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Some Aspects of the Source Description of Gravitation*

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We contribute to the source description of gravitation due to Schwinger by developing the necessary gravitational modifications of the couplings of various Boson fields to their sources and by providing a numerical supplement to his qualitative discussion of two-particle exchange.

I. INTRODUCTION

In a recent paper [1] Schwinger has formulated a source description of gravitation; in this paper we discuss two aspects of this description. The first aspect is the construction of the interaction skeleton. In particular, we discuss the *source terms* which provide the gravitational modifications of the couplings of the boson fields to their sources. The second aspect is the calculation of the two-particle exchanges implied by the first primitive interactions. Schwinger [1], [2] has discussed the principles involved and we merely provide the numerical details.

In Section II we describe the noninteracting behaviour of the bosons that we study in this paper; they are spinless particles with mass, photons, and gravitons. The first primitive interaction between the spinless particles and gravitons is introduced in Section III by coupling the gravitational field to the stress tensor of the particle. In this weak field situation we introduce the accompanying gravitational modification of the coupling of this matter field to its source; this source term guarantees that the primitive interaction is invariant under gravitational gauge transformations. In addition, this Section contains the calculation of some emission amplitudes using various forms of the primitive interaction. Section IV contains the corresponding results for the first primitive photon-graviton interaction.

The extension from the primitive interaction to the interaction skeleton which includes the graviton-graviton interaction is made in Section V through the principle of general coordinate invariance. It is here that the general problem

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of finding the gravitational modifications or source terms is discussed. We present a (deceptively) simple prescription for calculating a more complex modification from a simpler one. In Section VI we make explicit the first modification of the coupling of the gravitational field to its source. This section also contains a calculation of the two-graviton emission amplitude that is implied by the first primitive graviton-graviton interaction. In Section VII we develop the interaction skeleton to describe two-particle scattering and further illustrate our prescription for the source terms.

Having calculated various emission amplitudes, we use them in Section VIII to find the corresponding two-particle exchange contributions to the vacuum-persistance probability amplitude. The resulting modifications of both the photon and graviton propagators as well as those implied for the Coulomb and Newtonian potentials are presented. Finally, we verify that our results satisfy probability requirements.

II. Noninteracting Systems

We will review the source description of the various particles [1]–[3] under conditions of noninteraction to establish notations[1] and expressions that will be referred to frequently in the remainder of the paper. The simplest case is that of a spinless particle with mass m which is described by a real scalar function $K(x)$. The vacuum-persistance probability amplitude (briefly, the vacuum amplitude) is given by

$$\begin{gathered}\langle 0_+ \mid 0_- \rangle^K = \exp[iw_2(K)] \\ w_2(K) = (1/2) \int (dx)(dx')\, K(x)\, \Delta_+(x - x')\, K(x')\end{gathered} \tag{1}$$

and

$$\Delta_+(x - x') = \int \frac{(dp)}{(2\pi)^4} \frac{\exp[ip(x - x')]}{p^2 + m^2 - i\epsilon} \tag{2}$$

which involves the limiting process $\epsilon \to +0$. The multiparticle states are found by considering the situation in which $K = K_1 + K_2$ where K_2, effectively localized in time prior to K_1, creates particles which are subsequently absorbed by K_1. For this situation we require the property:

$$\Delta_+(x - x') = i \int d\omega_p \exp[ip(x - x')], \qquad x^0 > x^{0'}, \tag{3}$$

[1] The relativistic and quantum-mechanical notations are the same as those in Refs. [1]–[3], except that the Minkowski metric is written as $\eta_{\mu\nu}$. Unless otherwise stated, our units are $h/2\pi = c = 8\pi G = 1$, where G is the Newtonian gravitational constant.

where

$$(d\mathbf{p}) = 2p^0(2\pi)^3\, d\omega_p \tag{4}$$

defines $d\omega_p$ and $p^0 = (\mathbf{p}^2 + m^2)^{1/2}$. The vacuum amplitude becomes

$$\begin{aligned} \langle 0_+ \mid 0_- \rangle^K &= \langle 0_+ \mid 0_- \rangle^{K_1} (0_+ \mid 0_-)^K \langle 0_+ \mid 0_- \rangle^{K_2} \\ (0_+ \mid 0_-)^K &= \exp\left[-\int d\omega_p\, K_1(-p)\, K_2(p)\right], \end{aligned} \tag{5}$$

where

$$K(p) = \int (dx)\, K(x) \exp(-ipx). \tag{6}$$

By comparing Eq. (5) with the completeness relation

$$\langle 0_+ \mid 0_- \rangle^K = \sum_{\{n\}} \langle 0_+ \mid \{n\} \rangle^{K_1} \langle \{n\} \mid 0_- \rangle^{K_2}, \tag{7}$$

we can extract the multiparticle states; we find that

$$\begin{aligned} \langle \{n\} \mid 0_- \rangle^K &= \langle 0_+ \mid 0_- \rangle^K \prod_p (iK_p)^{n_p}/[n_p!]^{1/2} \\ \langle 0_+ \mid \{n\} \rangle^K &= \langle 0_+ \mid 0_- \rangle^K \prod_p (iK_p{}^*)^{n_p}/[n_p!]^{1/2} \end{aligned} \tag{8}$$

where $n_p = 0, 1, 2, \ldots$, and we have written

$$K_p = (d\omega_p)^{1/2}\, K(p). \tag{9}$$

The function $w_2(K)$ can be presented in the alternative form:

$$\begin{aligned} w_2(K) &= W_m^{(0)} + W_K^{(0)}, \\ W_m^{(0)} = \int (dx)\, L_m^{(0)}(x), &\qquad W_K^{(0)} = \int (dx)\, K(x)\, \phi(x), \end{aligned} \tag{10}$$

where

$$L_m^{(0)} = -(1/2)[\partial_\mu \phi\, \partial^\mu \phi + m^2 \phi\phi], \tag{11}$$

and it is understood that $w_2(K)$ is stationary with respect to variations of the auxiliary quantities $\phi(x)$. Thus, the resulting field equation is

$$(-\partial^2 + m^2)\, \phi(x) = K(x), \tag{12}$$

and when this is solved with the appropriate boundary conditions we obtain

$$\phi(x) = \int (dx')\, \Delta_+(x - x')\, K(x'). \tag{13}$$

Finally, when we substitute this expression into Eqs. (10), thereby eliminating the auxiliary quantities $\phi(x)$, we obtain the original form of $w_2(K)$ as expressed by Eq. (1).

We turn now to the description of photons by introducing the real vectorial function $J^\mu(x)$ which satisfies $\partial_\mu J^\mu = 0$. We start with the implicit differential form of $w_2(J)$:

$$\begin{gathered} w_2(J) = W_\gamma^{(0)} + W_J^{(0)}, \\ W_\gamma^{(0)} = \int (dx)\, L_\gamma^{(0)}(x), \qquad W_J^{(0)} = \int (dx)\, J^\mu(x)\, A_\mu(x), \end{gathered} \tag{14}$$

where

$$L_\gamma^{(0)} = -(1/4)\, F^{\mu\nu} F_{\mu\nu}\,, \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\,. \tag{15}$$

The requirement that $w_2(J)$ be stationary with respect to variations of the auxiliary fields $A_\mu(x)$ implies the field equations:

$$\partial_\nu(\partial^\mu A^\nu - \partial^\nu A^\mu) = J^\mu, \tag{16}$$

which have the solution (with the usual boundary conditions):

$$A_\mu(x) = \int (dx')\, D_{\mu\nu}(x - x')\, J^\nu(x') + \partial_\mu \lambda(x), \tag{17}$$

where

$$D_{\mu\nu}(x - x') = \eta_{\mu\nu} D_+(x - x'). \tag{18}$$

In Eq. (18), $\eta_{\mu\nu}$ is the Minkowski metric (with diagonal elements $-1, +1, +1, +1$) and $D_+(x - x')$ is the zero-mass version of $\Delta_+(x - x')$. The arbitrary gauge function $\lambda(x)$ in Eq. (17) disappears when we substitute that equation into Eqs. (14) to obtain the form:

$$w_2(J) = (1/2) \int (dx)(dx')\, J^\mu(x)\, D_{\mu\nu}(x - x')\, J^\nu(x'). \tag{19}$$

By considering the situation in which $J^\mu = J_1{}^\mu + J_2{}^\mu$, where the decomposition has the same meaning as before, we are led to the form

$$(0_+ \mid 0_-)^J = \exp\Big[-\int d\omega_p\, J_1{}^\mu(-p)\, \eta_{\mu\nu} J_2{}^\nu(p)\Big]. \tag{20}$$

To complete the derivation of the multiphoton state, we introduce the two real polarization vectors $e_\mu{}^a(p)$ associated with each p^μ. They satisfy

$$\begin{gathered} e_\mu{}^a(p)\, \eta^{\mu\nu} e_\nu{}^b(p) = \delta^{ab}, \qquad a, b = 1, 2, \\ p^\mu e_\mu{}^a(p) = 0, \qquad \bar{p}^\mu e_\mu{}^a(p) = 0, \end{gathered} \tag{21}$$

where

$$\bar{p}^\mu = p^\mu + 2n^\mu n^\nu p_\nu \tag{22}$$

and n^μ is a time-like unit vector: $n^2 = -1$. Then we may write

$$\eta_{\mu\nu} = \sum_a e_\mu{}^a(p)\, e_\nu{}^a(p) + (p_\mu \bar{p}_\nu + p_\nu \bar{p}_\mu)/(p\bar{p}), \tag{23}$$

and when this is substituted into Eq. (20) only the polarization sum survives since $p_\mu J^\mu(p) = 0$. We obtain

$$(0_+ \mid 0_-)^J = \exp\left[\sum_{pa} (iJ_{pa}^*)_1\, (iJ_{pa})_2\right], \tag{24}$$

where

$$J_{pa} = (d\omega_p)^{1/2}\, e_\mu{}^a(p)\, J^\mu(p), \qquad a = 1, 2, \tag{25}$$

is a convenient notation for writing the multiphoton states:

$$\begin{gathered} \langle\{n\} \mid 0_-\rangle^J = \langle 0_+ \mid 0_-\rangle^J \prod_{pa} (iJ_{pa})^{n_{pa}}/[n_{pa}!]^{1/2}, \\ \langle 0_+ \mid \{n\}\rangle^J = \langle 0_+ \mid 0_-\rangle^J \prod_{pa} (iJ_{pa}^*)^{n_{pa}}/[n_{pa}!]^{1/2}. \end{gathered} \tag{26}$$

Eqs. (26) were obtained from the completeness relation (7) adapted to the photon source $J^\mu(x)$.

The (hypothetical) graviton will be assumed to be a spin-2 particle with no rest mass. Its noninteracting behaviour is then described by

$$\begin{gathered} w_2(T) = W_g^{(2)} + W_T^{(1)}, \\ W_g^{(2)} = \int (dx)\, L_g^{(2)}(x), \qquad W_T^{(1)} = \int (dx)\, T^{\mu\nu}(x)\, h_{\mu\nu}(x), \end{gathered} \tag{27}$$

where the source function $T^{\mu\nu}(x)$ is a real symmetric tensor satisfying $\partial_\mu T^{\mu\nu} = 0$, and $h_{\mu\nu}(x)$ are the gravitational field variables. The Lagrange function for spin-2 massless particles may be written as

$$L_g^{(2)} = (1/2)\, \eta^{\mu\nu}[\Gamma^\kappa_{\mu\lambda}\Gamma^\lambda_{\nu\kappa} - \Gamma^\lambda_{\mu\nu}\Gamma^\kappa_{\lambda\kappa}], \tag{28}$$

where

$$\Gamma^{\lambda}_{\mu\nu} = \partial_\mu h_\nu{}^\lambda + \partial_\nu h_\mu{}^\lambda - \partial^\lambda h_{\mu\nu} . \tag{29}$$

The field equations implied by the stationary requirement on $w_2(T)$, with respect to variations of $h_{\mu\nu}(x)$, are

$$\begin{aligned} -\partial^2 h_{\mu\nu} + (1/2)(\partial_\mu h_\nu + \partial_\nu h_\mu) &= \bar{\eta}_{\mu\nu,\lambda\kappa} T^{\lambda\kappa}, \\ h_\mu = 2\partial^\lambda h_{\lambda\mu} - \partial_\mu h, \qquad h &= \eta^{\mu\nu} h_{\mu\nu} , \end{aligned} \tag{30}$$

and

$$\bar{\eta}_{\mu\nu,\lambda\kappa} = (1/2)[\eta_{\mu\lambda}\eta_{\nu\kappa} + \eta_{\mu\kappa}\eta_{\nu\lambda} - \eta_{\mu\nu}\eta_{\lambda\kappa}]. \tag{31}$$

The solution with the appropriate boundary conditions is

$$h_{\mu\nu}(x) = \int (dx')\, D_{\mu\nu,\lambda\kappa}(x - x')\, T^{\lambda\kappa}(x') - (1/2)[\partial_\mu \xi_\nu(x) + \partial_\nu \xi_\mu(x)], \tag{32}$$

where

$$D_{\mu\nu,\lambda\kappa}(x - x') = \bar{\eta}_{\mu\nu,\lambda\kappa} D_+(x - x').$$

The arbitrary vector gauge functions $\xi_\mu(x)$ disappear when we substitute Eq. (32) into Eqs. (27). The result is an explicit integral construction of $w_2(T)$:

$$w_2(T) = (1/2) \int (dx)(dx')\, T^{\mu\nu}(x)\, D_{\mu\nu,\lambda\kappa}(x - x')\, T^{\lambda\kappa}(x'). \tag{34}$$

The multigraviton states are found by specializing $w_2(T)$ to the familiar situation $T = T_1 + T_2$. We must again consider the form

$$(0_+ \mid 0_-)^T = \exp\left[- \int d\omega_k\, T_1^{\mu\nu}(-k)\, \bar{\eta}_{\mu\nu,\lambda\kappa} T_2^{\lambda\kappa}(k)\right], \tag{35}$$

where k is the four-momentum variable of the real graviton. Substituting Eq. (23), with p replaced by k, into Eq. (31) produces a form of $\bar{\eta}_{\mu\nu,\lambda\kappa}$ appropriate to Eq. (35). Thus, the fact that $k_\mu T^{\mu\nu}(k) = 0$ allows the replacement:

$$\bar{\eta}_{\mu\nu,\lambda\kappa} \to \sum_{r=1,2} e^r_{\mu\nu}(k)\, e^r_{\lambda\kappa}(k), \tag{36}$$

where

$$\begin{aligned} e^1_{\mu\nu}(k) &= (1/\surd 2)[e_\mu{}^1(k)\, e_\nu{}^1(k) - e_\mu{}^2(k)\, e_\nu{}^2(k)], \\ e^2_{\mu\nu}(k) &= (1/\surd 2)[e_\mu{}^1(k)\, e_\nu{}^2(k) + e_\mu{}^2(k)\, e_\nu{}^1(k)]. \end{aligned} \tag{37}$$

The properties of these polarization tensors, deduced from Eqs. (21) are:

$$\begin{gathered} e^{r}_{\mu\nu}(k)\,\eta^{\mu\lambda}\eta^{\nu\kappa}e^{s}_{\lambda\kappa}(k) = \delta^{rs}, \qquad r, s = 1, 2, \\ \eta^{\mu\nu}e^{r}_{\mu\nu}(k) = 0, \\ k^{\mu}e^{r}_{\mu\nu}(k) = 0, \qquad \bar{k}^{\mu}e^{r}_{\mu\nu}(k) = 0. \end{gathered} \tag{38}$$

By comparing the version of Eq. (35) containing the replacement (36) with the completeness relation we deduce the multigraviton states:

$$\begin{aligned} \langle\{n\} \mid 0_-\rangle^T &= \langle 0_+ \mid 0_-\rangle^T \prod_{kr} (iT_{kr})^{n_{kr}}/[n_{kr}!]^{1/2}, \\ \langle 0_+ \mid \{n\}\rangle^T &= \langle 0_+ \mid 0_-\rangle^T \prod_{kr} (iT^*_{kr})^{n_{kr}}/[n_{kr}!]^{1/2}, \end{aligned} \tag{39}$$

where

$$T_{kr} = (d\omega_k)^{1/2}\, e^{r}_{\mu\nu}(k)\, T^{\mu\nu}(k). \tag{40}$$

We summarize our discussion of the noninteracting systems by stating that the vacuum amplitude for the combined system of material particles, photons, and gravitons is given by

$$\begin{aligned} \langle 0_+ \mid 0_-\rangle^{KJT} &= \exp[iw_2(K, J, T)], \\ w_2(K, J, T) &= w_2(K) + w_2(J) + w_2(T). \end{aligned} \tag{41}$$

III. First Primitive Particle-Graviton Interaction

Schwinger has successfully described gravitational phenomena in the quasi-static macroscopic domain by coupling the gravitational field to the stress tensor of various systems [1]. Following his suggestion, we extend that prescription to interactions in the microscopic domain. The stress tensor of the system defined by $W^{(0)}_m$ is found by considering the response of this system to a general coordinate transformation $x^\mu \to x^\mu + \delta x^\mu$. Using

$$\delta'(\partial_\mu) = -(\partial_\mu\, \delta x^\nu)\, \partial_\nu\,, \qquad \delta'(dx) = (\partial_\mu\, \delta x^\mu)(dx), \tag{42}$$

and $\delta'\phi(x) = 0$, we find this response to be

$$\delta' W^{(0)}_m = \int (dx)\, T^{\mu\nu}_m(x)\, \partial_\mu\, \delta x_\nu\,, \tag{43}$$

where

$$T_m^{\mu\nu} = \partial^\mu\phi\, \partial^\nu\phi + \eta^{\mu\nu} L_m^{(0)}. \tag{44}$$

The assumption that the primitive interaction is given by

$$W_m^{(1)} = \int (dx)\, T_m^{\mu\nu}(x)\, h_{\mu\nu}(x), \tag{45}$$

is incorrect even in the weak field situation since we have ignored the source $K(x)$. Thus, the fact that the divergence of $T_m^{\mu\nu}(x)$ is given by

$$\partial_\mu T_m^{\mu\nu}(x) = -K(x)\, \partial^\nu\phi(x) \neq 0, \tag{46}$$

means that when we substitute expression (32) for $h_{\mu\nu}(x)$ into Eq. (45) the gauge functions $\xi_\mu(x)$ survive. This deficiency is eliminated by formulating the primitive interaction in the following way:

$$w_{2,1}(K, T) = W_m^{(1)} + W_K^{(1)}, \tag{47}$$

where

$$W_K^{(1)} = \int (dx)\, X_\lambda(x)\, K(x)\, \partial^\lambda\phi(x) \tag{48}$$

involves

$$X_\lambda(x) = -\int (dx')\, f_\lambda^{\mu\nu}(x - x')\, h_{\mu\nu}(x'). \tag{49}$$

This linear functional of the gravitational field must respond to the gauge transformations

$$h_{\mu\nu} \to h_{\mu\nu} - (1/2)(\partial_\mu\xi_\nu + \partial_\nu\xi_\mu), \tag{50}$$

according to

$$X_\lambda \to X_\lambda + \xi_\lambda\,. \tag{51}$$

This response requires that $f_\lambda^{\mu\nu}(x' - x)$ satisfy

$$-\partial_\mu f_\lambda^{\mu\nu}(x' - x) = \delta_\lambda{}^\nu\, \delta(x' - x). \tag{52}$$

The construction of the function $f_\lambda^{\mu\nu}(x' - x)$ is greatly facilitated by considering the analogous situation that arises in electrodynamics [2]. The latter situation requires the construction of a function $f^\mu(x' - x)$ that satisfies

$$-\partial_\mu f^\mu(x' - x) = \delta(x' - x). \tag{53}$$

Thus, we can satisfy Eq. (52) with the construction:

$$f_{\lambda}^{\mu\nu}(x'-x) = f^{\mu}(x'-x)\,\delta_{\lambda}{}^{\nu} + f^{\nu}(x'-x)\,\delta_{\lambda}{}^{\mu} + \int (dx'')[\partial''_{\lambda} f^{\nu}(x'-x'')]\, f^{\mu}(x''-x). \tag{54}$$

Now an alternative way of presenting the primitive interaction is:

$$w_{2,1}(K, T) = \int (dx)\, t_m^{\mu\nu}(x)\, h_{\mu\nu}(x), \tag{55}$$

where

$$t_m^{\mu\nu}(x) = T_m^{\mu\nu}(x) - \int (dx')\, K(x')\, \partial^{\lambda}\phi(x')\, f_{\lambda}^{\mu\nu}(x'-x) \tag{56}$$

satisfies

$$\partial_{\mu} t_m^{\mu\nu}(x) = 0. \tag{57}$$

The two terms in Eq. (56) imply that both the particle and the source are involved in the mechanism of graviton radiation. This means that $f_{\lambda}^{\mu\nu}(x'-x)$ must be local in time. But this requirement is satisfied by making $f^{\mu}(x'-x)$ local in time and such a function is already available from the source description of electrodynamics. In our context the source function $K(x)$ supplies a time-like vector,[2] as represented by $-\partial_K$, from which we construct

$$f^{\mu}(x'-x) = -(\nabla^{\mu}/\nabla^2)\,\delta(x'-x), \tag{58}$$

where

$$\nabla^{\mu} = \partial^{\mu} - \partial_K{}^{\mu}(\partial\partial_K)/\partial_K{}^2. \tag{59}$$

This version of $f^{\mu}(x'-x)$ defines a particular type of effective source and we will limit ourselves to this choice.[3] To understand better the nature of the source characterized by this function, we will now consider the emission amplitudes that are implied by this choice.

By treating $t_m^{\mu\nu}(x)$ as a weak effective graviton source, we find that the emission amplitude for a graviton of four-momentum k and polarization index r is

$$\langle 1_{kr} \mid 0_{-}\rangle^K = i(d\omega_k)^{1/2}\, e_{\mu\nu}^{r}(k) \int (dx)\, e^{-ikx} t_m^{\mu\nu}(x). \tag{60}$$

[2] A more precise definition is given at the end of Section VII.

[3] Our intention here is to parallel Schwinger's discussion of the akin situation in electrodynamics. See [2] for additional details including a verification of time locality.

The presence of the factor $e^r_{\mu\nu}(k)$ allows some simplifications which may be presented in the form of replacements:

$$T_m^{\mu\nu}(x) \to \partial^\mu \phi(x)\, \partial^\nu \phi(x), \tag{61}$$

and

$$\int (dx)\, e^{-ikx} f_\lambda^{\mu\nu}(x' - x) \to i S_\lambda^{\mu\nu}(k; \partial_K)\, e^{-ikx'}, \tag{62}$$

where

$$S_\lambda^{\mu\nu}(k; \partial_K) = (\delta_\lambda{}^\mu\, \partial_K{}^\nu + \delta_\lambda{}^\nu\, \partial_K{}^\mu)/(k\, \partial_K) - k_\lambda\, \partial_K{}^\mu\, \partial_K{}^\nu/(k\, \partial_K)^2. \tag{63}$$

All told this leads to the replacement of $t_m^{\mu\nu}(x)$ by

$$\partial^\mu \phi(x)\, \partial^\nu \phi(x) - i S_\lambda^{\mu\nu}(k; \partial_K)\, K(x)\, \partial^\lambda \phi(x). \tag{64}$$

Now consider the situation where K is a superposition of a particle-detection source and an extended source as indicated by the replacement $K \to K_1 + K$. The relevant part of expression (64) for the emission of a particle and a graviton is

$$2\partial^\mu \phi_1(x)\, \partial^\nu \phi(x) - i S_\lambda^{\mu\nu}(k; \partial_K)\, K(x)\, \partial^\lambda \phi_1(x), \tag{65}$$

where $\phi_1(x)$ is the field associated with the detection source K_1 :

$$\begin{aligned} \phi_1(x) &= \int (dx')\, K_1(x')\, \Delta_+(x' - x) \\ &= \sum_p (d\omega_p)^{1/2}\, (iK_p{}^*)_1\, e^{-ipx}. \end{aligned} \tag{66}$$

If the particle is emitted with a four-momentum p then the source $K(x)$ must supply the four-momentum $P = p + k$; thus,

$$S_\lambda^{\mu\nu}(k; \partial_K) \to (\delta_\lambda{}^\mu p^\nu + \delta_\lambda{}^\nu p^\mu)/(kp) - k_\lambda p^\mu p^\nu/(kp)^2. \tag{67}$$

The emission amplitude is

$$\langle 1_{kr} 1_p \mid 0_- \rangle^K = i(d\omega_p\, d\omega_k)^{1/2}\, e^r_{\mu\nu}(k) \left[\frac{2p^\mu p^\nu}{P^2 + m^2} - \frac{p^\mu p^\nu}{(kp)}\right] K(P). \tag{68}$$

However, the fact that $P^2 = -m^2 + 2(kp)$ means that the second term in brackets cancels the first term. Contrast this result with the one obtained by replacing $-i\partial_K$ in $f_\lambda^{\mu\nu}(x' - x)$ by a constant vector parallel to the time axis. For this type of source the second term in Eq. (68) does not appear [2] and the first term appears intact.

The denominator of the first term becomes very small for soft gravitons; hence, our original choice of $f_{\lambda}^{\mu\nu}(x' - x)$ avoids this strong radiation that would accompany the creation of the material particle.

The probability amplitude for the emission of two particles from an extended graviton source does not involve the source term $W_K^{(1)}$. To make this statement convincing we recast the primitive interaction into the form:

$$w_{2,1}(K, T) = \int (dx)\, T_m^{\mu\nu}(x)\, H_{\mu\nu}(x), \tag{69}$$

where

$$H_{\mu\nu}(x) = h_{\mu\nu}(x) + (1/2)[\partial_\mu X_\nu(x) + \partial_\nu X_\mu(x)] \tag{70}$$

is invariant under the gauge transformations (50). When we specialize the source K to the arrangement $K_1 + K_2$, the relevant part of $T_m^{\mu\nu}$ involves only the field ϕ_1 presented in Eqs. (66). Then the emission amplitude for two particles of four-momenta p and p' reads

$$\langle 1_p 1_{p'} \mid 0_- \rangle^T = i(d\omega_p\, d\omega_{p'})^{1/2}\, P^{\mu\nu}(p, p')(1/2)\, k^2 H_{\mu\nu}(k), \tag{71}$$

where

$$k = p + p', \qquad (1/2)\, k^2 = (pp') - m^2, \tag{72}$$

and

$$P^{\mu\nu}(p, p') = \eta^{\mu\nu} - (p^\mu p'^\nu + p^\nu p'^\mu)/[(pp') - m^2]. \tag{73}$$

We can replace $H_{\mu\nu}(k)$ by the more general field $h_{\mu\nu}(k)$ because

$$k_\mu P^{\mu\nu}(p, p') = 0, \tag{74}$$

and this means that the source term $W_K^{(1)}$ makes no contribution. The result stated in Eq. (71) implies a modification of the graviton propagator but we postpone that calculation until we have the corresponding results for two-photon and two-graviton emission.

IV. First Primitive Photon-Graviton Interaction

In this section we consider the first primitive interaction between photons and gravitons. Following the procedure of the previous section, we postulate this interaction:

$$w_{2,1}(J, T) = \int (dx)\, T_\gamma^{\mu\nu}(x)\, H_{\mu\nu}(x), \tag{75}$$

where

$$T_\gamma^{\mu\nu} = F^{\mu\lambda}\eta_{\lambda\kappa}F^{\nu\kappa} - (1/4)\,\eta^{\mu\nu}F^{\lambda\kappa}F_{\lambda\kappa} \tag{76}$$

is the stress tensor associated with the system defined by $W_\gamma^{(0)}$; its derivation requires the use of Eqs. (42) together with

$$\delta' A_\mu(x) = -\partial_\mu\,\delta x^\nu A_\nu(x). \tag{77}$$

Using Eq. (70) and the fact that

$$\partial_\mu T_\gamma^{\mu\nu}(x) = -F^{\nu\mu}(x)\,J_\mu(x), \tag{78}$$

we can write the primitive interaction in the alternative form

$$w_{2,1}(J, T) = W_\gamma^{(1)} + W_J^{(1)}, \tag{79a}$$

$$W_\gamma^{(1)} = \int (dx)\,T_\gamma^{\mu\nu}h_{\mu\nu}, \qquad W_J^{(1)} = \int (dx)\,X_\mu F^{\mu\nu}J_\nu. \tag{79b}$$

To calculate the one-graviton emission amplitude implied by this interaction we use the form:

$$w_{2,1}(J, T) = \int (dx)\,t_\gamma^{\mu\nu}(x)\,h_{\mu\nu}(x), \tag{80}$$

where

$$t_\gamma^{\mu\nu}(x) = T_\gamma^{\mu\nu}(x) - \int (dx')\,F^{\lambda\kappa}(x')\,J_\kappa(x')\,f_\lambda^{\mu\nu}(x' - x) \tag{81}$$

satisfies

$$\partial_\mu t_\gamma^{\mu\nu}(x) = 0. \tag{82}$$

The probability amplitude has the same structure as Eq. (60):

$$\langle 1_{kr} \mid 0_- \rangle^J = i(d\omega_k)^{1/2}\,e_{\mu\nu}^r(k) \int (dx)\,e^{-ikx}t_\gamma^{\mu\nu}(x). \tag{83}$$

In this case the presence of the polarization tensor allows the replacement of $t_\gamma^{\mu\nu}(x)$ by

$$F^{\mu\lambda}(x)\,\eta_{\lambda\kappa}F^{\nu\kappa}(x) - iS_\lambda^{\mu\nu}(k;\partial_J)\,J_\kappa(x)\,F^{\lambda\kappa}(x), \tag{84}$$

which indicates that the time-like vector $-i\partial_J$, which is involved in the definition of $f_\lambda^{\mu\nu}(x' - x)$, is now supplied by the source $J^\mu(x)$.

The probability amplitude for the emission of a graviton and a photon is developed by considering $J_\mu(x)$ to be the superposition of a photon-detection source and an extended photon source as represented by the substitution $J^\mu \to J_1^\mu + J^\mu$. The relevant part of $t_\gamma^{\mu\nu}(x)$ is

$$2F_1^{\mu\lambda}(x)\,\eta_{\lambda\kappa}F^{\nu\kappa}(x) - iS_\lambda^{\mu\nu}(k;\,\partial_J)\,J_\kappa(x)\,F_1^{\lambda\kappa}(x) \tag{85}$$

where $F_1^{\mu\nu}(x)$ is the field associated with the detection source:

$$F_{1\mu\nu}(x) = \int (d\omega_p)\,J_1{}^\lambda(-p)[\eta_{\lambda\nu}p_\mu - \eta_{\lambda\mu}p_\nu]\,e^{-ipx}. \tag{86}$$

By using the decomposition of $\eta_{\mu\nu}$ in Eq. (23) we can write

$$F_{1\mu\nu}(x) = \sum_{pa} (d\omega_p)^{1/2}\,(J_{pa}^*)_1\,e^{-ipx}[e_\nu{}^a(p)\,p_\mu - e_\mu{}^a(p)\,p_\nu]. \tag{87}$$

The emission amplitude may be written in the unified form:

$$\langle 1_{kr}1_{pa} \mid 0_-\rangle^J = i(d\omega_k\,d\omega_p)^{1/2}\,e_{\mu\nu}^r(k)\,e_\lambda{}^a(p)\,U^{\mu\nu,\lambda\kappa}(k,\,p)\,A_\kappa(P), \tag{88}$$

where $P = p + k$ is the total four-momentum supplied by the source, when we relate $J_\kappa(x)$ in Eq. (85) to the field $A_\kappa(x)$ through Eq. (16). The first term in Eq. (85) makes the following contribution to $U^{\mu\nu,\lambda\kappa}(k, p)$:

$$2p^\mu p^\nu\eta^{\lambda\kappa} + (\eta^{\nu\lambda}\eta^{\nu\kappa} + \eta^{\mu\kappa}\eta^{\nu\lambda})(pk) - (p^\mu\eta^{\nu\lambda} + p^\nu\eta^{\mu\lambda})\,p^\kappa - (p^\mu\eta^{\nu\kappa} + p^\nu\eta^{\mu\kappa})\,k^\lambda, \tag{89}$$

and the second or source term contributes:

$$-2p^\mu p^\nu\eta^{\lambda\kappa} + (p^\mu\eta^{\nu\lambda} + p^\nu\eta^{\mu\lambda})(p^\kappa - k^\kappa) + 2p^\mu p^\nu k^\lambda k^\kappa/(kp). \tag{90}$$

The total contribution reduces to the simple form:

$$U^{\mu\nu,\lambda\kappa}(k, p) = (1/2)\,P^2[P^{\mu\lambda}(k, p)\,P^{\nu\kappa}(k, p) + P^{\mu\kappa}(k, p)\,P^{\nu\lambda}(k, p)] \tag{91}$$

where

$$P^{\mu\nu}(k, p) = \eta^{\mu\nu} - (k^\mu p^\nu + k^\nu p^\mu)/(kp), \tag{92}$$

is actually Eq. (73) specialized to this zero-mass situation. The properties

$$k_\mu P^{\mu\nu}(k, p) = 0, \qquad p_\mu P^{\mu\nu}(k, p) = 0, \tag{93}$$

not only guarantee the invariance of Eq. (88) under the gravitational gauge transformations (50) but also under the usual electromagnetic gauge transformations.

The two-photon emission amplitude will be calculated from Eq. (75) specialized to two photon-detection sources which may be indicated by the replacement:

$$T_\gamma^{\mu\nu} \to F_1^{\mu\lambda}\eta_{\lambda\kappa}F_1^{\nu\kappa} + \eta^{\mu\nu}L_\gamma^{(0)}(F_1). \tag{94}$$

By using Eq. (87) for the fields $F_1^{\mu\nu}$ we find that the probability amplitude for the emission of two photons of four-momenta p and p' and polarization indices a and b, respectively, is

$$\langle 1_{pa} 1_{p'b} \mid 0_- \rangle^T = -i(d\omega_p\, d\omega_{p'})^{1/2}\, k^2 e_\mu{}^a(p)\, e_\nu{}^b(p')\, \bar{P}^{\mu\nu,\lambda\kappa}(p, p')\, H_{\lambda\kappa}(k), \quad (95)$$

where

$$\bar{P}^{\mu\nu,\lambda\kappa}(p, p') = (1/2)[P^{\mu\lambda}(p, p')\, P^{\nu\kappa}(p, p') + P^{\mu\kappa}(p, p')\, P^{\nu\lambda}(p, p') - P^{\mu\nu}(p, p')\, P^{\lambda\kappa}(p, p')], \quad (96)$$

and $k = p + p'$. We may replace $H_{\lambda\kappa}(k)$ in Eq. (95) by the field $h_{\lambda\kappa}(k)$ because

$$k_\lambda \bar{P}^{\mu\nu,\lambda\kappa}(p, p') = 0. \quad (97)$$

We close this section by pointing out that

$$P_\mu{}^\lambda(p, p')\, P_\lambda{}^\nu(p, p') = P_\mu{}^\nu(p, p'), \quad (98)$$

and

$$\bar{P}_{\mu\nu}^{\lambda\kappa}(p, p')\, \bar{P}_{\lambda\kappa}^{\sigma\tau}(p, p') = \bar{P}_{\mu\nu}^{\sigma\tau}(p, p'), \quad (99)$$

which are useful in calculating two-particle exchanges.

V. Interaction Skeleton

The introduction of the first primitive interaction may be viewed as the introduction of new, effective sources. For example, $t_m^{\mu\nu}(x)$ is an effective graviton source introduced by the first primitive particle-graviton interaction. By coupling these sources to each other we obtain more complicated interactions which imply additional effective sources. In this section we will develop the description of a class of processes belonging to this proliferation; these are the so-called skeleton interactions which may be analyzed as the exchange of a single particle between appropriately defined effective sources. This development cannot be divorced from the concomitant introduction of more complicated primitive interactions including the graviton-graviton interaction. The unifying principle is that of general coordinate invariance and our starting point is Schwinger's introduction of that principle in the context of a source description [1].

The response of $W_m^{(0)}$ to infinitesimal coordinate transformations can be represented by the variation:

$$\delta_c W_m^{(0)} = \int (dx)[\delta W_m^{(0)}/\delta\phi]\, \delta_c\phi, \quad (100)$$

where

$$\delta_c \phi = -\delta x_\nu \, \partial^\nu \phi. \tag{101}$$

The response of $W_m^{(1)}$ to infinitesimal gravitational gauge transformations is

$$\delta_g W_m^{(1)} = \int (dx) \, \partial_\mu [\delta W_m^{(1)} / \delta h_{\mu\nu}] \, \delta \xi_\nu \, . \tag{102}$$

Reference to Eqs. (10) and (11) for $W_m^{(0)}$ and Eqs. (44) and (45) for $W_m^{(1)}$ reveals the identity

$$\partial_\mu [\delta W_m^{(1)} / \delta h_{\mu\nu}] = [\delta W_m^{(0)} / \delta \phi] \, \partial^\nu \phi. \tag{103}$$

By making the connection $\delta \xi_\nu = \delta x_\nu$, we introduce a higher symmetry which links the gravitational gauge transformations with arbitrary coordinate transformations as expressed by

$$\delta_g h_{\mu\nu} = -(1/2)(\partial_\mu \, \delta x_\nu + \partial_\nu \, \delta x_\mu). \tag{104}$$

To obtain the full response of $W_m^{(1)}$ to the transformation: $\bar{x}^\mu = x^\mu + \delta x^\mu$, we must supplement Eq. (102) with

$$\delta_c W_m^{(1)} = \int (dx) [\delta W_m^{(1)} / \delta \phi] \, \delta_c \phi + \int (dx) [\delta W_m^{(1)} / \delta h_{\mu\nu}] \, \delta_c h_{\mu\nu} \, , \tag{105}$$

where

$$\delta_c h_{\mu\nu} = \delta' h_{\mu\nu} - \delta x^\lambda \, \partial_\lambda h_{\mu\nu} \tag{106}$$

involves the tensor transformation property

$$\delta' h_{\mu\nu} = -h_{\mu\lambda} \, \partial_\nu \, \delta x^\lambda - h_{\nu\lambda} \, \partial_\mu \, \delta x^\lambda. \tag{107}$$

The new interaction $W_m^{(2)}$ is introduced by requiring that its response to infinitesimal gravitational gauge transformations cancel the response in Eq. (105),

$$\delta_g W_m^{(2)} + \delta_c W_m^{(1)} = 0. \tag{108}$$

In general, $W_m^{(n)}$ will be connected to $W_m^{(n-1)}$ by

$$\delta_g W_m^{(n)} + \delta_c W^{(n-1)} = 0, \qquad n = 1, 2, \ldots \, . \tag{109}$$

This guiding principle will aid the development of the series

$$W_m = W_m^{(0)} + W_m^{(1)} + W_m^{(2)} + \cdots, \tag{110}$$

where the superscripts indicate the number of times the gravitational field appears as a factor in the integrands. The function W_m will be invariant under the variations

$$\delta\phi = \delta_c\phi, \qquad \delta h_{\mu\nu} = \delta_g h_{\mu\nu} + \delta_c h_{\mu\nu} . \tag{111}$$

The definitions $g_{\mu\nu} = \eta_{\mu\nu} + 2h_{\mu\nu}$ simplifies the last variation,

$$\delta g_{\mu\nu} = -g_{\mu\lambda}\,\partial_\nu\,\delta x^\lambda - g_{\nu\lambda}\,\partial_\mu\,\delta x^\lambda - \delta x^\lambda\,\partial_\lambda g_{\mu\nu} . \tag{112}$$

This familiar equation suggests the following expression for the function W_m :

$$W_m = \int (dx)(-g)^{1/2}(-1/2)[\partial_\mu\phi g^{\mu\nu}\,\partial_\nu\phi + m^2\phi^2], \tag{113}$$

where

$$g(x) = \det g_{\mu\nu}(x), \qquad g_{\mu\lambda}(x)\,g^{\lambda\nu}(x) = \delta_\mu{}^\nu. \tag{114}$$

By making use of the expansions

$$\begin{aligned} -g(x) &= 1 + 2h(x) + \cdots, \\ g^{\mu\nu}(x) &= \eta^{\mu\nu} - 2h^{\mu\nu}(x) + \cdots, \end{aligned} \tag{115}$$

we can reproduce the previously defined functions $W_m^{(0)}$ and $W_m^{(1)}$. We have followed this heuristic development of the principle of general coordinate invariance since it will help us in constructing the necessary modifications of the coupling of the matter field ϕ to the source K.

To satisfy the principle of general coordinate invariance, we must replace the structure $W_K^{(0)} + W_K^{(1)}$ by the more general development:

$$W_K = W_K^{(0)} + W_K^{(1)} + W_K^{(2)} + \cdots, \tag{116}$$

where the superscripts have the same meaning as in Eq. (110) and the source terms are related by

$$\delta_g W_K^{(n)} + \delta_c W_K^{(n-1)} = 0, \qquad n = 1, 2,\ldots . \tag{117}$$

The variations are those connected with Eq. (109). The equation labelled $n = 1$ is already satisfied by the structure of $W_K^{(1)}$ defined in Eq. (48). The equation labelled $n = 2$ introduces $W_K^{(2)}$ and implies the differential equation

$$-\partial_\mu[\delta W_K^{(2)}/\delta h_{\mu\nu}] = \Gamma^\nu_{\sigma\tau}[\delta W_K^{(1)}/\delta h_{\sigma\tau}] + 2h_\sigma{}^\nu\,\partial_\tau[\delta W_K^{(1)}/\delta h_{\sigma\tau}] - [\delta W_K^{(1)}/\delta\phi]\,\partial^\nu\phi. \tag{118}$$

The right side of this equation is a known, linear functional of the gravitational field which we write as

$$Z^{\nu}(x) = \int (dx')\, Z^{\nu,\lambda\kappa}(x, x')\, h_{\lambda\kappa}(x'). \tag{119}$$

It should be noted that the structure of $W_K^{(1)}$ implies that $Z^{\nu,\lambda\kappa}(x, x')$ is local in time. The form of Eq. (118) suggests that we write

$$W_K^{(2)}(h) = (1/2) \int (dx)(dx')\, K^{\mu\nu,\lambda\kappa}(x, x')\, h_{\mu\nu}(x)\, h_{\lambda\kappa}(x'), \tag{120}$$

which implies the integrability conditions

$$K^{\mu\nu,\lambda\kappa}(x, x') = K^{\lambda\kappa,\mu\nu}(x', x). \tag{121}$$

The differential equation (118) implies the condition

$$-\partial_{\mu} K^{\mu\nu,\lambda\kappa}(x, x') = Z^{\nu,\lambda\kappa}(x, x'). \tag{122}$$

In addition to these requirements, $K^{\mu\nu,\lambda\kappa}$ should be linear in the source K and linear in the field ϕ since we only seek that modification of $W_K^{(0)}$ due to the presence of a strong gravitational field. This suggests that we relate $K^{\mu\nu,\lambda\kappa}$ directly to its derivatives. An algebraic construction of $W_K^{(2)}$ accomplishes this and at the same time by-passes the problem of satisfying the integrability conditions. The function $W_K^{(2)}$ is introduced by making reference only to its response to gravitational gauge transformations and this suggests that we make use of the gauge-invariant field $H_{\mu\nu}$ defined in Eq. (70). These, and other,[4] considerations lead us to the prescription:

$$W_K^{(2)}(H) = 0$$

or

$$(1/2) \int (dx)(dx')\, K^{\mu\nu,\lambda\kappa}(x, x')\, H_{\mu\nu}(x)\, H_{\lambda\kappa}(x') = 0. \tag{123}$$

The modification $W_K^{(1)}$ already satisfies a similar equation. By eliminating the field $H_{\sigma\tau}$ in favor of $h_{\sigma\tau}$, we obtain

$$\begin{aligned} W_K^{(2)}(h) = &- \int (dx)(dx')\, X_{\nu}(x)\, Z^{\nu,\lambda\kappa}(x, x')\, h_{\lambda\kappa}(x') \\ &+ (1/2) \int (dx)(dx')\, \partial_{\lambda}' Z^{\nu,\lambda\kappa}(x, x')\, X_{\nu}(x)\, X_{\kappa}(x'). \end{aligned} \tag{124}$$

[4] The decisive reason for this prescription is given near the end of Section VII.

We eliminated $K^{\mu\nu,\lambda\kappa}(x, x')$ by using Eq. (122). This $W_K^{(2)}$ is local in time and responds to infinitesimal gravitational gauge transformations according to

$$\delta_g W_K^{(2)} = -\int (dx)\, Z^\nu(x)\, \delta x_\nu\,, \tag{125}$$

since the variation of $h_{\lambda\kappa}(x')$ in the first term in Eq. (124) is cancelled by the variation of the second term in that equation. This results from the gauge invariance of $H_{\mu\nu}$ and, of course, is not limited to the case $n = 2$. Thus, in general, we have the prescription

$$W_K^{(n)}(H) = 0, \qquad n = 1, 2,\ldots\,. \tag{126}$$

Eqs. (126) and (116) determine each of the source functions $W_K^{(n)}(h)$ by the method just outlined for the case $n = 2$.

Exactly the same development can be carried out to describe more complicated photon-graviton interactions. The result is that

$$\begin{aligned} W_\gamma &= -(1/4) \int (dx)(-g)^{1/2} F_{\mu\nu} g^{\mu\lambda} g^{\nu\kappa} F_{\lambda\kappa} \\ &= W_\gamma^{(0)} + W_\gamma^{(1)} + W_\gamma^{(2)} + \cdots \end{aligned} \tag{127}$$

reproduces our original expression for $W_\gamma^{(0)} + W_\gamma^{(1)}$ and builds further interactions by the principle of general coordinate invariance. Furthermore, the series

$$W_J = W_J^{(0)} + W_J^{(1)} + W_J^{(2)} + \cdots \tag{128}$$

has each of its terms $W_J^{(n)}$ determined by the equations

$$\left.\begin{aligned} \delta_g W_J^{(n)} + \delta_c W_J^{(n-1)} &= 0, \\ W_J^{(n)}(H) &= 0, \end{aligned}\right. \qquad n = 1, 2,\ldots, \tag{129}$$

which involves the use of the variation

$$\delta_c A_\mu = -A_\nu\, \partial_\mu\, \delta x^\nu - \delta x^\nu\, \partial_\nu A_\mu\,. \tag{130}$$

We have introduced the first primitive interactions by coupling the gravitational field to the stress tensors of the systems. But the system of noninteracting gravitons defined by Eqs. (27) and (28) also possesses a stress tensor. Thus, we are led to the introduction of a graviton-graviton interaction. In fact, to satisfy the principle of general coordinate invariance we must develop a series of interactions:

$$W_g = W_g^{(2)} + W_g^{(3)} + W_g^{(4)} + \cdots\,. \tag{131}$$

A function W_g which is invariant under general coordinate transformations and contains the previously introduced $W_g^{(2)}$ is

$$W_g = (1/2) \int (dx)(-g)^{1/2} g^{\mu\nu}[\Gamma^{\lambda}_{\mu\kappa}\Gamma^{\kappa}_{\nu\lambda} - \Gamma^{\lambda}_{\mu\nu}\Gamma^{\kappa}_{\lambda\kappa}], \tag{132}$$

where

$$\begin{aligned} \Gamma^{\kappa}_{\mu\nu} &= (1/2)\, g^{\kappa\lambda}(\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu}) \\ &= g^{\kappa\lambda}\Gamma_{\mu\nu\lambda}\,. \end{aligned} \tag{133}$$

The expansions in Eqs. (115) will generate the series in Eq. (131). In particular, we will obtain $W_T^{(3)}$ which is a major portion of the first primitive graviton-graviton interaction; we will discuss this in the next section. Finally, we introduce the modifications of $W_T^{(1)}$,

$$W_T = W_T^{(1)} + W_T^{(2)} + W_T^{(3)} + \cdots \tag{134}$$

which will be generated by the equations

$$\begin{aligned} \delta_g W_T^{(n)} + \delta_c W_T^{(n-1)} &= 0, \qquad n = 2, 3, \ldots, \\ W_T^{(n)}(H) &= 0, \end{aligned} \tag{135}$$

and the only variations involved are those in Eqs. (104) and (106). Tye other part of the first primitive interaction, $W_T^{(2)}$, will be constructed in the next section.

The theory we have so far may be summarized in the statement

$$\langle 0_+ \mid 0_- \rangle^{KJT} = \exp[iw(K, J, T)], \tag{136}$$

where

$$w(K, J, T) = W_m + W_\gamma + W_g + W_K + W_J + W_T\,. \tag{137}$$

When the consequences of the implied action principle are worked out, we obtain not only the series of primitive interactions in which more and more gravitons participate but also all the skeleton interactions found by compounding these primitive interactions. This development is illustrated in Section VII.

VI. First Primitive Graviton-Graviton Interaction

In this section we continue the evaluation of emission amplitudes implied by the first primitive interactions. For this purpose we employ the weak field treatment which characterized the calculations of Sections III and IV. The first primitive graviton-graviton interaction is given by

$$w_3(T) = W_g^{(3)}(h) + W_T^{(2)}(h). \tag{138}$$

The evaluation of the functional $W_T^{(2)}$ follows from the conditions

$$\begin{aligned} \delta_g W_T^{(2)}(h) + \delta_c W_T^{(1)}(h) &= 0, \\ W_T^{(2)}(H) &= 0. \end{aligned} \tag{139}$$

The first of these equations provides the differential equation

$$-\partial_\mu[\delta W_T^{(2)}(h)/\delta h_{\mu\nu}(x)] = \int (dx')\, \Gamma_T^{\nu,\lambda\kappa}(x, x')\, h_{\lambda\kappa}(x'), \tag{140}$$

where

$$\Gamma_T^{\nu,\lambda\kappa}(x, x') = [\eta^{\nu\lambda} T^{\kappa\sigma}(x) + \eta^{\nu\kappa} T^{\lambda\sigma}(x) - \eta^{\nu\sigma} T^{\lambda\kappa}(x)]\, \partial_\sigma\, \delta(x - x'). \tag{141}$$

The second of Eqs. (139) leads to a structure that is similar to Eq. (124):

$$\begin{aligned} W_T^{(2)}(h) = &- \int (dx)(dx')\, X_\nu(x)\, \Gamma_T^{\nu,\lambda\kappa}(x, x')\, h_{\lambda\kappa}(x') \\ &+ (1/2) \int (dx)(dx')\, \partial_\lambda{}' \Gamma_T^{\nu,\lambda\kappa}(x, x')\, X_\nu(x)\, X_\kappa(x'), \end{aligned} \tag{142}$$

or, performing the integration over x',

$$W_T^{(2)}(h) = \int (dx)\, T^{\mu\nu}[(1/2)\, \partial_\mu X^\lambda \partial_\nu X_\lambda - \Gamma^\lambda_{\mu\nu} X_\lambda]. \tag{143}$$

Under infinitesimal gravitational gauge transformations

$$\delta_g \Gamma^\lambda_{\mu\nu} = -\partial_\mu \partial_\nu\, \delta x^\lambda, \qquad \delta_g X_\lambda = \delta x_\lambda\, ; \tag{144}$$

thus, we see from Eq. (143) that the response of $W_T^{(2)}(h)$ to the same transformations is

$$\delta_g W_T^{(2)}(h) = - \int (dx)\, T^{\mu\nu} \Gamma^\lambda_{\mu\nu}\, \delta x_\lambda\, . \tag{145}$$

This in turn cancels the response

$$\delta_c W_T^{(1)}(h) = \int (dx)\, T^{\mu\nu} \delta_c h_{\mu\nu}\, , \tag{146}$$

where $\delta_c h_{\mu\nu}$ is defined by Eqs. (106) and (107) and may be written in the form

$$\delta_c h_{\mu\nu} = \Gamma_{\mu\nu\lambda}\, \delta x^\lambda - \partial_\nu(h_{\mu\lambda}\, \delta x^\lambda) - \partial_\mu(h_{\nu\lambda}\, \delta x^\lambda). \tag{147}$$

We have verified that $W_T^{(2)}(h)$ satisfies the first of Eqs. (139). The functional $W_g^{(3)}(h)$ satisfies a similar equation,

$$\delta_g W_g^{(3)} + \delta_c W_g^{(2)} = 0, \tag{148}$$

which is a consequence of the invariance of $W_g(h)$ under general coordinate transformations. Eq. (148) implies the differential equation

$$-\partial_\mu[\delta W_g^{(3)}/\delta h_{\mu\nu}] = \Gamma^\nu_{\lambda\kappa}[\delta W_g^{(2)}/\delta h_{\lambda\kappa}]. \tag{149}$$

It then follows that the structure

$$t_g^{\mu\nu}(x) = \delta W_g^{(3)}/\delta h_{\mu\nu}(x) + \delta W_T^{(2)}/\delta h_{\mu\nu}(x), \tag{150}$$

satisfies

$$\partial_\mu t_g^{\mu\nu}(x) = 0,$$

in virtue of the weak field equations

$$\delta W_g^{(2)}/\delta h_{\mu\nu}(x) + T^{\mu\nu}(x) = 0. \tag{152}$$

Then the response of $w_3(T)$ to a change in the graviton source may be written as

$$\delta_T w_3(T) = \int (dx)(dx')\, \delta T^{\lambda\kappa}(x')\, D_{\lambda\kappa,\mu\nu}(x'-x)\, t_g^{\mu\nu}(x), \tag{153}$$

which identifies $t_g^{\mu\nu}(x)$ as the effective graviton source implied by the first primitive graviton-graviton interaction. The emission of a graviton, of four-momentum k and polarization index r, from this source is then described by

$$\langle 1_{kr} \mid 0_- \rangle^T = i(d\omega_k)^{1/2}\, e^r_{\mu\nu}(k) \int (dx)\, e^{-ikx} t_g^{\mu\nu}(x). \tag{154}$$

It is interesting and useful to relate the first or field term in $t_g^{\mu\nu}(x)$ to the stress tensor $T_g^{\mu\nu}$ of the noninteracting graviton system described by $W_g^{(2)}$. The latter is defined by

$$\delta' W_g^{(2)} = \int (dx)\, T_g^{\mu\nu}(x)\, \partial_\mu\, \delta x_\nu\,, \tag{155}$$

which implies the use of the transformation Eqs. (42) and (107). Using the expressions (27) and (28), we find that

$$\begin{aligned} T_g^{\mu\nu} = \eta^{\mu\nu} L_g^{(2)} &- (1/2)[\bar{\pi}^{\mu,\lambda\kappa}\partial^\nu h_{\lambda\kappa} + \bar{\pi}^{\nu,\lambda\kappa}\partial^\mu h_{\lambda\kappa}] \\ &+ \partial_\rho[\bar{\Sigma}^{\mu,\rho\nu} + \bar{\Sigma}^{\nu,\rho\mu}] + (h_\kappa{}^\mu \delta_\lambda{}^\nu + h_\kappa{}^\nu \delta_\lambda{}^\mu)\, \partial_\rho \bar{\pi}^{\rho,\lambda\kappa}, \end{aligned} \tag{156}$$

where

$$\bar{\Sigma}^{\mu,\rho\nu} = \bar{\pi}^{\mu,\lambda\kappa}(\delta_\lambda{}^\rho h_\kappa{}^\nu - \delta_\lambda{}^\nu h_\kappa{}^\rho), \tag{157}$$

and

$$\begin{aligned} \bar{\pi}^{\mu,\lambda\kappa} &= \partial L_g^{(2)}/\partial(\partial_\mu h_{\lambda\kappa}) \\ &= \Gamma^{\lambda\kappa\mu} - \bar{\eta}^{\mu\nu,\lambda\kappa}\partial_\nu h - (1/2)\,\eta^{\lambda\kappa}h^\mu. \end{aligned} \tag{158}$$

In the appendix we develop an identity for the purpose of calculating the functional derivatives of $W_g^{(2)}$ and $W_g^{(3)}$. Using this identity we find that

$$\delta W_g^{(3)}/\delta h_{\mu\nu} = T_g^{\mu\nu} + \partial_\lambda\partial_\kappa \Lambda^{\mu\nu,\lambda\kappa}, \tag{159}$$

where

$$\Lambda^{\mu\nu,\lambda\kappa} = \eta^{\mu\kappa}\gamma^{\nu\lambda} - \eta^{\mu\nu}\gamma^{\lambda\kappa} + \eta^{\nu\lambda}\gamma^{\mu\kappa} - \eta^{\kappa\lambda}\gamma^{\mu\nu}, \tag{160}$$

and $\gamma^{\mu\nu}$ is the quadratic term in the expansion

$$(-g)^{1/2}\, g^{\mu\nu} = \eta^{\mu\nu} - 2(h^{\mu\nu} - (1/2)\,\eta^{\mu\nu}h) + 2\gamma^{\mu\nu} + \cdots. \tag{161}$$

Chang [4] has pointed out that the properties

$$\Lambda^{\mu\nu,\lambda\kappa} = \Lambda^{\lambda\kappa,\mu\nu} = -\Lambda^{\mu\kappa,\lambda\nu} = -\Lambda^{\lambda\nu,\mu\kappa}, \tag{162}$$

imply that this term makes no contribution to the integrals of energy-momentum and angular momentum. Furthermore, this term does not contribute to emission (or absorption) amplitudes since

$$e^r_{\mu\nu}(k) \int (dx)\, e^{-ikx}\partial_\lambda\partial_\kappa \Lambda^{\mu\nu,\lambda\kappa}(x) = 0, \tag{163}$$

which follows from $k^2 = 0$ and the properties of $e^r_{\mu\nu}(k)$ in Eqs. (38). Thus, there is essentially no difference between $T_g^{\mu\nu}$ and $\delta W_g^{(3)}/\delta h_{\mu\nu}$.

We now turn our attention to the calculation of the two-graviton emission amplitude implied by $w_3(T)$. This amplitude can be calculated from the form of $w_3(T)$ in Eq. (138). The result is simple but the intermediate steps are tedious. There is another approach which is much simpler and leads, of course, to the same result. In the latter approach we take advantage of the weak field situation by performing the gauge transformation

$$h_{\mu\nu} \to H_{\mu\nu} = h_{\mu\nu} + (1/2)(\partial_\mu X_\nu + \partial_\nu X_\mu). \tag{164}$$

The source term then disappears and the primitive interaction reduces to

$$w_3(T) = W_g^{(3)}(H). \tag{165}$$

The field $H_{\mu\nu}(x)$ may be written in the form

$$H_{\mu\nu}(x) = \int (dx')\, C_{\mu\nu}^{\lambda\kappa}(x - x')\, h_{\lambda\kappa}(x'), \tag{166}$$

where

$$C_{\mu\nu}^{\lambda\kappa}(x - x') = \delta_{(\mu}^{\lambda}\delta_{\nu)}^{\kappa}\, \delta(x - x') - \partial_{(\mu} f_{\nu)}^{\lambda\kappa}(x - x') \tag{167}$$

involves the symmetrization $A_{(\mu}B_{\nu)} = (1/2)(A_\mu B_\nu + A_\nu B_\mu)$. The property

$$\partial_\lambda' C_{\mu\nu}^{\lambda\kappa}(x - x') = 0 \tag{168}$$

guarantees that the emitted gravitons have the correct polarization. Thus, the field associated with a detecting source becomes

$$[H_{\mu\nu}(x)]_1 = \sum_{k,r} \{i(d\omega_k)^{1/2}\, (T_{kr}^*)_1\, e^{-ikx} e_{\lambda\kappa}^r(k)[\delta_{(\mu}^{\lambda}\delta_{\nu)}^{\kappa} - k_{(\mu} S_{\nu)}^{\lambda\kappa}(k;\partial_T)]\}, \tag{169}$$

which indicates that the source supplies the time-like vector $-i\partial_T$. If the emitted gravitons have four-momenta k and k', we may make the replacement

$$S_\nu^{\lambda\kappa}(k;\partial_T) \to (\delta_\nu{}^\lambda k'^\kappa + \delta_\nu{}^\kappa k'^\lambda)/(kk') - k_\nu k'^\lambda k'^\kappa/(kk')^2. \tag{170}$$

This leads to the simple form

$$[H_{\mu\nu}(x)]_1 = \sum_{kr} i(d\omega_k)^{1/2}\, (T_{kr}^*)_1\, E_{\mu\nu}^r(k;k')\, e^{-ikx}, \tag{171}$$

where

$$E_{\mu\nu}^r(k;k') = e_{\lambda\kappa}^r(k)\, P_{(\mu}^{\lambda}(k,k')\, P_{\nu)}^{\kappa}(k,k'), \tag{172}$$

and $P_{\mu\nu}(k, k')$ is the projection operator in Eq. (92). The two-graviton emission amplitude then reads

$$\langle 1_{kr} 1_{k's} \mid 0_- \rangle^T = i(d\omega_k\, d\omega_{k'})^{1/2} \int (dx)(dx')\, e^{-ikx} e^{-ik'x'}$$
$$\times\, E_{\mu\nu}^r(k;k')\, E_{\lambda\kappa}^s(k';k)\, \Omega^{\mu\nu,\lambda\kappa}(x,x'), \tag{173}$$

where

$$\Omega^{\mu\nu,\lambda\kappa}(x,x') = [\delta^2 W_g^{(3)}/\delta h_{\mu\nu}(x)\, \delta h_{\lambda\kappa}(x')]_H, \tag{174}$$

and the subscript "H" reminds us that $\Omega^{\mu\nu,\lambda\kappa}$ is still a linear functional of the gauge-invariant field. The next step is to functionally differentiate Eq. (159) using the expression for $T_g^{\mu\nu}$ in Eq. (156). However, the properties of the polarization tensors in Eqs. (38) and the projection operators in Eqs. (93) imply

$$\begin{aligned} k^\mu E^r_{\mu\nu}(k; k') &= 0 = k'^\mu E^r_{\mu\nu}(k; k'), \\ k^\nu E^r_{\mu\nu}(k; k') &= 0 = k'^\nu E^r_{\mu\nu}(k; k'), \\ \eta^{\mu\nu} E^r_{\mu\nu}(k; k') &= 0. \end{aligned} \tag{175}$$

This means that very little actually has to be calculated. The result is

$$\langle 1_{kr} 1_{k's} \mid 0_- \rangle^T = i(d\omega_k \, d\omega_{k'})^{1/2} \, E^r_{\mu\lambda}(k; k') \, \eta^{\lambda\kappa} E_{\kappa\nu}(k'; k) \, V^{\mu\nu}(k, k'), \tag{176}$$

where

$$V^{\mu\nu}(k, k') = [4(kk') \, P^{\mu\sigma}(k, k') \, P^{\nu\tau}(k, k') + \eta^{\mu\nu} Q^{\sigma\tau}(k, k')] \, H_{\sigma\tau}(k + k') \tag{177}$$

and

$$Q^{\sigma\tau}(k, k') = 2(k'^\sigma k'^\tau + k^\sigma k^\tau) + (k^\sigma k'^\tau + k^\tau k'^\sigma) - 3\eta^{\sigma\tau}(kk'). \tag{178}$$

The property

$$(k + k')_\sigma \, Q^{\sigma\tau}(k, k') = 0 \tag{179}$$

means that we can make the replacement $H_{\sigma\tau} \to h_{\sigma\tau}$ in Eq. (177).

Before closing this section we note one more property of the gauge-invariant polarization tensors:

$$\sum_{r=1,2} E^r_{\mu\nu}(k; k') \, E^r_{\lambda\kappa}(k; k') = \bar{P}_{\mu\nu,\lambda\kappa}(k, k'), \tag{180}$$

where the projection operator $\bar{P}_{\mu\nu,\lambda\kappa}$ is defined in Eq. (96).

VII. Source Terms and Two-Particle Scattering

So far in our development we have applied the prescription for the source terms to the first primitive interactions only. In these simple situations we have shown that the source terms need not be calculated if we use the gauge-invariant fields $H_{\mu\nu}$. In this section we expand the interaction skeleton sufficiently to describe two-particle scattering and demonstrate again that the source terms need not be calculated. This demonstration will clarify the nature of the source terms and make their prescription plausible.

We restrict the system of interest to interacting photons and gravitons since no new features would occur if we included the spinless particles. The interaction skeleton reduces to

$$w(J, T) = W_\gamma + W_J + W_g + W_T. \tag{181}$$

The various primitive interactions are made explicit by expanding the right hand side of (181) according to Eqs. (115):

$$w(J, T) = W_\gamma^{(0)} + W_g^{(2)} + W_R, \tag{182}$$

where W_R represents the remaining terms in the expansion. The action principle then provides the field equations:

$$\begin{aligned} -\delta W_\gamma^{(0)}/\delta A_\mu &= \delta W_R/\delta A_\mu, \\ -\delta W_g^{(2)}/\delta h_{\mu\nu} &= \delta W_R/\delta h_{\mu\nu}. \end{aligned} \tag{183}$$

This system of equations will be solved by introducing an iteration procedure:

$$\begin{aligned} A_\mu &= A_\mu' + A_\mu'' + \cdots, \\ h_{\mu\nu} &= h_{\mu\nu}' + h_{\mu\nu}'' + \cdots, \end{aligned} \tag{184}$$

where A_μ' and $h_{\mu\nu}'$ are the noninteracting fields introduced in Section II and satisfy

$$\begin{aligned} -[\delta W_\gamma^{(0)}/\delta A_\mu]' &= J^\mu, \\ -[\delta W_g^{(2)}/\delta h_{\mu\nu}]' &= T^{\mu\nu} \end{aligned} \tag{185}$$

respectively (the primes mean the functional derivatives are evaluated with the noninteracting fields). The fields A_μ'' and $h_{\mu\nu}''$ represent corrections from the first primitive interactions and satisfy

$$\begin{aligned} -[\delta W_\gamma^{(0)}/\delta A_\mu]'' &= j^\mu = \{\delta[W_\gamma^{(1)} + W_J^{(1)}]/\delta A_\mu\}', \\ -[\delta W_g^{(2)}/\delta h_{\mu\nu}]'' &= t^{\mu\nu} = t_\gamma^{\mu\nu} + t_g^{\mu\nu}. \end{aligned} \tag{186}$$

The elements of $t^{\mu\nu}$ are also evaluated with the noninteracting fields and are the same as those discussed previously.

The iteration procedure (184) is also applied to Eq. (182) and produces the series:

$$w(J, T) = w_2(J) + w_2(T) + w_{2,1}(J, T) + w_3(T) + w_4(J, T) + \cdots, \tag{187}$$

where

$$w_4(J, T) = [W_\gamma^{(2)} + W_J^{(2)}]' + [W_g^{(4)} + W_T^{(3)}]' + \frac{1}{2}\int (dx)[t^{\mu\nu}h_{\mu\nu}'' + j^\mu A_\mu''] \tag{188}$$

describes the various two-particle scattering processes. To make this more explicit we eliminate the fields A''_μ and $h''_{\mu\nu}$; Eqs. (186) have the solutions

$$\begin{aligned} A''_\mu(x) &= \int (dx')\, D_{\mu\nu}(x-x')\, j^\nu(x') + \text{gradients}, \\ h''_{\mu\nu}(x) &= \int (dx')\, D_{\mu\nu,\lambda\kappa}(x-x')\, t^{\lambda\kappa}(x') + \text{gradients}, \end{aligned} \tag{189}$$

respectively. The gradient structures are immaterial to the evaluation of Eq. (188) since the extended sources j^μ and $t^{\mu\nu}$ are conserved. The resulting expression is

$$w_4(J, T) = w_4(T) + w_4(J) + w_{2,2}(J, T), \tag{190}$$

where

$$w_4(T) = [W_g^{(4)} + W_T^{(3)}]' + \tfrac{1}{2} t_g^{\mu\nu} D_{\mu\nu,\lambda\kappa} t_g^{\lambda\kappa} \tag{191}$$

describes graviton-graviton scattering,

$$w_4(J) = \tfrac{1}{2} t_\gamma^{\mu\nu} D_{\mu\nu,\lambda\kappa} t_\gamma^{\lambda\kappa} \tag{192}$$

describes photon-photon scattering, and

$$w_{2,2}(J, T) = [W_\gamma^{(2)} + W_J^{(2)}]' + \tfrac{1}{2} j^\mu D_{\mu\nu} j^\nu + t_\gamma^{\mu\nu} D_{\mu\nu,\lambda\kappa} t_g^{\lambda\kappa} \tag{193}$$

describes photon-graviton scattering. Throughout Eqs. (191)–(193) we have employed a short hand notation in which the tensor indices represent the coordinates as well, e.g.,

$$\frac{1}{2} j^\mu D_{\mu\nu} j^\nu = \frac{1}{2} \int (dx)(dx')\, j^\mu(x)\, D_{\mu\nu}(x-x')\, j^\nu(x'). \tag{194}$$

The last step in this development is the elimination of the noninteracting fields in favor of the sources using the familiar solutions of the field Eqs. (185). An unambiguous elimination requires expressions (191)–(193) to be invariant under the Abelian gauge transformations. The electromagnetic gauge transformations present no problem, so we consider only the gravitational gauge transformations

$$\delta h'_{\mu\nu} = -(1/2)(\partial_\mu\, \delta x_\nu + \partial_\nu\, \delta x_\mu). \tag{195}$$

The response of expression (193) to these transformations may be written as

$$\delta w_{2,2}(J, T) = \delta[W_\gamma^{(2)} + W_J^{(2)}]' + \int (dx)[j^\mu\, \delta A''_\mu + t_\gamma^{\mu\nu}\, \delta h''_{\mu\nu}], \tag{196}$$

in which

$$\begin{aligned} \delta A''_\mu &= F'_{\mu\nu}\,\delta x^\nu + \text{gradients}, \\ \delta h''_{\mu\nu} &= \Gamma'_{\mu\nu\lambda}\,\delta x^\lambda + \text{gradients}. \end{aligned} \tag{197}$$

The form of Eqs. (197) may be obtained from the solutions (189) or by iteration of Eqs. (130) and (147). The other variation in Eq. (196) is evaluated using

$$\partial_\mu\{\delta[W_\gamma^{(2)} + W_J^{(2)}]/\delta h_{\mu\nu}\}' = -\Gamma''^{\nu}_{\lambda\kappa} t_\gamma^{\lambda\kappa} - F''^{\nu}_{\mu} j^\mu \tag{198}$$

which follows from the invariance of $W_\gamma + W_J$ under general coordinate invariance as discussed in Section V. Thus, the total response of $w_{2,2}(J, T)$ is zero. The same procedure applied to $w_4(T)$ and $w_4(J)$ shows that they are also gauge-invariant.

But the invariance of $w_4(J, T)$ means that we can carry out the transformation

$$h'_{\mu\nu}(x) \to H_{\mu\nu}(x) = \int (dx')\, C^{\lambda\kappa}_{\mu\nu}(x - x')\, h'_{\lambda\kappa}(x'), \tag{199}$$

which eliminates the source terms $W_J^{(2)}$ and $W_T^{(3)}$. Furthermore, when the extended sources are evaluated with the gauge-invariant fields they become:

$$\begin{aligned} [j^\mu(x)]_H &= [\delta W_\gamma^{(1)}/\delta A_\mu]_H', \\ [t_\gamma^{\mu\nu}(x)]_H &= \int (dx')[\delta W_\gamma^{(1)}/\delta h_{\lambda\kappa}(x')]'\, C^{\mu\nu}_{\lambda\kappa}(x' - x), \\ [t_g^{\mu\nu}(x)]_H &= \int (dx')[\delta W_g^{(3)}/\delta h_{\lambda\kappa}(x')]_H'\, C^{\mu\nu}_{\lambda\kappa}(x' - x). \end{aligned} \tag{200}$$

Thus, all reference to the source terms is eliminated. The form of the expressions (200) suggests the following transformation of the propagation function:

$$D_{\mu\nu,\lambda\kappa} \to C^{\sigma\tau}_{\mu\nu} C^{\alpha\beta}_{\lambda\kappa} D_{\sigma\tau,\alpha\beta}\,, \tag{201}$$

which employs the notation illustrated in Eq. (194). The transformations (199) and (201) absorb all the projectors $C^{\lambda\kappa}_{\mu\nu}$ and in fact define a particular gauge. Thus, the source terms allow the consistent introduction of a particular gauge and disappear once this purpose is achieved.

Finally, we must refine the definition of the gauge function $f^\mu(x - x')$ used in connection with the first primitive interactions so that it is applicable to the present situation. Thus, if the transformation (199) is to eliminate both source terms $W_J^{(2)}$ and $W_T^{(3)}$ then the symbols $-i\partial_J$ and $-i\partial_T$ must be given the same numerical assignment which we now represent by N_S. By choosing N_S as the four-vector representing the total energy and momentum emitted (absorbed) by the production (detection) sources, we remain consistent with previous assignments and provide a definition that may be applied to more complicated situations. This vector is parallel to the time axis in the rest frame of the particle system, but the resulting gauge function does not lack covariance of form.

VIII. Simple Two-Particle Exchange

The introduction of the primitive interactions requires that we extend the definition of the source functions to values of momenta that are not on the mass shells. This allows two extended sources to interact through the exchange of more than one particle. These new interactions are not among the terms of the infinite series represented by $w(K, J, T)$ in Eq. (137); rather they appear as modifications of those terms. In this section we will discuss two-particle exchanges between extended sources that lead to modifications of the terms in $w_2(K, J, T)$ defined by Eq. (41).

The probability amplitude for the emission of a photon and graviton from a weak extended photon source has been stated in Eqs. (88) and (91). It can be presented in the equivalent form

$$\langle 1_{kr} 1_{pa} \mid 0_- \rangle^J = i(d\omega_k\, d\omega_p)^{1/2}\, e^r_{\mu\nu}(k)\, e_\lambda{}^a(p)\, P^{\mu\lambda}(k, p)\, P^{\nu\kappa}(k, p)\, J_\kappa(P), \quad (202)$$

which makes explicit reference to the source J. The corresponding absorption amplitude is

$$\langle 0_+ \mid 1_{kr} 1_{pa} \rangle = i(d\omega_k\, d\omega_p)^{1/2}\, e^r_{\mu\nu}(k)\, e_\lambda{}^a(p)\, P^{\mu\lambda}(k, p)\, P^{\nu\kappa}(k, p)\, J_\kappa(-P). \quad (203)$$

The contribution to the vacuum amplitude associated with photon and graviton exchange between sources, J_1 and J_2, is

$$\sum_{kr,pa} \langle 0_+ \mid 1_{kr} 1_{pa} \rangle^{1} \langle 1_{kr} 1_{pa} \mid 0_- \rangle^{J_2} = -\int d\omega_p\, d\omega_k\, J_1{}^\mu(-P)\, P_{\mu\nu}(k, p)\, J_2{}^\nu(P), \quad (204)$$

in which the polarization sums have been removed by the use of Eqs. (23) and (36). To make this expression take on the appearance of being a modification of $w_2(J)$, we introduce a unit factor in the form

$$(2\pi)^3 \int dM^2\, d\omega_P\, \delta(p + k - P), \qquad -P^2 = M^2 > 0. \quad (205)$$

Then the two-particle summation becomes

$$-\int dM^2\, d\omega_P\, J_1{}^\mu(-P)\, I_{\mu\nu}(P)\, J_2{}^\nu(P), \quad (206)$$

where

$$I^{\mu\nu}(P) = (2\pi)^3 \int d\omega_p \, d\omega_k \, \delta(p + k - P) \, P^{\mu\nu}(k, p) \tag{207}$$

satisfies

$$P_\mu I^{\mu\nu}(P) = 0. \tag{208}$$

This property of $I^{\mu\nu}(P)$ means that it only has spatial components in the rest frame of the timelike vector P. A quick calculation in this frame produces the result

$$I^{\mu\nu}(P) = (1/24\pi^2) \, N^{\mu\nu}(P), \tag{209}$$

where

$$N^{\mu\nu}(P) = \eta^{\mu\nu} + P^\mu P^\nu / M^2. \tag{210}$$

The contribution to the vacuum amplitude becomes

$$-(G/3\pi) \int dM^2 \, d\omega_P \, J_1^\mu(-P) \, \eta_{\mu\nu} J_2^\nu(P). \tag{211}$$

We have introduced the Newtonian gravitational constant G by lifting the restriction $8\pi G = 1$.

The result in expression (211) is restricted to a specific spatio-temporal arrangement of the sources. But the source functions $J_1^\mu(-P)$ and $J_2^\mu(P)$, defined for $-P^2 > 0$, are only elements of the general source function $J^\mu(x)$ which acts in space-time. Another element, which refers to $-P^2 = 0$, is involved in

$$i \int (dx)(dx') \, J_1^\mu(x) \, D_{\mu\nu}(x - x') \, J_2^\nu(x') = - \int d\omega_P \, J_1^\mu(-P) \, \eta_{\mu\nu} J_2^\nu(P) \tag{212}$$

which is the contribution to the vacuum amplitude associated with photon propagation between weak production and detection sources. The various elements are united in the generalization

$$w_2(J) \to (1/2) \int (dx)(dx') \, J^\mu(x) \, \bar{D}_{\mu\nu}(x - x') \, J^\nu(x'). \tag{213}$$

Our previous results are contained in the modified propagation function

$$\bar{D}_{\mu\nu}(x - x') = \int \frac{(dp)}{(2\pi)^4} \, e^{ip(x-x')} \bar{D}_{\mu\nu}(p), \tag{214}$$

where

$$\bar{D}_{\mu\nu}(p) = \frac{\eta_{\mu\nu}}{p^2 - i\epsilon} - p^2 G \int_{\to 0}^{\infty} \frac{dM}{M} \frac{(2/3\pi) \, \eta_{\mu\nu}}{p^2 + M^2 - i\epsilon} \, . \tag{215}$$

The lower limit of the integral indicates some procedure must be introduced to deal with the infrared problem. An alternative version of the propagation function which involves only the first term of the expression

$$-p^2 \int_{\to 0}^{\infty} \frac{dM}{M} \frac{1}{p^2 + M^2 - i\epsilon} = \int_{\to 0}^{\infty} M\, dM \left[\frac{1}{p^2 + M^2 - i\epsilon} - \frac{1}{M^2}\right] \quad (216)$$

has been rejected since it does not exist. The second term in Eq. (216) is independent of p^2; thus, its contribution to the vacuum amplitude is a phase factor which has no physical consequences since it effects neither the vacuum-persistance probability nor the particle-mediated coupling of different sources.

The generalization in Eq. (213) implies a modification of the Coulomb potential:

$$\bar{\mathscr{D}}(x - x') = - \int_{-\infty}^{+\infty} dx^0\, \bar{D}_{00}(x - x'). \quad (217)$$

Both versions of the propagation function give the same result [1]:

$$4\pi\bar{\mathscr{D}}(x - x') = 1/r + 2G/3\pi r^3, \quad (218)$$

provided $r = |\mathbf{x} - \mathbf{x}'|$ is not zero. Of course, the quadrupole term is not within the range of contemporary experimentation.

The contributions to the vacuum amplitude associated with two-particle exchange between extended graviton sources lead to modifications of the function $w_2(T)$. We provide a numerical supplement to Schwinger's qualitative discussion of these modifications [1]. The modified graviton propagator has the structure

$$\bar{D}_{\mu\nu,\lambda\kappa}(k) = \frac{\bar{\eta}_{\mu\nu,\lambda\kappa}}{k^2 - i\epsilon} - k^2 G \int_{\to 0}^{\infty} \frac{dM}{M} \frac{\sigma_{\mu\nu,\lambda\kappa}(M)}{k^2 + M^2 - i\epsilon}, \quad (219)$$

which is analogous in form to the photon propagator defined by Eq. (215). The function $w_2(T)$ is then replaced by

$$\bar{w}_2(T) = \frac{\kappa}{2} \int (dx)(dx')\, T^{\mu\nu}(x)\, \bar{D}_{\mu\nu,\lambda\kappa}(x - x')\, T^{\lambda\kappa}(x'), \quad (220)$$

where

$$\bar{D}_{\mu\nu,\lambda\kappa}(x - x') = \int \frac{(dk)}{(2\pi)^4} e^{ik(x-x')} \bar{D}_{\mu\nu,\lambda\kappa}(k), \quad (221)$$

and $\kappa = 8\pi G$ indicates that we are now using the units of Ref. [1]. The relationship between $\sigma_{\mu\nu,\lambda\kappa}(M)$ and the various two-particle exchange contributions to the vacuum amplitude is

$$-\frac{\kappa^2}{16\pi} \int dM^2\, d\omega_k\, T_1^{\mu\nu}(-k)\, \sigma_{\mu\nu,\lambda\kappa}(M)\, T_2^{\lambda\kappa}(k) = v^{(0)} + v^{(1)} + v^{(2)} \quad (222)$$

in which $-k^2 = M^2$. The contributions to the vacuum amplitude have been designated as follows: $v^{(0)}$ is the summation over the material particle states,

$$v^{(0)} = (1/2) \sum_{p,p'} \langle 0_+ \mid 1_p 1_{p'} \rangle^{T_1} \langle 1_p 1_{p'} \mid 0_- \rangle^{T_2}, \tag{223}$$

$v^{(1)}$ is the summation over the photon states,

$$v^{(1)} = (1/2) \sum_{pa,p'b} \langle 0_+ \mid 1_{pa} 1_{p'b} \rangle^{T_1} \langle 1_{pa} 1_{p'b} \mid 0_- \rangle^{T_2}, \tag{224}$$

and $v^{(2)}$ is the summation over the graviton states,

$$v^{(2)} = (1/2) \sum_{pr,p's} \langle 0_+ \mid 1_{pr} 1_{p's} \rangle^{T_1} \langle 1_{pr} 1_{p's} \mid 0_- \rangle^{T_2}. \tag{225}$$

Since the contributions are additive, we write

$$\sigma_{\mu\nu,\lambda\kappa}(M) = \sum_{i=0}^{2} \sigma^{(i)}_{\mu\nu,\lambda\kappa}(M). \tag{226}$$

The probability amplitude for emission of two material particles from a weak source has already been stated in a form equivalent to

$$\langle 1_p 1_{p'} \mid 0_- \rangle^T = i \frac{\kappa}{2} (d\omega_p \, d\omega_{p'})^{1/2} P_{\mu\nu}(p, p') \, \overline{T}^{\mu\nu}(k) \tag{227}$$

where

$$\overline{T}^{\mu\nu} = \bar{\eta}^{\mu\nu,\lambda\kappa} T_{\lambda\kappa} \, . \tag{228}$$

The two-particle summation becomes

$$v^{(0)} = -\frac{\kappa^2}{8} \int d\omega_p \, d\omega_{p'} [\overline{T}_1^{\mu\nu}(-k) \, P_{\mu\nu}(p, p') \, P_{\lambda\kappa}(p, p') \, \overline{T}_2^{\lambda\kappa}(k)]. \tag{229}$$

We concentrate our attention on the total momentum k by introducing a unit factor in the form

$$(2\pi)^3 \int dM^2 \, d\omega_k \, \delta(p + p' - k), \qquad -k^2 = M^2 > (2m)^2. \tag{230}$$

Then

$$v^{(0)} = -\frac{\kappa^2}{16\pi} \int dM^2 \, d\omega_k \, \overline{T}_1^{\mu\nu}(-k) \, I^{(0)}_{\mu\nu,\lambda\kappa}(k) \, \overline{T}_2^{\lambda\kappa}(k), \tag{231}$$

where

$$I^{(0)}_{\mu\nu,\lambda\kappa}(k) = (2\pi)^4 \int d\omega_p \, d\omega_{p'} \, \delta(p + p' - k) \, P_{\mu\nu}(p, p') \, P_{\lambda\kappa}(p, p'). \tag{232}$$

The property of $P_{\mu\nu}(p, p')$ in Eq. (74) implies that $I^{(0)}_{\mu\nu,\lambda\kappa}(k)$ has only spatial components in the rest frame of the time-like vector $\mathbf{k}$. The calculation of $I^{(0)}_{\mu\nu,\lambda\kappa}(k)$ in this frame involves a sum over all solid angles which is facilitated by the use of

$$\int \frac{d\Omega}{4\pi} u^k u^l u^m u^n = \frac{1}{15} [\eta^{kl}\eta^{mn} + \eta^{kn}\eta^{lm} + \eta^{km}\eta^{ln}], \tag{233}$$

where u^k, $k = 1, 2, 3$, are the components of a unit vector. The result is

$$I^{(0)}_{\mu\nu,\lambda\kappa}(k) = A^{(0)}(M) \, 4N_{\mu\nu}(k) \, N_{\lambda\kappa}(k) + B^{(0)}(M) \, \Xi_{\mu\nu,\lambda\kappa}(k), \tag{234}$$

where

$$\begin{aligned} A^{(0)}(M) &= \frac{1}{72\pi} \left[1 - \left(\frac{2m}{M}\right)^2\right]^{1/2} \left[1 + \frac{2m^2}{M^2}\right]^2 \\ B^{(0)}(M) &= \frac{1}{60\pi} \left[1 - \left(\frac{2m}{M}\right)^2\right]^{1/2} \left[1 - \left(\frac{2m}{M}\right)^2\right]^2, \end{aligned} \tag{235}$$

and

$$\Xi_{\mu\nu,\lambda\kappa}(k) = (1/2)[N_{\mu\lambda}(k) \, N_{\nu\kappa}(k) + N_{\mu\kappa}(k) \, N_{\nu\lambda}(k) - (2/3) \, N_{\mu\nu}(k) \, N_{\lambda\kappa}(k)]. \tag{236}$$

$N_{\mu\nu}$ is defined by Eq. (210). Substitution of Eq. (234) into Eq. (231) gives

$$v^{(0)} = -\frac{\kappa^2}{16\pi} \int dM^2 \, d\omega_k \, T_1^{\mu\nu}(-k) \, \sigma^{(0)}_{\mu\nu,\lambda\kappa}(M) \, T_2^{\lambda\kappa}(k), \tag{237}$$

where

$$\sigma^{(0)}_{\mu\nu,\lambda\kappa}(M) = A^{(0)}(M) \, \eta_{\mu\nu}\eta_{\lambda\kappa} + B^{(0)}(M) \, \xi_{\mu\nu,\lambda\kappa'}, \tag{238}$$

and

$$\xi_{\mu\nu,\lambda\kappa} = (1/2)[\eta_{\mu\lambda}\eta_{\nu\kappa} + \eta_{\mu\kappa}\eta_{\nu\lambda} - (2/3) \, \eta_{\mu\nu}\eta_{\lambda\kappa}]. \tag{239}$$

The probability amplitude for emission of two photons from a weak source, Eq. (95), can be written in the form

$$\langle 1_{pa} 1_{p'b} \mid 0_- \rangle^T = -i\kappa (d\omega_p \, d\omega_{p'})^{1/2} \, e_\mu{}^a(p) \, e_\nu{}^b(p') \, \bar{P}^{\mu\nu,\lambda\kappa}(p, p') \, \bar{T}_{\lambda\kappa}(k). \tag{240}$$

Then the two-photon summation becomes

$$v^{(1)} = -\frac{\kappa^2}{2} \int d\omega_p \, d\omega_{p'} \, \bar{T}_1^{\mu\nu}(-k) \, \bar{P}_{\mu\nu,\lambda\kappa}(p, p') \, \bar{T}_2^{\lambda\kappa}(k). \tag{241}$$

Again we introduce a unit factor in the form (230), but in this case $-k^2 = M^2 > 0$. A calculation in the rest frame of the vector $\mathbf{k}$ verifies that

$$I^{(1)}_{\mu\nu,\lambda\kappa}(k) = 64\pi^4 \int d\omega_p\, d\omega_{p'}\, \delta(p + p' - k)\, \bar{P}_{\mu\nu,\lambda\kappa}(p, p') \tag{242}$$

is given by

$$I^{(1)}_{\mu\nu,\lambda\kappa}(k) = (1/5\pi)\, \Xi_{\mu\nu,\lambda\kappa}(k). \tag{243}$$

The property

$$\eta^{\mu\nu} I^{(1)}_{\mu\nu,\lambda\kappa}(k) = 0 \tag{244}$$

can be traced back to a similar property possessed by $T^{\mu\nu}_{\gamma}$ which is defined by Eq. (76). The desired result is

$$\sigma^{(1)}_{\mu\nu,\lambda\kappa}(M) = A^{(1)}(M)\, \eta_{\mu\nu}\eta_{\lambda\kappa} + B^{(1)}(M)\, \xi_{\mu\nu,\lambda\kappa}\,, \tag{245}$$

where

$$A^{(1)}(M) = 0, \qquad B^{(1)}(M) = 1/5\pi. \tag{246}$$

The probability amplitude for emission of two gravitons has been stated in Eq. (176). For present purposes, it should be multiplied by κ and the k's changed to p's. Then the two graviton summation becomes

$$v^{(2)} = -\frac{\kappa^2}{4} \int d\omega_p\, d\omega_{p'} [V_1^{\mu\nu}(p, p')\, P_{\mu\nu}(p, p')\, P_{\lambda\kappa}(p, p')\, V_2^{\lambda\kappa}(p, p')], \tag{247}$$

in which the polarization summations were performed by using Eq. (180). A more explicit form is

$$v^{(2)} = -\kappa^2 \int d\omega_p\, d\omega_{p'} [\bar{T}_1^{\mu\nu}(-k)\, G_{\mu\nu}(p, p')\, G_{\lambda\kappa}(p, p')\, \bar{T}_2^{\lambda\kappa}(k)], \tag{248}$$

$$G_{\mu\nu}(p, p') = (1/2)[\eta_{\mu\nu} + (p_\mu p_\nu' + p_\nu p_\mu')/(pp') - 2(p_\mu p_\nu + p_\mu' p_\nu')/(pp')], \tag{249}$$

and $k = p + p'$. We now introduce the unit factor and consider the integral

$$I^{(2)}_{\mu\nu,\lambda\kappa}(k) = 128\pi^4 \int d\omega_p\, d\omega_{p'}\, \delta(p + p' - k)\, G_{\mu\nu}(p, p')\, G_{\lambda\kappa}(p, p'). \tag{250}$$

The property $k^\mu G_{\mu\nu}(p, p') = 0$ means that $I^{(2)}_{\mu\nu,\lambda\kappa}(k)$ only has spatial components in the rest frame of the time-like vector $\mathbf{k}$. A calculation in this frame produces

$$I^{(2)}_{\mu\nu,\lambda\kappa}(k) = \frac{1}{\pi}\, N_{\mu\nu}(k)\, N_{\lambda\kappa}(k) + \frac{3}{10\pi}\, \Xi_{\mu\nu,\lambda\kappa}(k). \tag{251}$$

Thus the desired result is

$$\sigma^{(2)}_{\mu\nu,\lambda\kappa}(M) = A^{(2)}(M)\,\eta_{\mu\nu}\eta_{\lambda\kappa} + B^{(2)}(M)\,\xi_{\mu\nu,\lambda\kappa}\,, \tag{252}$$

where

$$A^{(2)}(M) = \frac{1}{4\pi}\,, \qquad B^{(2)}(M) = \frac{3}{10\pi}\,. \tag{253}$$

The modified Newtonian potential due to the exchange of the massless particles is

$$4\pi\bar{\mathscr{D}}_g(r) = 1/r + 4G/15\pi r^3 + 9G/10\pi r^3, \tag{254}$$

where the second and third terms refer to photon and graviton exchange, respectively.

Finally, we must show that our results satisfy probability requirements. The forms

$$T^*_{\mu\nu}(k)\,\eta^{\mu\nu}\eta^{\lambda\kappa}T_{\lambda\kappa}(k), \qquad T^*_{\mu\nu}(k)\,\xi^{\mu\nu,\lambda\kappa}T_{\lambda\kappa}(k), \tag{255}$$

are positive definite; thus the probability requirements are

$$A^{(i)}(M) > 0, \qquad B^{(i)}(M) > 0, \qquad i = 0, 1, 2. \tag{256}$$

Reference to Eqs. (235), (246), and (253) shows that our results satisfy these requirements.

IX. Conclusion

A brief summary is achieved by remarking that we have developed Schwinger's theory to about the same extent that Feynman developed his theory in his Warsaw lectures [5]. That is, we have a gauge covariant scheme for calculating skeleton interactions and simple two-particle exchanges that satisfy probability requirements. Can we extend our results to more complicated multiparticle exchanges as DeWitt [6] has extended Feynman's results? This is an interesting question that deserves further attention.

Appendix

We will sketch the derivation of a well-known identity and adapt it to the calculation of those functional derivatives used in this paper. Except for the emphasis on the field $h_{\mu\nu}$, our discussion follows that of Gupta [7].

The fact that Lagrange function

$$L_g/(-g)^{1/2} = (1/2)\,g^{\mu\nu}[\Gamma^\lambda_{\mu\kappa}\Gamma^\kappa_{\nu\lambda} - \Gamma^\lambda_{\mu\nu}\Gamma^\kappa_{\lambda\kappa}] \tag{A.1}$$

is invariant under linear coordinate transformations implies the identity

$$\delta_\nu{}^\mu L_g = 2\pi^{\kappa,\lambda\mu}\,\partial_\kappa h_{\lambda\nu} + \pi^{\mu,\lambda\kappa}\,\partial_\nu h_{\lambda\kappa} + (\eta_{\nu\lambda} + 2h_{\nu\lambda})\,\partial L_g/\partial h_{\mu\lambda}\,, \tag{A.2}$$

where

$$\pi^{\lambda,\mu\nu} = \partial L_g/\partial(\partial_\lambda h_{\mu\nu}). \tag{A.3}$$

The use of this identity allows the quantity

$$\tau^{\mu\nu} = \eta^{\mu\nu} L_g - (1/2)[\pi^{\mu,\lambda\kappa}\,\partial^\nu h_{\lambda\kappa} + \pi^{\nu,\lambda\kappa}\,\partial^\mu h_{\lambda\kappa}] \tag{A.4}$$

to be written as

$$\tau^{\mu\nu} = [\delta_\kappa{}^\mu\delta_\lambda{}^\nu + h_\kappa{}^\mu\delta_\lambda{}^\nu + h_\kappa{}^\nu\delta_\lambda{}^\mu]\,\delta W_g/\delta h_{\lambda\kappa} + \partial_\rho[\eta^{\nu\lambda}R_\lambda^{\rho\mu} + \eta^{\mu\lambda}R_\lambda^{\rho\nu}], \tag{A.5}$$

where

$$R_\lambda^{\rho\mu} = (1/2)\,\pi^{\rho,\mu\kappa}(\eta_{\kappa\lambda} + 2h_{\kappa\lambda}). \tag{A.6}$$

Now we form the quantity

$$\theta^{\mu\nu} = \tau^{\mu\nu} + \partial_\rho[\Sigma^{\mu,\rho\nu} + \Sigma^{\nu,\rho\mu}] - [\delta_\kappa{}^\mu\delta_\lambda{}^\nu + h_\kappa{}^\mu\delta_\lambda{}^\nu + h_\kappa{}^\nu\delta_\lambda{}^\mu]\,\delta W_g/\delta h_{\lambda\kappa}\,, \tag{A.7}$$

where

$$\begin{aligned} \Sigma^{\mu,\rho\nu} &= \pi^{\mu,\lambda\kappa}(\delta_\lambda{}^\rho h_\kappa{}^\nu - \delta_\lambda{}^\nu h_\kappa{}^\rho) \\ &= \eta^{\nu\lambda}R_\lambda^{\mu\rho} - \eta^{\rho\lambda}R_\lambda^{\mu\nu}. \end{aligned} \tag{A.8}$$

An alternative version of $\theta^{\mu\nu}$ is:

$$\theta^{\mu\nu} = \partial_\rho[\eta^{\mu\lambda}(R_\lambda^{\rho\nu} + R_\lambda^{\nu\rho}) + \eta^{\nu\lambda}(R_\lambda^{\mu\rho} + R_\lambda^{\rho\mu}) - \eta^{\rho\lambda}(R_\lambda^{\mu\nu} + R_\lambda^{\nu\mu})]. \tag{A.9}$$

Evaluating $R_\lambda^{\mu\nu}$ from (A.6) and (A.1), and substituting in (A.9), we find that

$$\theta^{\mu\nu} = (1/2)\,\partial_\lambda\partial_\kappa[(-g)^{1/2}\,(\eta^{\mu\nu}g^{\lambda\kappa} - \eta^{\mu\kappa}g^{\nu\lambda} + \eta^{\lambda\kappa}g^{\mu\nu} - \eta^{\nu\lambda}g^{\mu\kappa})]. \tag{A.10}$$

By equating the two expressions for $\theta^{\mu\nu}$ in Eqs. (A.7) and (A.10), and using Eq. (A.4) for $\tau^{\mu\nu}$, we obtain

$$\begin{aligned} \delta W_g/h_{\mu\nu} = {} & \eta^{\mu\nu}L_g - (1/2)[\pi^{\mu,\lambda\kappa}\,\partial^\nu h_{\lambda\kappa} + \pi^{\nu,\lambda\kappa}\,\partial^\mu h_{\lambda\kappa}] \\ & + \partial_\rho[\Sigma^{\mu,\rho\nu} + \Sigma^{\nu,\rho\mu}] - (h_\kappa{}^\mu\delta_\lambda{}^\nu + h_\kappa{}^\nu\delta_\lambda{}^\mu)\,\delta W_g/\delta h_{\lambda\kappa} \\ & + (1/2)\,\partial_\lambda\partial_\kappa[(-g)^{1/2}\,(\eta^{\mu\kappa}g^{\nu\lambda} - \eta^{\mu\nu}g^{\lambda\kappa} + \eta^{\nu\lambda}g^{\mu\kappa} - \eta^{\lambda\kappa}g^{\mu\nu})]. \end{aligned} \tag{A.11}$$

When we apply the expansions in Eqs. (115) to this identity we easily obtain the functional derivatives of $W_g^{(2)}$ and $W_g^{(3)}$.

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