972).

[24]See, for example, Agnese and Calvini (1975), Bregman (1973), Charap and Tait (1974), Freund (1974), Kasuya (1975), Lord (1972), Omote (1971), Utiyama (1973), and references given there.

[25]The spin contact interaction of U_4 theory, in analogy with weak interaction, presumably should become strong at sufficiently high energies. Thus it should be worth keeping in any case.

[26]Fermi constant $G_F \approx 10^{-49}$ erg cm^3.

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APPENDIX: VARIATION OF THE CURVATURE SCALAR DENSITY

Start with a manifold (L_4, g) with contravariant metric tensor density $\mathfrak{g}^{ij} := e g^{ij}$ and curvature scalar density $\mathfrak{R} := \mathfrak{g}^{ij} R_{ij}$. The variation of $\mathfrak{R}$ is given by

$$\delta\mathfrak{R} = \delta\tilde{\mathfrak{R}} = R_{ij}\,\delta\mathfrak{g}^{ij} + \mathfrak{g}^{ij}\,\delta R_{ij}. \tag{A1}$$

Using Eq. (2.4), we get the generalized Palatini formula

$$\delta R_{ij} = 2\nabla_{[k}\,\delta\Gamma^{k}_{i]j} + 2S_{ki}^{\cdot\cdot\,l}\,\delta\Gamma^{k}_{lj}. \tag{A2}$$

Furthermore

$$R_{ij}\,\delta\mathfrak{g}^{ij} = -eG^{ij}\,\delta g_{ij}. \tag{A3}$$

Substitute Eqs. (A3) and (A2) into Eq. (A1) and get, apart from an uninteresting divergence,

$$(1/e)\delta\mathfrak{R} = -G^{ij}\,\delta g_{ij} + 2P_{k}^{\cdot\,ji}\,\delta\Gamma^{k}_{ij}, \tag{A4}$$

with

$$P_{k}^{\cdot\,ji} := g^{jl}\,T_{kl}^{\cdot\cdot\,i} + \delta^{i}_{[k}\left(Q_{l]}^{\cdot\,jl} - 2Q_{l]}g^{jl}\right); \quad Q_l := Q_{lk}^{\cdot\cdot\,k}/4.$$

In a U_4, Eq. (A4) reduces to

$$(1/e)\delta\mathfrak{R} = -G^{ij}\,\delta g_{ij} + 2T_{k}^{\cdot\,ji}\,\delta\Gamma^{k}_{ij}. \tag{A5}$$

The variation of the connection $\delta\Gamma$ in (A5) can be expressed in terms of δg and δS (or δK) via Eq. (2.11). Using the indentity (2.14), we finally have

$$(1/e)\delta\mathfrak{R}/\delta g_{ij} = -G_{ij} + \overset{*}{\nabla}_k\left(T^{ijk} - T^{jki} + T^{kij}\right), \tag{A6}$$

$$(1/e)\delta\mathfrak{R}/\delta S_{ij}^{\cdot\cdot\,k} = -2\left(T_{k}^{\cdot\,ij} - T^{ij}_{\cdot\cdot\,k} + T^{j\,\cdot\,i}_{\cdot\,k}\right). \tag{A7}$$

From Eq. (A7) we can derive the equivalent relation

$$(1/e)\delta\mathfrak{R}/\delta K_{ij}^{\cdot\cdot\,k} = -2T_{k}^{\cdot\,ji}. \tag{A8}$$

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