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Asymptotic behaviors of the heat kernel in covariant perturbation theory

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The trace of the heat kernel is expanded in a basis of nonlocal curvature invariants of nth order. The coefficients of this expansion (the nonlocal form factors) are calculated to third order in the curvature inclusive. The early-time and late-time asymptotic behaviors of the trace of the heat kernel are presented with this accuracy. The late-time behavior gives the criterion of analyticity of the effective action in quantum field theory. The latter point is exemplified by deriving the effective action in two dimensions.

I. INTRODUCTION

Heat kernel is a universal tool in theoretical and mathematical physics. One can point out, in particular, its applications to quantum theory of gauge fields, quantum gravity,[1–8] theory of strings,[9] and mathematical theory of differential operators on nontrivial manifolds.[10–18] The significance of this object follows from the fact that the vast scope of problems boils down to the analysis of the operator quantity

$$K(s)=\exp sH \tag{1.1}$$

which is associated with some differential operator H and has a kernel

$$K(s|x,y)\equiv\exp sH\delta(x,y) \tag{1.2}$$

solving the heat equation

$$\frac{\partial}{\partial s}K(s|x,y)=H\,K(s|x,y),\quad K(s|x,y)\,|_{s=0}=\delta\,(x,y). \tag{1.3}$$

Here we shall focus on quantum field theory, in which case H coincides with the Hessian of the classical action (times a local matrix[6]). In quantum field theory, the heat kernel (1.1) governs the semiclassical loop expansion to all orders, and in the covariant diagrammatic technique[2,3,6–8,19–22] it becomes indispensable. In particular, it generates the main ingredient of this expansion—the propagator of the theory,

$$\frac{1}{H}=-\int_0^\infty ds\,K(s), \tag{1.4}$$

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and at one-loop order leads to the effective action in terms of the functional trace of the heat kernel Tr $K(s)$

$$-W=\frac{1}{2}\int_0^\infty \frac{ds}{s}\,\mathrm{Tr}\,K(s), \tag{1.5}$$

$$\mathrm{Tr}\,K(s)\equiv\int d^{2\omega}x\,\mathrm{tr}\,K(s|x,y)\big|_{y=x}. \tag{1.6}$$

Here tr, as distinct from Tr in (1.5), denotes the matrix trace with respect to the discrete indices of $K(s|x,y)$ (which arise for any matrix-valued operator H, corresponding to the fields of nonzero spin) and 2ω is the space–time dimension.

Apart from special backgrounds (see, e.g., Ref. 16), the heat kernel cannot be calculated exactly, and for the effective action in quantum field theory, one needs it as a *functional* of background fields.[5,8] For an approximate calculation, two regular schemes[23] have thus far been used: the technique of asymptotic expansion at early time[3,10–13,24] ($s\rightarrow 0$) and covariant perturbation theory.[19–22]

As seen from (1.5), the asymptotic expansion of (1.6) in $s\rightarrow 0$ allows one to single out the ultraviolet divergences generated by the lower integration limit in s. One can also obtain the local effects of vacuum polarization by massive quantum fields[25] corresponding to the asymptotic expansion of the effective action in inverse powers of large mass parameter $m^2\rightarrow\infty$. In massive theories, the integral (1.5) acquires the multiplier $e^{-m^2 s}$ exponentially suppressing large scales of s and generating the $1/m^2$–expansion. The coefficients of this expansion are the space–time integrals of local invariants $a_n(x,x)$, $n=0,1,2,\ldots$, of growing power in curvature and its derivatives, called DeWitt or HAMIDEW coefficients. The local (Schwinger–DeWitt) expansion is, therefore, an asymptotic approximation of *small and slowly varying* background fields. The term HAMIDEW, which means Hadamard–Minakshisundaram–DeWitt, has been proposed by Gibbons[26] to praise the joint efforts of mathematicians and physicists in the pioneering studies of the early-time expansion of the heat kernel.[3,10,11] These studies contained the explicit calculation of $a_n(x,x)$ for $n=0$, 1, and 2. The coefficient $a_3(x,x)$ has been worked out by Gilkey,[13] while the highest-order coefficient available now for a generic theory, $a_4(x,x)$, was obtained by Avramidi[24] (see also Ref. 27).

This expansion, very efficient for obtaining covariant renormalizations and anomalies, becomes, however, unreliable for large and/or rapidly varying fields and completely fails in massless theories, because, in the absence of a damping factor $e^{-m^2 s}$, the early-time expansion of Tr $K(s)$ in (1.5) is nonintegrable at the upper limit $s\rightarrow\infty$. The calculational technique which solves this problem in the case of rapidly varying fields is covariant perturbation theory.[19–22] It corresponds to a partial summation of the Schwinger–DeWitt series by summing all terms with a given power of the curvature and any number of derivatives.[28] This is still an expansion of Tr $K(s)$ in powers of the curvature but the coefficients of this expansion are nonlocal. In contrast to the Schwinger–DeWitt expansion, in this technique the convergence of the integral (1.5) at $s\rightarrow\infty$ for massless theories is controlled by the late-time behavior of these nonlocal coefficients, which altogether comprise the late-time behavior of the heat kernel.

Covariant perturbation theory is already capable of reproducing the effects of nonlocal vacuum polarization and particle creation by rapidly varying fields. It should, therefore, contain the Hawking radiation effect[29] and its backreaction on the metric in the gravitational collapse problem.[30–32] This was in fact the original motivation for studying this theory. This motivation has recently been strengthened in the work[33] where it is shown that loop expansion of field theory can be trusted in the spacetime domain near null infinity where the massless vacuum particles are radiated.

Covariant perturbation theory was proposed in Ref. 19 and then applied for the calculation of the heat kernel and effective action to second order in the curvature.[20] However, as pointed out in papers 32, 8, and 33, one has to go as far as third order in the curvature (in the action) to obtain the component of radiation that remains stable after the black hole is formed. The third-order calculation has recently been completed[22] for both the heat kernel and one-loop effective action. The basis of third-order curvature invariants was built,[22,34] and all nonlocal coefficients of these invariants both in the heat kernel and effective action were calculated as functions of three commuting operator arguments. Several integral representations for these functions were obtained. The results for the effective action checked by deriving the trace anomaly in two and four dimensions. Here we focus on the results for the heat kernel.

The structure of nonlocal coefficients in the heat kernel is discussed below but the full results for these coefficients[22] are usable only in the format of the computer algebra program *Mathematica*.[35] Therefore, they will not be presented here. Below we present only the asymptotic behaviors of the heat kernel at early and late times, which are most important, in view of the discussion above, for the theory of massless quantum fields, and for the spectral analysis on Riemann manifolds.

For a technical discussion of covariant perturbation theory and relevant physical problems we refer the reader to papers[19–21] and the recent paper[22,34] where this theory along with some of its applications is reviewed. Although we consider only the trace of the heat kernel, the knowledge of the functional trace (1.6) is sufficient, owing to the variational method in Ref. 36, for obtaining also the coincidence limit of the heat kernel tr $K(s|x,y)|_{y=x}$ and coincidence limits of its derivatives with respect to one of the space–time arguments. Finally, by using the method in Refs. 15 and 37, covariant perturbation theory can be extended to the calculation of heat kernel with separated points—the object very important for multiloop diagrams[7] and for field theory at finite temperature.[38] To second order in the curvature, this object was calculated in Ref. 39.

Like the Schwinger–DeWitt technique, covariant perturbation theory fails in the case of large fields. Apart from some special cases,[40] the problem of large fields remains unsolved but one may think of an approximation scheme complimentary to covariant perturbation theory, where one starts with large and slowly varying fields. The lowest-order approximation of such a scheme would be the case of covariantly constant background fields. This approximation has recently been considered (for the vanishing Riemann curvature, however) in Ref. 41 with a result generalizing the old Schwinger's result[2] to the non-Abelian vector fields. In a rather nontrivial way, the result in Ref. 41 can also be extended to the case of nonvanishing, covariantly constant, Riemann curvature.[42]

The plan of the paper is as follows. In Sec. II we introduce the notation, and review the general setting of covariant perturbation theory for the heat kernel in the case of a generic second-order Hamiltonian. Next we comment on the full results obtainede for the trace of the heat kernel to third order in the curvature. Section III presents the early-time asymptotic behavior of the trace of the heat kernel as obtained from these results. We carry out a comparison with the Schwinger–DeWitt series and, as a by-product, obtain a workable expression for the cubic terms of the HAMIDEW coefficient a_4. Finally, Sec. IV contains the main result of the present paper: the late-time asymptotic behavior of the trace of the heat kernel to third order in the curvature. As mentioned above, the capability of covariant perturbation theory of producing this behavior is the principal advantage of the method.

II. COVARIANT PERTURBATION THEORY FOR THE TRACE OF THE HEAT KERNEL

The subject of calculation in covariant perturbation theory of[19–22] is the heat kernel (1.1) where H is the generic second-order operator

$$H=g^{\mu\nu}\nabla_\mu\nabla_\nu\hat{1}+(\hat{P}-\tfrac{1}{6}R\hat{1}),\quad g^{\mu\nu}\nabla_\mu\nabla_\nu\equiv\Box \tag{2.1}$$

acting in a linear space of fields $\varphi^A(x)$. Here A stands for any set of discrete indices, and the hat indicates that the quantity is a matrix acting on a vector φ^A: $\hat{1}=\delta^A{}_B$, $\hat{P}=P^A{}_B$, etc. We have $\mathrm{tr}\ \hat{1}=\delta^A{}_A$, $\mathrm{tr}\ \hat{P}=P^A{}_A$, etc. In (2.1), $g_{\mu\nu}$ is a positive-definite metric characterized by its Riemann curvature[43] $R^{\alpha\beta\mu\nu}$, ∇_μ is a covariant derivative (with respect to an arbitrary connection) characterized by its commutator curvature

$$(\nabla_\mu\nabla_\nu-\nabla_\nu\nabla_\mu)\varphi^A=\mathscr{R}^A{}_{B\mu\nu}\varphi^B, \quad \mathscr{R}^A{}_{B\mu\nu}\equiv\hat{\mathscr{R}}_{\mu\nu}, \tag{2.2}$$

and $\hat{P}$ is an arbitrary matrix. The redefinition of the potential in (2.1) by inclusion of the term in the Ricci scalar R is a matter of convenience, while the absence of the term linear in ∇_μ can be always achieved by a redefinition of the connection entering the covariant derivative. The only important limitation in the operator (2.1) is the structure of its second-order derivatives which form a covariant Laplacian $\Box$. There exist, however, efficient methods[6] by which a more general problem can be reduced to the case of (2.1) (see also Ref. 17).

The present paper, like the preceding ones,[19–22] deals only with the version of covariant perturbation theory appropriate for noncompact asymptotically flat and empty manifolds. The interest in this setting follows from the fact that it results in the Euclidean effective action which, for certain quantum states, is sufficient for obtaining the scattering amplitudes and expectation-value equations of Lorentzian field theory.[19] Under the assumed conditions, the covariant Laplacian $\Box$ has a unique Green's function corresponding to zero boundary conditions at infinity. The masslessness of the operator (2.1) means that, like the Riemann and commutator curvatures, the potential $\hat{P}$ falls off at infinity. For the precise conditions of this fall off see Ref. 20.

For the set of the field strengths (curvatures)

$$R^{\alpha\beta\mu\nu}, \quad \hat{\mathscr{R}}_{\mu\nu}, \quad \hat{P} \tag{2.3}$$

characterizing the background we use the collective notation $\mathfrak{R}$. The calculations in covariant perturbation theory are carried out with accuracy $\mathrm{O}[\mathfrak{R}^N]$, i.e., up to terms of Nth and higher power in the curvatures (2.3). It is worth noting that, since the calculations are covariant, any term containing the metric is in fact of infinite power in the curvature, and $\mathrm{O}[\mathfrak{R}^N]$ means terms containing N or more curvatures *explicitly*.

The trace of the heat kernel is an invariant functional of background fields which belongs to the class of invariants considered in Ref. 34. It is, therefore, expandable in the basis of nonlocal invariants of Nth order built in Ref. 34. To third order, the expansion is of the form

$$\begin{aligned}\mathrm{Tr}\ K(s)=\frac{1}{(4\pi s)^\omega}\int dx\ g^{1/2}\ \mathrm{tr}\Bigg\{\ &\hat{1}+s\hat{P}+s^2\sum_{i=1}^{5} f_i(-s\Box_2)\ \mathfrak{R}_1\mathfrak{R}_2(i)\\ &+s^3\sum_{i=1}^{11} F_i(-s\Box_1,-s\Box_2,-s\Box_3)\ \mathfrak{R}_1\mathfrak{R}_2\mathfrak{R}_3(i)\\ &+s^4\sum_{i=12}^{25} F_i(-s\Box_1,-s\Box_2,-s\Box_3)\ \mathfrak{R}_1\mathfrak{R}_2\mathfrak{R}_3(i)\\ &+s^5\sum_{i=26}^{28} F_i(-s\Box_1,-s\Box_2,-s\Box_3)\ \mathfrak{R}_1\mathfrak{R}_2\mathfrak{R}_3(i)\\ &+s^6\ F_{29}(-s\Box_1,-s\Box_2,-s\Box_3)\ \mathfrak{R}_1\mathfrak{R}_2\mathfrak{R}_3(29)\ +\mathrm{O}[\ \mathfrak{R}^4\ ]\Bigg\}.\end{aligned} \tag{2.4}$$

The full list of quadratic $\mathfrak{R}_1\mathfrak{R}_2(i)$, $i=1$ to 5, and cubic $\mathfrak{R}_1\mathfrak{R}_2\mathfrak{R}_3(i)$, $i=1$ to 29, curvature invariants here, as well as the conventions concerning the action of box operator arguments $(\Box_1,\Box_2,\Box_3)$ on the curvatures $(\mathfrak{R}_1,\mathfrak{R}_2,\mathfrak{R}_3)$ labeled by the corresponding numbers, are presented and discussed in much detail in Ref. 34. The form factors of this expansion $f_i(\xi)$, $i=1$ to 5, and $F_i(\xi_1,\xi_2,\xi_3)$, $i=1$ to 29, all express through the basic second-order and third-order form factors

$$f(\xi)=\int_{\alpha\geqslant 0} d^2\alpha\ \delta(1-\alpha_1-\alpha_2)\exp(-\alpha_1\alpha_2\xi)=\int_0^1 d\alpha\ \mathrm{e}^{-\alpha(1-\alpha)\xi}, \tag{2.5}$$

$$F(\xi_1,\xi_2,\xi_3)=\int_{\alpha\geqslant 0} d^3\alpha\ \delta(1-\alpha_1-\alpha_2-\alpha_3)\exp(-\alpha_1\alpha_2\xi_3-\alpha_2\alpha_3\xi_1-\alpha_1\alpha_3\xi_2). \tag{2.6}$$

The structure of these expressions is as follows:

$$f_i(\xi)=r_i(\xi)\ f(\xi)+v_i(\xi), \tag{2.7}$$

$$F_i(\xi_1,\xi_2,\xi_3)=R_i(\xi_1,\xi_2,\xi_3)\ F(\xi_1,\xi_2,\xi_3)+\sum_{n=1}^{3}\ U_i^n(\xi_1,\xi_2,\xi_3)\ f(\xi_n)+V_i(\xi_1,\xi_2,\xi_3), \tag{2.8}$$

where $r_i(\xi)$, $v_i(\xi)$, $R_i(\xi_1,\xi_2,\xi_3)$, $U_i^n(\xi_1,\xi_2,\xi_3)$, and $V_i(\xi_1,\xi_2,\xi_3)$ are certain rational functions of their arguments. The functions r_i and v_i have in the denominator the powers of ξ, while the denominators of R_i, U_i^n, and V_i are formed by the powers of the universal quantity $\xi_1\xi_2\xi_3\Delta$, $\Delta\equiv\xi_1^2+\xi_2^2+\xi_3^2-2\xi_1\xi_2-2\xi_2\xi_3-2\xi_3\xi_1$, and also contain the factors $(\xi_j-\xi_k)$, $j\neq k$ (in the denominators of U_i^n). Despite this fact, all the form factors are analytic in their arguments at $\xi_j=0$, $\Delta=0$ and $\xi_j=\xi_k$, and the mechanism of maintaining this analyticity is based on linear differential equations which the functions (2.5)–(2.6) satisfy.[22] Here we present some of these equations in the form which will be used in Sec. IV for the derivation of the effective action for massless conformal field in two dimensions:

$$\frac{d}{d\xi}f(\xi)=-\frac{1}{4}f(\xi)-\frac{1}{2}\frac{f(\xi)-1}{\xi}, \tag{2.9}$$

$$\frac{d^2}{d\xi^2}f(\xi)=\frac{1}{16}f(\xi)+\frac{1}{4}\frac{f(\xi)-1}{\xi}+\frac{3}{4}\frac{f(\xi)-1+\frac{1}{6}\xi}{\xi^2}, \tag{2.10}$$

$$\begin{aligned} s\frac{\partial}{\partial s}F(-s\Box_1,-s\Box_2,-s\Box_3)=&-\left(s\frac{\Box_1\Box_2\Box_3}{D}+1\right)F(-s\Box_1,-s\Box_2,-s\Box_3)\\ &-\frac{\Box_1(\Box_3+\Box_2-\Box_1)}{2D}f(-s\Box_1)-\frac{\Box_2(\Box_3+\Box_1-\Box_2)}{2D}\\ &\times f(-s\Box_2)-\frac{\Box_3(\Box_1+\Box_2-\Box_3)}{2D}f(-s\Box_3), \end{aligned} \tag{2.11}$$

$$D\equiv\Box_1^2+\Box_2^2+\Box_3^2-2\Box_1\Box_2-2\Box_1\Box_3-2\Box_2\Box_3. \tag{2.12}$$

The explicit results for the second-order form factors are given in Ref. 20 (see also Ref. 44). The explicit results for the third-order form factors are given in Ref. 22 and take pages. They are, however, manageable in the format of the computer algebra program *Mathematica*. This program was used for a number of derivations in Ref. 22, and for obtaining the asymptotic behaviors presented below.

III. THE EARLY-TIME BEHAVIOR OF THE TRACE OF THE HEAT KERNEL AND THE SCHWINGER–DEWITT EXPANSION

The early-time behavior of the heat kernel follows from the tables of the paper in Ref. 22 and the behavior of the basic form factors (2.5) and (2.6)

$$f(-s\Box)=1+\tfrac{1}{6}s\Box+\tfrac{1}{60}s^2\Box^2+O(s^3),\quad s\to 0, \tag{3.1}$$

$$F(-s\Box_1,-s\Box_2,-s\Box_3)=\tfrac{1}{2}+\tfrac{1}{24}s(\Box_1+\Box_2+\Box_3)+O(s^2),\quad s\to 0. \tag{3.2}$$

It is striking that, in the resulting early-time expansion, the third-order form factors are nonlocal and, for some of them, the expansion starts with a negative power of s.[22] One can also see that such a behavior is inherent only in the gravitational form factors, and, moreover, the nonlocal operators $1/\Box$ in their asymptotic expressions act only on the gravitational curvatures. As discussed in Refs. 20 and 34, these features will persist at all higher orders in $\mathfrak{R}$, and the cause is that the basis set of curvatures for the heat kernel does not contain the Riemann tensor which gets excluded via the Bianchi identitites in terms of the Ricci tensor. Below we show that restoring the Riemann tensor restores the locality of the early-time expansion.

The early-time expansion for $\mathrm{Tr}\,K(s)$ is of the form[3]

$$\mathrm{Tr}\,K(s)=\frac{1}{(4\pi s)^{\omega}}\sum_{n=0}^{\infty}s^n\int dx\,g^{1/2}\,\mathrm{tr}\,\hat{a}_n(x,x), \tag{3.3}$$

where $\hat{a}_n(x,x)$ are the DeWitt coefficients with coincident arguments. All $\hat{a}_n(x,x)$ are *local* functions of the background fields entering the operator (2.1). There exist independent methods for obtaining these coefficients, and, for $n=0,1,2,3,4$, the $\hat{a}_n(x,x)$ have been calculated explicitly.[3,4,6,10,–13,24,27] A comparison with these known expressions, carried out below, provides a powerful check of the results of covariant perturbation theory.

By using the behaviors (3.1)–(3.2) and the tables of form factors of Ref. 22, one arrives at Eq. (3.3) with the following results for the (integrated) DeWitt coefficients a_0 to a_4:

$$\int dx\,g^{1/2}\,\mathrm{tr}\,\hat{a}_0(x,x)=\int dx\,g^{1/2}\,\mathrm{tr}\,\hat{1}, \tag{3.4}$$

$$\int dx\,g^{1/2}\,\mathrm{tr}\,\hat{a}_1(x,x)=\int dx\,g^{1/2}\,\mathrm{tr}\,\hat{P}, \tag{3.5}$$

$$\int dx\,g^{1/2}\,\mathrm{tr}\,\hat{a}_2(x,x)=\int dx\,g^{1/2}\,\mathrm{tr}\Bigg\{\frac{1}{2}\hat{P}_1\hat{P}_2+\frac{1}{12}\hat{\mathcal{R}}_{1\mu\nu}\hat{\mathcal{R}}_2^{\mu\nu}+\frac{1}{60}R_{1\,\mu\nu}R_2^{\mu\nu}\hat{1}-\frac{1}{180}R_1R_2\hat{1}$$
$$+\frac{\Box_3}{360\Box_1\Box_2}R_1R_2R_3\hat{1}+\frac{1}{90}\left(-\frac{2}{\Box_3}+\frac{\Box_3}{\Box_1\Box_2}\right)R^{\mu}_{1\alpha}R^{\alpha}_{2\ \beta}R^{\beta}_{3\ \mu}\hat{1}$$
$$+\frac{1}{180}\left(\frac{2}{\Box_2}-\frac{\Box_3}{\Box_1\Box_2}\right)R_1^{\mu\nu}R_{2\,\mu\nu}R_3\hat{1}$$

$$+\frac{1}{180}\left(-\frac{2}{\Box_1\Box_2}-\frac{3}{\Box_2\Box_3}\right)R_1^{\alpha\beta}\nabla_\alpha R_2\nabla_\beta R_3\hat{1}$$

$$+\frac{1}{45\Box_1\Box_2}\nabla^\mu R_1^{\nu\alpha}\nabla_\nu R_{2\ \mu\alpha}R_3\hat{1}+\frac{1}{45\Box_2\Box_3}R_1^{\mu\nu}\nabla_\mu R_2^{\alpha\beta}\nabla_\nu R_{3\ \alpha\beta}\hat{1}$$

$$+\frac{1}{45}\left(\frac{2}{\Box_1\Box_2}-\frac{1}{\Box_2\Box_3}\right)R_1^{\mu\nu}\nabla_\alpha R_{2\ \beta\mu}\nabla^\beta R^\alpha_{3\nu}\hat{1}$$

$$-\frac{1}{45\Box_1\Box_2\Box_3}\nabla_\alpha\nabla_\beta R_1^{\mu\nu}\nabla_\mu\nabla_\nu R_2^{\alpha\beta}R_3\hat{1}$$

$$-\frac{2}{45\Box_1\Box_2\Box_3}\nabla_\mu R_1^{\alpha\lambda}\nabla_\nu R^\beta_{2\lambda}\nabla_\alpha\nabla_\beta R_3^{\mu\nu}\hat{1}\Bigg\}+\mathrm{O}[\mathfrak{R}^4], \tag{3.6}$$

$$\int dx\, g^{1/2}\,\mathrm{tr}\,\hat{a}_3(x,x)=\int dx\, g^{1/2}\,\mathrm{tr}\Bigg\{\frac{\Box_2}{12}\hat{P}_1\hat{P}_2+\frac{\Box_2}{120}\hat{\mathcal{R}}_{1\mu\nu}\hat{\mathcal{R}}_2^{\mu\nu}+\frac{\Box_2}{180}\hat{P}_1R_2$$

$$+\frac{\Box_2}{840}R_{1\ \mu\nu}R_2^{\mu\nu}\hat{1}-\frac{\Box_2}{3780}R_1R_2\hat{1}+\frac{1}{6}\hat{P}_1\hat{P}_2\hat{P}_3-\frac{1}{45}\hat{\mathcal{R}}^\mu_{1\alpha}\hat{\mathcal{R}}^\alpha_{2\beta}\hat{\mathcal{R}}^\beta_{3\mu}$$

$$+\frac{1}{12}\hat{\mathcal{R}}_1^{\mu\nu}\hat{\mathcal{R}}_{2\ \mu\nu}\hat{P}_3+\frac{1}{180}\left(1+\frac{2\Box_1}{\Box_2}-\frac{4\Box_3}{\Box_2}+\frac{\Box_3^2}{\Box_1\Box_2}\right)R_1^{\mu\nu}R_{2\ \mu\nu}\hat{P}_3$$

$$+\left(\frac{1}{45}+\frac{\Box_3}{18\Box_1}\right)R_1^{\alpha\beta}\hat{\mathcal{R}}_{2\ \alpha}{}^\mu\hat{\mathcal{R}}_{3\ \beta\mu}+\frac{1}{3360}\left(-\frac{28}{27}+\frac{4\Box_1}{3\Box_3}+\frac{\Box_3^2}{\Box_1\Box_2}\right)$$

$$\times R_1R_2R_3\hat{1}+\frac{1}{840}\left(-\frac{4}{9}-\frac{2\Box_1}{\Box_3}+\frac{\Box_3^2}{\Box_1\Box_2}\right)R^\mu_{1\alpha}R^\alpha_{2\beta}R^\beta_{3\mu}\hat{1}$$

$$+\frac{1}{15120}\left(1-\frac{\Box_1}{\Box_2}+\frac{8\Box_3}{\Box_2}-\frac{7\Box_3^2}{2\Box_1\Box_2}\right)R_1^{\mu\nu}R_{2\ \mu\nu}R_3\hat{1}$$

$$+\frac{1}{45}\left(\frac{2}{\Box_2}-\frac{\Box_3}{\Box_1\Box_2}\right)\nabla^\mu R_1^{\nu\alpha}\nabla_\nu R_{2\ \mu\alpha}\hat{P}_3+\frac{1}{18\Box_1}R_{1\ \alpha\beta}\nabla_\mu\hat{\mathcal{R}}_2^{\mu\alpha}\nabla_\nu\hat{\mathcal{R}}_3^{\nu\beta}$$

$$-\frac{1}{36\Box_1}R_1^{\alpha\beta}\nabla_\alpha\hat{\mathcal{R}}_2^{\mu\nu}\nabla_\beta\hat{\mathcal{R}}_{3\ \mu\nu}+\frac{1}{9\Box_1}R_1^{\mu\nu}\nabla_\mu\nabla_\lambda\hat{\mathcal{R}}_2^{\lambda\alpha}\hat{\mathcal{R}}_{3\ \alpha\nu}$$

$$+\frac{1}{2520}\left(\frac{1}{\Box_1}-\frac{2}{\Box_2}-\frac{4\Box_1}{\Box_2\Box_3}-\frac{2\Box_3}{\Box_1\Box_2}\right)R_1^{\alpha\beta}\nabla_\alpha R_2\nabla_\beta R_3\hat{1}$$

$$+\frac{1}{1080}\left(\frac{\Box_3}{\Box_1\Box_2}-\frac{2}{7\Box_2}\right)\nabla^\mu R_1^{\nu\alpha}\nabla_\nu R_{2\ \mu\alpha}R_3\hat{1}$$

$$+\frac{1}{504}\left(\frac{\Box_1}{\Box_2\Box_3}-\frac{6}{5\Box_2}\right)R_1^{\mu\nu}\nabla_\mu R_2^{\alpha\beta}\nabla_\nu R_{3\ \alpha\beta}\hat{1}$$

$$+\frac{1}{420}\left(\frac{2\Box_3}{\Box_1\Box_2}-\frac{4}{3\Box_1}-\frac{\Box_1}{\Box_2\Box_3}\right)R_1^{\mu\nu}\nabla_\alpha R_{2\ \beta\mu}\nabla^\beta R_{3\nu}^{\alpha}\hat{1}$$

$$+\frac{1}{45\Box_1\Box_2}\nabla_\alpha\nabla_\beta R_1^{\mu\nu}\nabla_\mu\nabla_\nu R_2^{\alpha\beta}\hat{P}_3$$

$$+\frac{1}{756}\left(-\frac{1}{\Box_1\Box_2}-\frac{3}{\Box_2\Box_3}\right)\nabla_\alpha\nabla_\beta R_1^{\mu\nu}\nabla_\mu\nabla_\nu R_2^{\alpha\beta}R_3\hat{1}$$

$$+\frac{1}{315}\left(-\frac{1}{\Box_1\Box_2}-\frac{3}{\Box_2\Box_3}\right)\nabla_\mu R_1^{\alpha\lambda}\nabla_\nu R_{2\ \lambda}^{\beta}\nabla_\alpha\nabla_\beta R_3^{\mu\nu}\hat{1}$$

$$+\frac{1}{1890\Box_1\Box_2\Box_3}\nabla_\lambda\nabla_\sigma R_1^{\alpha\beta}\nabla_\alpha\nabla_\beta R_2^{\mu\nu}\nabla_\mu\nabla_\nu R_3^{\lambda\sigma}\hat{1}\Bigg\}+\mathrm{O}[\Re^4], \tag{3.7}$$

$$\int dx\, g^{1/2}\,\mathrm{tr}\,\hat{a}_4(x,x)=\int dx\, g^{1/2}\,\mathrm{tr}\left\{\frac{\Box_2^2}{120}\hat{P}_1\hat{P}_2+\frac{\Box_2^2}{1260}\hat{P}_1R_2+\frac{\Box_2^2}{1680}\hat{\mathcal{R}}_{1\mu\nu}\hat{\mathcal{R}}_2^{\mu\nu}\right.$$

$$+\frac{\Box_2^2}{15120}R_{1\ \mu\nu}R_2^{\mu\nu}\hat{1}+\frac{\Box_3}{24}\hat{P}_1\hat{P}_2\hat{P}_3-\frac{\Box_3}{630}\hat{\mathcal{R}}_{1\alpha}^{\mu}\hat{\mathcal{R}}_{2\beta}^{\alpha}\hat{\mathcal{R}}_{3\mu}^{\beta}$$

$$+\frac{\Box_1+\Box_2+2\Box_3}{180}\hat{\mathcal{R}}_1^{\mu\nu}\hat{\mathcal{R}}_{2\ \mu\nu}\hat{P}_3+\frac{2\Box_1-\Box_3}{15120}R_1R_2\hat{P}_3$$

$$+\frac{1}{1680}\left(\Box_1+\frac{\Box_1^2}{\Box_2}+\frac{2\Box_3}{3}+\frac{\Box_1\Box_3}{\Box_2}-\frac{5\Box_3^2}{\Box_2}+\frac{3\Box_3^3}{2\Box_1\Box_2}\right)R_1^{\mu\nu}R_{2\ \mu\nu}\hat{P}_3$$

$$+\frac{\Box_3}{720}\hat{P}_1\hat{P}_2R_3+\frac{1}{840}\left(\Box_1+4\Box_3+\frac{4\Box_2\Box_3}{\Box_1}+\frac{4\Box_3^2}{\Box_1}\right)R_1^{\alpha\beta}\hat{\mathcal{R}}_{2\ \alpha}^{\ \ \mu}\hat{\mathcal{R}}_{3\ \beta\mu}$$

$$+\frac{13\Box_1-2\Box_3}{30240}R_1\hat{\mathcal{R}}_2^{\mu\nu}\hat{\mathcal{R}}_{3\ \mu\nu}+\frac{1}{50400}\left(\frac{2\Box_1^2}{\Box_3}+\frac{\Box_1\Box_2}{\Box_3}-2\Box_3\right.$$

$$\left.+\frac{\Box_3^3}{\Box_1\Box_2}\right)R_1R_2R_3\hat{1}+\frac{1}{18900}\left(\frac{3\Box_3^3}{2\Box_1\Box_2}-\frac{2\Box_1^2}{\Box_3}-\frac{\Box_1\Box_2}{\Box_3}-\frac{3\Box_3}{2}\right)$$

$$\times R_{1\alpha}^{\mu}R_{2\ \beta}^{\alpha}R_{3\ \mu}^{\beta}\hat{1}+\frac{1}{151200}\left(\Box_1-\frac{\Box_1^2}{\Box_2}+6\Box_3+\frac{8\Box_1\Box_3}{\Box_2}\right.$$

$$\left.-\frac{13\Box_3^2}{\Box_2}+\frac{3\Box_3^3}{\Box_1\Box_2}\right)R_1^{\mu\nu}R_{2\ \mu\nu}R_3\hat{1}+\frac{1}{252}\hat{\mathcal{R}}_1^{\alpha\beta}\nabla^\mu\hat{\mathcal{R}}_{2\mu\alpha}\nabla^\nu\hat{\mathcal{R}}_{3\nu\beta}$$

$$+\frac{1}{60}\hat{\mathcal{R}}_1^{\mu\nu}\nabla_\mu\hat{P}_2\nabla_\nu\hat{P}_3+\frac{1}{180}\nabla_\mu\hat{\mathcal{R}}_1^{\mu\alpha}\nabla^\nu\hat{\mathcal{R}}_{2\ \nu\alpha}\hat{P}_3-\frac{1}{1890}R_1^{\mu\nu}\nabla_\mu R_2\nabla_\nu\hat{P}_3$$

$$+\frac{1}{420}\left(\frac{2}{3}+\frac{\Box_1}{\Box_2}+\frac{2\Box_3}{\Box_2}-\frac{3\Box_3^2}{2\Box_1\Box_2}\right)\nabla^\mu R_1^{\nu\alpha}\nabla_\nu R_{2\ \mu\alpha}\hat{P}_3$$

$$+\frac{1}{180}R_1^{\mu\nu}\nabla_\mu\nabla_\nu\hat{P}_2\hat{P}_3+\frac{1}{105}\left(\frac{1}{12}+\frac{\Box_3}{\Box_1}\right)R_{1\ \alpha\beta}\nabla_\mu\hat{\mathcal{R}}_2^{\mu\alpha}\nabla_\nu\hat{\mathcal{R}}_3^{\nu\beta}$$

$$+\frac{1}{210}\left(-\frac{1}{6}-\frac{\Box_3}{\Box_1}\right)R_1^{\alpha\beta}\nabla_\alpha\hat{\mathcal{R}}_2^{\mu\nu}\nabla_\beta\hat{\mathcal{R}}_{3\ \mu\nu}$$

$$-\frac{1}{7560}R_1\nabla_\alpha\hat{\mathcal{R}}_2^{\alpha\mu}\nabla^\beta\hat{\mathcal{R}}_{3\ \beta\mu}+\frac{1}{105}\left(\frac{1}{6}+\frac{\Box_2}{\Box_1}+\frac{\Box_3}{\Box_1}\right)R_1^{\mu\nu}\nabla_\mu\nabla_\lambda\hat{\mathcal{R}}_2^{\lambda\alpha}\hat{\mathcal{R}}_{3\ \alpha\nu}$$

$$+\frac{1}{25200}\left(\frac{1}{9}-\frac{3\Box_1}{\Box_2}-\frac{5\Box_1^2}{2\Box_2\Box_3}+\frac{\Box_3}{\Box_1}-\frac{\Box_3}{\Box_2}-\frac{\Box_3^2}{\Box_1\Box_2}\right)R_1^{\alpha\beta}\nabla_\alpha R_2\nabla_\beta R_3\hat{1}$$

$$+\frac{1}{37800}\left(\frac{7\Box_3}{\Box_2}-\frac{\Box_1}{\Box_2}-\frac{3\Box_3^2}{\Box_1\Box_2}\right)\nabla^\mu R_1^{\nu\alpha}\nabla_\nu R_{2\ \mu\alpha}R_3\hat{1}$$

$$+\frac{1}{12600}\left(-\frac{4}{3}-\frac{\Box_1}{\Box_2}+\frac{3\Box_1^2}{2\Box_2\Box_3}-\frac{2\Box_3}{\Box_2}\right)R_1^{\mu\nu}\nabla_\mu R_2^{\alpha\beta}\nabla_\nu R_{3\ \alpha\beta}\hat{1}$$

$$+\frac{1}{9450}\left(-3-\frac{\Box_1}{\Box_2}-\frac{3\Box_1^2}{2\Box_2\Box_3}-\frac{3\Box_3}{\Box_1}+\frac{\Box_3}{\Box_2}+\frac{3\Box_3^2}{\Box_1\Box_2}\right)R_1^{\mu\nu}\nabla_\alpha$$

$$\times R_{2\ \beta\mu}\nabla^\beta R_{3\nu}^\alpha\hat{1}+\frac{1}{420}\left(\frac{1}{\Box_2}+\frac{3\Box_3}{2\Box_1\Box_2}\right)\nabla_\alpha\nabla_\beta R_1^{\mu\nu}\nabla_\mu\nabla_\nu R_2^{\alpha\beta}\hat{P}_3$$

$$+\frac{1}{25200}\left(-\frac{20}{3\Box_2}-\frac{4}{\Box_3}-\frac{6\Box_1}{\Box_2\Box_3}+\frac{\Box_3}{\Box_1\Box_2}\right)\nabla_\alpha\nabla_\beta R_1^{\mu\nu}\nabla_\mu\nabla_\nu R_2^{\alpha\beta}R_3\hat{1}$$

$$+\frac{1}{6300}\left(-\frac{4}{\Box_2}-\frac{8}{3\Box_3}-\frac{4\Box_1}{\Box_2\Box_3}-\frac{\Box_3}{\Box_1\Box_2}\right)\nabla_\mu R_1^{\alpha\lambda}\nabla_\nu R_{2\lambda}^\beta\nabla_\alpha\nabla_\beta R_3^{\mu\nu}\hat{1}$$

$$+\frac{1}{6300\Box_1\Box_2}\nabla_\lambda\nabla_\sigma R_1^{\alpha\beta}\nabla_\alpha\nabla_\beta R_2^{\mu\nu}\nabla_\mu\nabla_\nu R_3^{\lambda\sigma}\hat{1}\Bigg\}+\mathrm{O}[\mathfrak{R}^4]. \tag{3.8}$$

The task is now to bring expressions (3.6)–(3.8) to a local form by restoring the Riemann tensor. The procedure that we use is as follows. For each a_n, we first consider a linear combination of all possible local invariants of the appropriate dimension with unknown coefficients. Next, in this combination, we exclude the Riemann tensor by the technique of Ref. 34 [see Eq. (2.2) of this reference], and equate the result to the nonlocal expression above. This gives a set of equations for the coefficients, which, in each case, has a *unique* solution. In the case of a_2, there is only one local invariant with explicit participation of the Riemann tensor: $\int dx\ g^{1/2}\ R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}$. In the case of a_3, there are seven (the integral over space–time is assumed):

$$\mathrm{tr}\ \hat{P}R^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta},\quad \mathrm{tr}\ \hat{\mathcal{R}}^{\alpha\beta}\hat{\mathcal{R}}^{\mu\nu}R_{\alpha\beta\mu\nu},\quad R^{\alpha\beta}{}_{\mu\nu}R^{\mu\nu}{}_{\sigma\rho}R^{\sigma\nu}{}_{\alpha\beta},\quad R^{\alpha\beta}{}_{\mu\nu}R^{\mu\nu}{}_{\sigma\rho}R^{\sigma\nu}{}_{\alpha\beta},$$
$$R^\alpha_\beta R_{\alpha\mu\nu\sigma}R^{\beta\mu\nu\sigma},\quad RR^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta},\quad R^{\alpha\mu}R^{\beta\nu}R_{\alpha\beta\mu\nu}, \tag{3.9}$$

and the coefficient of the sixth turns out to be zero. In the case of a_4, there are ten (counting only cubic):

$$\mathrm{tr}\,\Box\hat{P}R^{\alpha\beta\mu\nu}R_{\alpha\beta\mu\nu},\quad \mathrm{tr}\,\hat{P}\nabla_\mu\nabla_\alpha R_{\nu\beta}R^{\mu\nu\alpha\beta},\quad \mathrm{tr}\,\hat{\mathcal{R}}^{\alpha\beta}\Box\hat{\mathcal{R}}^{\mu\nu}R_{\alpha\beta\mu\nu},\quad \Box R^\alpha_\beta R_{\alpha\mu\nu\sigma}R^{\beta\mu\nu\sigma},$$
$$\Box RR^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta},\quad R_{\mu\nu}\nabla^\mu R_{\alpha\beta\sigma\rho}\nabla^\nu R^{\alpha\beta\sigma\rho},\quad R\nabla_\mu\nabla_\alpha R_{\nu\beta}R^{\mu\nu\alpha\beta},\tag{3.10}$$
$$\nabla_\mu R_{\nu\alpha}\nabla^\alpha R_{\rho\sigma}R^{\mu\rho\nu\sigma},\quad \nabla_\alpha R_{\beta\lambda}\nabla_\mu R^\lambda_\nu R^{\alpha\beta\mu\nu},\quad R^{\alpha\mu}\Box R^{\beta\nu}R_{\alpha\beta\mu\nu},$$

and the last one proves to be absent. The number of invariants with the Riemann tensor does not grow fast owing to the Bianchi identities and, particularly, their corollary which excludes $\Box R_{\alpha\beta\mu\nu}$ in a local way.

The final results are as follows. The expressions (3.4) and (3.5) are already in the local form. The expression (3.6) is brought to a local form

$$\int dx\, g^{1/2}\,\mathrm{tr}\,\hat{a}_2(x,x)=\int dx\, g^{1/2}\,\mathrm{tr}\left\{\frac{1}{2}\hat{P}\hat{P}+\frac{1}{12}\hat{\mathcal{R}}_{\mu\nu}\hat{\mathcal{R}}^{\mu\nu}+\left[\frac{1}{180}R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}-\frac{1}{180}R_{\mu\nu}R^{\mu\nu}\right]\hat{1}\right\}+\mathrm{O}[\,\Re^4\,],\tag{3.11}$$

by using the identity (6.36) of Ref. 34 which expresses for arbitrary space–time dimension 2ω the Gauss–Bonnet combination of Riemann and Ricci curvatures in terms of nonlocal invariants built of the Ricci tensor.

Finally, the expressions (3.7), (3.8) rewritten in terms of invariants (3.9) and (3.10) take the form

$$\begin{aligned}\int dx\, g^{1/2}\,\mathrm{tr}\,\hat{a}_3(x,x)=\int dx\, g^{1/2}\,\mathrm{tr}\Big\{&\frac{1}{12}\hat{P}\Box\hat{P}+\frac{1}{120}\hat{\mathcal{R}}_{\mu\nu}\Box\hat{\mathcal{R}}^{\mu\nu}+\frac{1}{180}\hat{P}\Box R+\Big[\frac{1}{840}R_{\mu\nu}\Box R^{\mu\nu}\\ &-\frac{1}{3780}R\Box R\Big]\hat{1}+\frac{1}{6}\hat{P}\hat{P}\hat{P}-\frac{1}{45}\hat{\mathcal{R}}^\mu{}_\alpha\hat{\mathcal{R}}^\alpha{}_\beta\hat{\mathcal{R}}^\beta{}_\mu+\frac{1}{12}\hat{P}\hat{\mathcal{R}}^{\alpha\beta}\hat{\mathcal{R}}_{\alpha\beta}\\ &+\frac{1}{72}R^{\mu\nu}{}_{\alpha\beta}\hat{\mathcal{R}}^{\alpha\beta}\hat{\mathcal{R}}_{\mu\nu}-\frac{1}{180}R^{\mu\nu}\hat{\mathcal{R}}^\alpha_\mu\hat{\mathcal{R}}_{\alpha\nu}+\frac{1}{180}R^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta}\hat{P}\\ &-\frac{1}{180}R^{\alpha\beta}R_{\alpha\beta}\hat{P}+\Big[-\frac{1}{1620}R^\alpha{}_\mu{}^\beta{}_\nu R^\mu{}_\sigma{}^\nu{}_\rho R^\sigma{}_\alpha{}^\rho{}_\beta\\ &+\frac{17}{45360}R^{\alpha\beta}{}_{\mu\nu}R^{\mu\nu}{}_{\sigma\rho}R^{\sigma\nu}{}_{\alpha\beta}+\frac{1}{7560}R_{\alpha\beta}R^\alpha{}_{\mu\nu\lambda}R^{\beta\mu\nu\lambda}\\ &+\frac{1}{945}R_{\alpha\beta}R^{\mu\nu}R^{\alpha\beta}_{\mu\nu}-\frac{4}{2835}R^\alpha_\beta R^\beta_\mu R^\mu_\alpha\Big]\hat{1}\Big\}+\mathrm{O}[\Re^4],\end{aligned}\tag{3.12}$$

$$\begin{aligned}\int dx\, g^{1/2}\,\mathrm{tr}\,\hat{a}_4(x,x)=\int dx\, g^{1/2}\,\mathrm{tr}\Big\{&\frac{1}{120}\hat{P}\Box^2\hat{P}+\frac{1}{1260}\hat{P}\Box^2R+\frac{1}{1680}\hat{\mathcal{R}}^{\mu\nu}\Box^2\hat{\mathcal{R}}_{\mu\nu}\\ &+\frac{1}{15120}R^{\mu\nu}\Box^2R_{\mu\nu}\hat{1}+\frac{1}{24}\Box\hat{P}\hat{P}\hat{P}-\frac{1}{630}\Box\hat{\mathcal{R}}^\mu{}_\alpha\hat{\mathcal{R}}^\alpha{}_\beta\hat{\mathcal{R}}^\beta{}_\mu\\ &+\frac{1}{252}\hat{\mathcal{R}}^{\alpha\beta}\nabla^\mu\hat{\mathcal{R}}_{\mu\alpha}\nabla^\nu\hat{\mathcal{R}}_{\nu\beta}+\frac{1}{180}\Box\hat{\mathcal{R}}^{\mu\nu}\hat{\mathcal{R}}_{\mu\nu}\hat{P}\\ &+\frac{1}{180}\hat{\mathcal{R}}^{\mu\nu}\Box\hat{\mathcal{R}}_{\mu\nu}\hat{P}+\frac{1}{90}\hat{\mathcal{R}}^{\mu\nu}\hat{\mathcal{R}}_{\mu\nu}\Box\hat{P}+\frac{1}{180}\nabla_\mu\hat{\mathcal{R}}^{\mu\alpha}\nabla^\nu\hat{\mathcal{R}}_{\nu\alpha}\hat{P}\end{aligned}$$

$$+\frac{1}{720}\hat{P}\hat{P}\Box R+\frac{1}{180}R^{\mu\nu}\nabla_\mu\nabla_\nu\hat{P}\hat{P}+\frac{1}{60}\hat{\mathcal{R}}^{\mu\nu}\nabla_\mu\hat{P}\nabla_\nu\hat{P}$$

$$-\frac{1}{1890}R^{\mu\nu}\nabla_\mu R\nabla_\nu\hat{P}-\frac{1}{15120}\Box\hat{P}RR+\frac{1}{7560}\hat{P}R\Box R$$

$$-\frac{1}{1260}\nabla^\mu R^{\nu\alpha}\nabla_\nu R_{\mu\alpha}\hat{P}-\frac{1}{840}R^{\mu\nu}\Box R_{\mu\nu}\hat{P}-\frac{1}{5040}R^{\mu\nu}R_{\mu\nu}\Box\hat{P}$$

$$+\frac{1}{1120}R^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta}\Box\hat{P}+\frac{1}{420}R^{\mu\nu\alpha\beta}\nabla_\mu\nabla_\alpha R_{\nu\beta}\hat{P}$$

$$+\frac{13}{30240}\Box R\hat{\mathcal{R}}^{\mu\nu}\hat{\mathcal{R}}_{\mu\nu}-\frac{1}{15120}R\hat{\mathcal{R}}^{\mu\nu}\Box\hat{\mathcal{R}}_{\mu\nu}$$

$$-\frac{1}{7560}R\nabla_\alpha\hat{\mathcal{R}}^{\alpha\mu}\nabla^\beta\hat{\mathcal{R}}_{\beta\mu}+\frac{1}{840}\Box R^{\alpha\beta}\hat{\mathcal{R}}_\alpha{}^\mu\hat{\mathcal{R}}_{\beta\mu}$$

$$-\frac{1}{1260}R^{\alpha\beta}\nabla_\alpha\hat{\mathcal{R}}^{\mu\nu}\nabla_\beta\hat{\mathcal{R}}_{\mu\nu}+\frac{1}{630}R^{\mu\nu}\nabla_\mu\nabla_\lambda\hat{\mathcal{R}}^{\lambda\alpha}\hat{\mathcal{R}}_{\alpha\nu}$$

$$+\frac{1}{1260}R_{\alpha\beta}\nabla_\mu\hat{\mathcal{R}}^{\mu\alpha}\nabla_\nu\hat{\mathcal{R}}^{\nu\beta}+\frac{1}{420}R_{\mu\nu\alpha\beta}\hat{\mathcal{R}}^{\alpha\beta}\Box\hat{\mathcal{R}}^{\mu\nu}$$

$$+\Big[\frac{1}{50400}R^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta}\Box R+\frac{1}{6300}\Box R_{\alpha\beta}R^\alpha{}_{\mu\nu\lambda}R^{\beta\mu\nu\lambda}$$

$$-\frac{1}{25200}R_{\lambda\sigma}\nabla^\lambda R^{\mu\nu\alpha\beta}\nabla^\sigma R_{\mu\nu\alpha\beta}-\frac{1}{37800}R^{\mu\nu\alpha\beta}\nabla_\mu\nabla_\alpha R_{\nu\beta}R$$

$$-\frac{1}{6300}R^{\mu\alpha\nu\beta}\nabla_\mu R_{\nu\lambda}\nabla^\lambda R_{\alpha\beta}-\frac{2}{4725}R^{\alpha\beta\mu\nu}\nabla_\alpha R_{\beta\lambda}\nabla_\mu R^\lambda_\nu$$

$$+\frac{1}{37800}R^{\mu\nu}\nabla_\alpha R_{\beta\mu}\nabla^\beta R^\alpha_\nu-\frac{1}{9450}R^{\mu\nu}\nabla_\mu R^{\alpha\beta}\nabla_\nu R_{\alpha\beta}$$

$$-\frac{1}{18900}\nabla^\mu R^{\nu\alpha}\nabla_\nu R_{\mu\alpha}R+\frac{29}{453600}R^{\alpha\beta}\nabla_\alpha R\nabla_\beta R$$

$$+\frac{1}{37800}RR^{\mu\nu}\Box R_{\mu\nu}-\frac{1}{75600}\Box RR^{\mu\nu}R_{\mu\nu}-\frac{1}{7560}\Box R^\mu_\alpha R^\alpha_\beta R^\beta_\mu$$

$$-\frac{1}{100800}\Box RRR\Big]\hat{1}\Big\}+\mathrm{O}[\mathfrak{R}^4]. \tag{3.13}$$

The expressions (3.4), (3.5), and (3.11) for a_0, a_1, and a_2 coincide with the results obtained by other methods.[3,4,6,10–13] It is easy to compare expression (3.12) for a_3 with the result in[13,24] since they differ only by the substitution[45]

$$\int dx\, g^{1/2}\, \mathrm{tr}\nabla_\alpha \hat{\mathcal{R}}^{\alpha\mu}\nabla^\beta \hat{\mathcal{R}}_{\beta\mu} = \int dx\, g^{1/2}\, \mathrm{tr}\Big(-\frac{1}{2}\hat{\mathcal{R}}_{\mu\nu}\Box\hat{\mathcal{R}}^{\mu\nu} + 2\hat{\mathcal{R}}^{\mu}{}_{\alpha}\hat{\mathcal{R}}^{\alpha}{}_{\beta}\hat{\mathcal{R}}^{\beta}{}_{\mu} + R^{\mu\nu}\hat{\mathcal{R}}^{\alpha}{}_{\mu}\hat{\mathcal{R}}_{\alpha\nu} - \frac{1}{2}R^{\mu\nu}{}_{\alpha\beta}\hat{\mathcal{R}}^{\alpha\beta}\hat{\mathcal{R}}_{\mu\nu}\Big), \quad (3.14)$$

and it is very difficult to compare expression (3.13) for a_4 with the result in Ref. 24. An algebra of the Bianchi identities, Jacobi identities for the commutator curvature (2.2) and integration by parts, which must be used for this purpose, can be found in Refs. 34 and 22. The coincidence *does* take place with accuracy $O[\mathfrak{R}^4]$ but expression (3.13) is a result of such drastic simplifications that it should be considered as *new*. It goes without saying that, although all the equations (3.11)–(3.13) are presently obtained with accuracy $O[\mathfrak{R}^4]$, the results for a_2 and a_3 are exact. It is also worth emphasizing that the further expansion of the form factors gives the terms of given quadratic and cubic orders in the curvature of *all* tr $\hat{a}_n(x,x)$.

IV. THE LATE-TIME BEHAVIOR OF THE TRACE OF THE HEAT KERNEL

As discussed above, the late-time asymptotic behavior is the most important result of covariant perturbation theory for the heat kernel, because it gives a universal criterion of the analyticity of the effective action (1.5) in the curvature for massless models and, thus, determines the range of applicability of this theory. Derivation of the late-time behavior of the form factors in the heat kernel was given in Ref. 20 to all orders in the curvature. For the basic form factors (2.5) and (2.6) this behavior is

$$f(-s\Box) = -\frac{1}{s}\frac{2}{\Box} + O\left(\frac{1}{s^2}\right), \quad s\to\infty \quad (4.1)$$

$$F(-s\Box_1,-s\Box_2,-s\Box_3) = \frac{1}{s^2}\left(\frac{1}{\Box_1\Box_2}+\frac{1}{\Box_1\Box_3}+\frac{1}{\Box_2\Box_3}\right) + O\left(\frac{1}{s^3}\right), \quad s\to\infty. \quad (4.2)$$

The late-time behaviors of all second-order and third-order form factors follow then from expressions (4.1)–(4.2) and the tables of the paper in Ref. 22. One has

$$f_i(-s\Box) = \frac{b_i}{s}\frac{1}{\Box} + O\left(\frac{1}{s^2}\right), \quad i=1 \text{ to } 5,$$

$$b_1=-1/6, \quad b_2=1/18, \quad b_3=1/3, \quad b_4=-1, \quad b_5=-1/2, \quad (4.3)$$

$$F_i(-s\Box_1,-s\Box_2,-s\Box_3) = \frac{a_i}{s^2}\left(\frac{1}{\Box_1\Box_2}+\frac{1}{\Box_1\Box_3}+\frac{1}{\Box_2\Box_3}\right) + O\left(\frac{1}{s^3}\right), \quad i=1 \text{ to } 7, 9, 10,$$

$$a_1=1/3, \quad a_2=-2/3, \quad a_3=0, \quad a_4=1/36, \quad a_5=0, \quad a_6=-1/6, \quad a_7=0,$$

$$a_9=-1/648, \quad a_{10}=0, \quad (4.4)$$

$$F_8(-s\Box_1,-s\Box_2,-s\Box_3) = \frac{1}{s^2}\left(\frac{1}{\Box_1\Box_2}+\frac{1}{\Box_1\Box_3}\right) + O\left(\frac{1}{s^3}\right), \quad (4.5)$$

$$F_{11}(-s\Box_1,-s\Box_2,-s\Box_3) = \frac{1}{s^2}\frac{1}{12}\left(\frac{1}{\Box_1\Box_2}-\frac{1}{\Box_1\Box_3}-\frac{1}{\Box_2\Box_3}\right) + O\left(\frac{1}{s^3}\right), \quad (4.6)$$

$$sF_i(-s\Box_1,-s\Box_2,-s\Box_3)=\frac{1}{s^2}\frac{a_i}{\Box_1\Box_2\Box_3}+\mathrm{O}\left(\frac{1}{s^3}\right),\quad i=12 \text{ to } 25,$$

$$a_{12}=-2,\ a_{13}=-2,\ a_{14}=-2,\ a_{15}=0,\ a_{16}=0,\ a_{17}=0,\ a_{18}=2,\ a_{19}=-1,$$

$$a_{20}=1/3,\ a_{21}=4,\ a_{22}=1/6,\ a_{23}=-1/3,\ a_{24}=-1/3,\ a_{25}=0, \tag{4.7}$$

$$s^2F_i(-s\Box_1,-s\Box_2,-s\Box_3)=\mathrm{O}\left(\frac{1}{s^3}\right),\quad i=26,\ 27,\ 28,\ 29. \tag{4.8}$$

The result is that the behavior of the trace of the heat kernel at large s is $s^{-\omega+1}$, and the coefficient of this asymptotic behavior is obtained to third order in the curvature. As shown in Ref. 20, this behavior holds at all orders in the curvature except zeroth. This power asymptotic behavior is characteristic of a noncompact manifold. For a compact manifold it will be replaced by the exponential behavior $\mathrm{Tr}\, K(s) \propto \exp(-\lambda_{\min}s)$, $s\rightarrow\infty$, where $\lambda_{\min}$ is the minimum eigenvalue of the operator $(-H)$ in (1.1). By applying the modification of covariant perturbation theory, appropriate for compact manifolds, one should be able to obtain this minimum eigenvalue as a nonlocal expansion in powers of the curvature (or a deviation of the curvature from the reference one).[46]

As seen from the expressions above, not all basis structures contribute to the leading asymptotic behavior. The asymptotic form of $\mathrm{Tr}\, K(s)$ is as follows:

$$\begin{aligned}
\mathrm{Tr}\, K(s)=\frac{s}{(4\pi s)^\omega}\int dx\, g^{1/2}\, \mathrm{tr}\Bigg\{&\hat{P}-\hat{P}\frac{1}{\Box}\hat{P}-\frac{1}{2}\hat{\mathcal{R}}_{\mu\nu}\frac{1}{\Box}\hat{\mathcal{R}}^{\mu\nu}+\frac{1}{3}\hat{P}\frac{1}{\Box}R-\frac{1}{6}R_{\mu\nu}\frac{1}{\Box}R^{\mu\nu}\hat{1}\\
&+\frac{1}{18}R\frac{1}{\Box}R\hat{1}+\hat{P}\frac{1}{\Box}\hat{P}\frac{1}{\Box}\hat{P}-2\hat{\mathcal{R}}^{\mu}{}_{\alpha}\frac{1}{\Box}\hat{\mathcal{R}}^{\alpha}{}_{\beta}\frac{1}{\Box}\hat{\mathcal{R}}^{\beta}{}_{\mu}+\frac{1}{36}\frac{1}{\Box}R\frac{1}{\Box}R\hat{P}+\frac{1}{18}R\frac{1}{\Box}R\frac{1}{\Box}\hat{P}\\
&-\frac{1}{6}\frac{1}{\Box}\hat{P}\frac{1}{\Box}\hat{P}R-\frac{1}{3}\hat{P}\frac{1}{\Box}\hat{P}\frac{1}{\Box}R+2\frac{1}{\Box}R^{\alpha\beta}\frac{1}{\Box}\hat{\mathcal{R}}_{\alpha}{}^{\mu}\hat{\mathcal{R}}_{\beta\mu}-\frac{1}{216}R\frac{1}{\Box}R\frac{1}{\Box}R\hat{1}\\
&+\frac{1}{12}\frac{1}{\Box}R^{\mu\nu}\frac{1}{\Box}R_{\mu\nu}R\hat{1}-\frac{1}{6}R^{\mu\nu}\frac{1}{\Box}R_{\mu\nu}\frac{1}{\Box}R\hat{1}-2\frac{1}{\Box}\hat{\mathcal{R}}^{\alpha\beta}\nabla^{\mu}\frac{1}{\Box}\hat{\mathcal{R}}_{\mu\alpha}\nabla^{\nu}\frac{1}{\Box}\hat{\mathcal{R}}_{\nu\beta}\\
&-2\frac{1}{\Box}\hat{\mathcal{R}}^{\mu\nu}\nabla_{\mu}\frac{1}{\Box}\hat{P}\nabla_{\nu}\frac{1}{\Box}\hat{P}-2\nabla_{\mu}\frac{1}{\Box}\hat{\mathcal{R}}^{\mu\alpha}\nabla^{\nu}\frac{1}{\Box}\hat{\mathcal{R}}_{\nu\alpha}\frac{1}{\Box}\hat{P}\\
&+2\frac{1}{\Box}R_{\alpha\beta}\nabla_{\mu}\frac{1}{\Box}\hat{\mathcal{R}}^{\mu\alpha}\nabla_{\nu}\frac{1}{\Box}\hat{\mathcal{R}}^{\nu\beta}-\frac{1}{\Box}R^{\alpha\beta}\nabla_{\alpha}\frac{1}{\Box}\hat{\mathcal{R}}^{\mu\nu}\nabla_{\beta}\frac{1}{\Box}\hat{\mathcal{R}}_{\mu\nu}\\
&+\frac{1}{3}\frac{1}{\Box}R\nabla_{\alpha}\frac{1}{\Box}\hat{\mathcal{R}}^{\alpha\mu}\nabla^{\beta}\frac{1}{\Box}\hat{\mathcal{R}}_{\beta\mu}+4\frac{1}{\Box}R^{\mu\nu}\nabla_{\mu}\nabla_{\lambda}\frac{1}{\Box}\hat{\mathcal{R}}^{\lambda\alpha}\frac{1}{\Box}\hat{\mathcal{R}}_{\alpha\nu}\\
&+\frac{1}{6}\frac{1}{\Box}R^{\alpha\beta}\nabla_{\alpha}\frac{1}{\Box}R\nabla_{\beta}\frac{1}{\Box}R\hat{1}-\frac{1}{3}\nabla^{\mu}\frac{1}{\Box}R^{\nu\alpha}\nabla_{\nu}\frac{1}{\Box}R_{\mu\alpha}\frac{1}{\Box}R\hat{1}\\
&-\frac{1}{3}\frac{1}{\Box}R^{\mu\nu}\nabla_{\mu}\frac{1}{\Box}R^{\alpha\beta}\nabla_{\nu}\frac{1}{\Box}R_{\alpha\beta}\hat{1}+\mathrm{O}[\mathfrak{R}^4]\Bigg\}+\mathrm{O}\left(\frac{1}{s^\omega}\right),\quad s\rightarrow\infty.
\end{aligned} \tag{4.9}$$

As discussed in the Introduction, the behavior (4.9) is very important for the effective action in massless theories. It controls the convergence of the integral (1.5) at the upper limit and serves as a criterion of analyticity of the effective action in the curvature. For manifolds of dimension

$$2\omega>2$$

this integral converges at the upper limit at each order in the curvature. Therefore, the effective action in four (and higher) dimensions is always analytic in the curvature whereas in two dimensions, $\omega=1$, it is generally not. An exceptional case is a conformal invariant two-dimensional scalar field with

$$\mathrm{tr}\,\hat{1}=1,\ \hat{\mathscr{R}}_{\mu\nu}=0,\ \hat{P}=R/6,\ \ R_{\mu\nu}=g_{\mu\nu}R/2,\ g^{1/2}R=\text{a total derivative},\ \omega=1, \tag{4.10}$$

for which the effective action is expandable in powers of the curvature because the integral (1.5) converges at the upper limit at each order of this expansion owing to specific cancellations in the leading asymptotic behavior (4.9).[20,22] However, generally, for a two-dimensional theory the effective action is nonanalytic in curvature, which implies that its calculation requires a further summation of the curvature series in the heat kernel – a technique which is beyond the presently discussed calculational scheme (see Introduction and Ref. 41).

For a two-dimensional theory (4.10) the full set of curvature invariants of Eq. (2.4) reduces to the following two structures: R_1R_2 and $R_1R_2R_3$, and Tr $K(s)$ takes the form of an expansion in powers of the Ricci scalar only:

$$\mathrm{Tr}\,K(s)=\frac{1}{4\pi s}\int dx\, g^{1/2}\Bigg\{1+s^2\sum_{i=1}^{5} c_i f_i(-s\Box_2)R_1R_2$$
$$+s^3\sum_{i=1}^{29} C_i F_i(-s\Box_1,-s\Box_2,-s\Box_3)R_1R_2R_3+\mathrm{O}[R^4]\Bigg\},\quad \omega=1 \tag{4.11}$$

where

$$c_1=1/2,\quad c_2=1,\quad c_3=1/6,\quad c_4=1/36,\quad c_5=0,$$

$$C_1=1/216,\quad C_4=1/6,\quad C_5=1/12,\quad C_6=1/36,\quad C_9=1,\quad C_{10}=1/4,\quad C_{11}=1/2,$$

$$C_{15}=\frac{s}{24}(\Box_1-\Box_2-\Box_3),\quad C_{16}=\frac{s}{48}(\Box_3-\Box_2-\Box_1),\quad C_{17}=\frac{s}{72}\Box_2,$$

$$C_{22}=\frac{s}{4}(\Box_1-\Box_2-\Box_3),\quad C_{23}=\frac{s}{8}(\Box_3-\Box_2-\Box_1),\quad C_{24}=\frac{s}{8}(\Box_1-\Box_2-\Box_3), \tag{4.12}$$

$$C_{25}=\frac{s}{16}(\Box_1-\Box_2-\Box_3),\quad C_{26}=\frac{s^2}{24}\Box_1\Box_2,\quad C_{27}=\frac{s^2}{4}\Box_1\Box_2,$$

$$C_{28}=\frac{s^2}{16}\Box_3(\Box_3-\Box_2-\Box_1),\quad C_{29}=\frac{s^3}{8}\Box_1\Box_2\Box_3,$$

$$C_2=C_3=C_7=C_8=C_{12}=C_{13}=C_{14}=C_{18}=C_{19}=C_{20}=C_{21}=0.$$

By using in (4.11) the asymptotic behaviors (4.3)–(4.8), one can now check that, at $s\to\infty$, the leading terms $1/s$ in Tr $K(s)/s$ cancel at both second order and third order in the curvature so that

$$\frac{1}{s}\,\mathrm{Tr}\,K(s)=\mathrm{O}\left(\frac{1}{s^2}\right),\quad s\to\infty. \tag{4.13}$$

As a result, the integral (1.5) converges at the upper limit. The convergence at the lower limit in the curvature-dependent terms holds trivially. Only the term of zeroth order in the curvature is ultraviolet divergent but, in the effective action of a massless theory, this term gets subtracted by a contribution of a local functional measure.[47] The actual calculation of this integral can be performed by using in (4.11) the table of form factors of Ref. 22 and the differential equations (2.9)–(2.11) for basic form factors, which allow one to convert Tr $K(s)/s$ into a total derivative in s:

$$\frac{1}{s}\mathrm{Tr}\,K(s)=\frac{1}{4\pi}\frac{d}{ds}\int dx\, g^{1/2}\left\{-\frac{1}{s}+l\,(s,\Box_2)R_1R_2+h(s,\Box_1,\Box_2,\Box_3)R_1R_2R_3+\mathrm{O}[R^4]\right\},$$
$$\omega=1, \tag{4.14}$$

where

$$l\,(s,\Box)=\frac{1}{\Box}\left[\frac{1}{8}f(-s\Box)-\frac{1}{4}\frac{f(-s\Box)-1}{s\Box}\right], \tag{4.15}$$

$$\begin{aligned}h(s,\Box_1,\Box_2,\Box_3)={}&sF(-s\Box_1,-s\Box_2,-s\Box_3)\frac{\Box_1\Box_2\Box_3}{3D^2}+\frac{f(-s\Box_1)}{8D^2\Box_1\Box_2}(\Box_1^4-2\Box_1^3\Box_3+2\Box_1\Box_3^3\\&-\Box_3^4-2\Box_1^3\Box_2+3\Box_2\Box_3^3-8\Box_1^2\Box_2\Box_3+8\Box_1\Box_2\Box_3^2-10\Box_1\Box_2^2\Box_3\\&-2\Box_2^2\Box_3^2)-\left[\frac{f(-s\Box_1)-1}{s\Box_1}\right]\frac{1}{4D\Box_1\Box_2}(\Box_1^2+4\Box_1\Box_2+\Box_2\Box_3-\Box_3^2)\\&-\frac{1}{\Box_2-\Box_3}\frac{\Box_2}{\Box_1}\left\{\frac{1}{8}\left[\frac{f(-s\Box_2)}{\Box_2}-\frac{f(-s\Box_3)}{\Box_3}\right]\right.\\&\left.-\frac{1}{4}\left[\frac{f(-s\Box_2)-1}{s\Box_2^2}-\frac{f(-s\Box_3)-1}{s\Box_3^2}\right]\right\}.\end{aligned} \tag{4.16}$$

Insertion of (4.14) (with the subtracted term of zeroth order in R) in (1.5) gives for the effective action:

$$W=\frac{1}{8\pi}\int dx\, g^{1/2}\,\{\,l\,(0,\Box_2)R_1R_2+h(0,\Box_1,\Box_2,\Box_3)R_1R_2R_3+\mathrm{O}[R^4]\,\},\quad \omega=1, \tag{4.17}$$

where use is made of the fact that the functions l and h vanish at $s\to\infty$. With the asymptotic behaviors (3.1) and (3.2), and the explicit expressions above, we obtain for $l\,(0,\Box)$ and the completely symmetrized in $\Box_1,\Box_2,\Box_3$ function $h(0,\Box_1,\Box_2,\Box_3)$ [which only contribute to (4.17)]

$$l\,(0,\Box)=\frac{1}{12}\frac{1}{\Box},\quad h^{\mathrm{sym}}(0,\Box_1,\Box_2,\Box_3)=0, \tag{4.18}$$

whence

$$W=\frac{1}{96\pi}\int dx\; g^{1/2}\, R\, \frac{1}{\Box} R+\mathrm{O}[R^4], \quad \omega=1. \tag{4.19}$$

Here the term of second order in the curvature reproduces the result of paper[20] (and the results of Refs. 9 and 30 obtained by integrating the trace anomaly).

Thus the third–order contribution in W really vanishes, and the mechanism of this vanishing is that, under special conditions like (4.10), the third-order contribution in $s^{-1}\mathrm{Tr}\, K(s)$ becomes a total derivative of a function vanishing at both $s=0$ and $s=\infty$. This mechanism underlies all "miraculous" cancellations of nonlocal terms including the trace anomaly in four dimensions.

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