8 The Unruh effect

Summary: Uniformly accelerated motion. The Rindler spacetime in 1+1 dimensions. Quantization of massless scalar field. The Rindler and the Minkowski vacua. Density of particles. The Unruh temperature.

The Unruh effect predicts the detection of particles in vacuum by an accelerated observer. In this chapter we consider the simplest case when the observer moves with a constant acceleration through the Minkowski spacetime. Even though the field is in the vacuum state, the observer finds a distribution of particles characteristic of a thermal bath of blackbody radiation.

8.1 Kinematics of uniformly accelerated motion

First we consider the trajectory of an object moving with a constant acceleration in the Minkowski spacetime. A model of this situation is a spaceship with an infinite energy supply and a propulsion engine that exerts a constant force (but moves with the ship). The resulting motion of the spaceship is such that the acceleration of the ship in its own frame of reference (the *proper acceleration*) is constant. This is the natural definition of a uniformly accelerated motion in a relativistic theory. (An object cannot move with $d\mathbf{v}/dt =$ const for all time because velocities must be smaller than the speed of light, $|\mathbf{v}| < 1$.)

We now introduce the reference frames that will play a major role in our considerations: the laboratory frame, the proper frame, and the comoving frame. The laboratory frame is the usual inertial reference frame with the coordinates (t, x, y, z). The proper frame is the accelerated system of reference that moves together with the observer; we shall also call it the accelerated frame. The comoving frame defined at a time t_0 is the *inertial* frame in which the accelerated observer is instantaneously at rest at $t = t_0$. (Thus the term *comoving frame* actually refers to a different frame for each t_0.)

By definition, the observer's proper acceleration at time $t = t_0$ is the 3-acceleration measured in the comoving frame at time t_0. We consider a uniformly accelerated observer whose proper acceleration is time-independent and equal to a given 3-vector $\mathbf{a}$. The trajectory of such an observer may be described by a worldline $x^\mu(\tau)$, where τ is the proper time measured by the observer. The proper time parametrization implies the condition

$$u^\mu u_\mu = 1, \quad u^\mu \equiv \frac{dx^\mu}{d\tau}. \tag{8.1}$$

It is a standard result that the 4-acceleration in the laboratory frame,

$$a^\mu \equiv \frac{du^\mu}{d\tau} = \frac{d^2x^\mu}{d\tau^2},$$

is related to the three-dimensional proper acceleration $\mathbf{a}$ by

$$a^\mu a_\mu = -\left|\mathbf{a}\right|^2. \tag{8.2}$$

Derivation of Eq. (8.2). Let $u^\mu(\tau)$ be the observer's 4-velocity and let t_c be the time variable in the comoving frame defined at $\tau = \tau_0$; this is the time measured by an *inertial* observer moving with the constant velocity $u^\mu(\tau_0)$. We shall show that the 4-acceleration $a^\mu(\tau)$ in the comoving frame has components $\left(0, a^1, a^2, a^3\right)$, where a^i are the components of the acceleration 3-vector $\mathbf{a} \equiv d^2\mathbf{x}/dt_c^2$ measured in the comoving frame. It will then follow that Eq. (8.2) holds in the comoving frame, and hence it holds also in the laboratory frame since the Lorentz-invariant quantity $a^\mu a_\mu$ is the same in all frames.

Since the comoving frame moves with the velocity $u^\mu(\tau_0)$, the 4-vector $u^\mu(\tau_0)$ has the components $(1, 0, 0, 0)$ in that frame. The derivative of the identity $u^\mu(\tau)u_\mu(\tau) = 1$ with respect to τ yields $a^\mu(\tau)u_\mu(\tau) = 0$, therefore $a^0(\tau_0) = 0$ in the comoving frame. Since $dt_c = u^0(\tau)d\tau$ and $u^0(\tau_0) = 1$, we have

$$\frac{d^2x^\mu}{dt_c^2} = \frac{1}{u^0}\frac{d}{d\tau}\left[\frac{1}{u^0}\frac{dx^\mu}{d\tau}\right] = \frac{d^2x^\mu}{d\tau^2} + \frac{dx^\mu}{d\tau}\frac{d}{d\tau}\frac{1}{u^0}.$$

It remains to compute

$$\frac{d}{d\tau}\frac{1}{u^0(\tau_0)} = -\left[u^0(\tau_0)\right]^{-2}\left.\frac{du^0}{d\tau}\right|_{\tau=\tau_0} = -a^0(\tau_0) = 0,$$

and it follows that $d^2x^\mu/d\tau^2 = d^2x^\mu/dt_c^2 = \left(0, a^1, a^2, a^3\right)$ as required. (Self-test question: why is $a^\mu = du^\mu/d\tau \neq 0$ even though $u^\mu = (1, 0, 0, 0)$ in the comoving frame?)

We now derive the trajectory $x^\mu(\tau)$ of the accelerated observer. Without loss of generality, we may assume that the acceleration is parallel to the x axis, $\mathbf{a} \equiv (a, 0, 0)$, where $a > 0$, and that the observer moves only in the x direction. Then the coordinates y and z of the observer remain constant and only the functions $x(\tau)$, $t(\tau)$ need to be computed. From Eqs. (8.1)-(8.2) it is straightforward to derive the general solution

$$x(\tau) = x_0 - \frac{1}{a} + \frac{1}{a}\cosh a\tau, \quad t(\tau) = t_0 + \frac{1}{a}\sinh a\tau. \tag{8.3}$$

This trajectory has zero velocity at $\tau = 0$ (which implies $x = x_0$, $t = t_0$).

Derivation of Eq. (8.3). Since $a^\mu = du^\mu/d\tau$ and $u^2 = u^3 = 0$, the components u^0, u^1 of the velocity satisfy

$$\left(\frac{du^0}{d\tau}\right)^2 - \left(\frac{du^1}{d\tau}\right)^2 = -a^2,$$
$$\left(u^0\right)^2 - \left(u^1\right)^2 = 1.$$

We may assume that $u_0 > 0$ (the time τ grows together with t) and that $du^1/d\tau > 0$, since the acceleration is in the positive x direction. Then

$$u^0 = \sqrt{1 + (u^1)^2}; \quad \frac{du^1}{d\tau} = a\sqrt{1 + (u^1)^2}.$$

The solution with the initial condition $u^1(0) = 0$ is

$$u^1(\tau) \equiv \frac{dx}{d\tau} = \sinh a\tau, \quad u^0(\tau) \equiv \frac{dt}{d\tau} = \cosh a\tau.$$

After an integration we obtain Eq. (8.3).

The trajectory (8.3) has a simpler form if we choose the initial conditions $x(0) = a^{-1}$ and $t(0) = 0$. Then the worldline is a branch of the hyperbola $x^2 - t^2 = a^{-2}$ (see Fig. 8.1). At large $|t|$ the worldline approaches the lightcone. The observer comes in from $x = +\infty$, decelerates and stops at $x = a^{-1}$, and then accelerates back towards infinity. In the comoving frame of the observer, this motion takes infinite proper time, from $\tau = -\infty$ to $\tau = +\infty$.

From now on, we drop the coordinates y and z and work in the 1+1-dimensional spacetime (t, x).

8.1.1 Coordinates in the proper frame

To describe quantum fields as seen by an accelerated observer, we need to use the *proper coordinates* (τ, ξ), where τ is the proper time and ξ is the distance measured by the observer. The proper coordinate system (τ, ξ) is related to the laboratory frame (t, x) by some transformation functions $\tau(t, x)$ and $\xi(t, x)$ which we shall now determine.

The observer's trajectory $t(\tau), x(\tau)$ should correspond to the line $\xi = 0$ in the proper coordinates. Let the observer hold a rigid measuring stick of proper length ξ_0, so that the entire stick accelerates together with the observer. Then the stick is instantaneously at rest in the comoving frame and the far endpoint of the stick has the proper coordinates (τ, ξ_0) at time τ. We shall derive the relation between the coordinates (t, x) and (τ, ξ) by computing the laboratory coordinates (t, x) of the far end of the stick as functions of τ and ξ_0.

In the comoving frame at time τ, the stick is represented by the 4-vector $s^\mu_{(\text{com})} \equiv (0, \xi_0)$ connecting the endpoints $(\tau, 0)$ and (τ, ξ_0). This comoving frame is an inertial system of reference moving with the 4-velocity $u^\mu(\tau) = dx^\mu/d\tau$. Therefore the coordinates $s^\mu_{(\text{lab})}$ of the stick in the laboratory frame can be found by applying the inverse Lorentz transformation to the coordinates $s^\mu_{(\text{com})}$:

$$\begin{bmatrix} s^0_{(\text{lab})} \\ s^1_{(\text{lab})} \end{bmatrix} = \frac{1}{\sqrt{1 - v^2}} \begin{pmatrix} 1 & v \\ v & 1 \end{pmatrix} \begin{bmatrix} s^0_{(\text{com})} \\ s^1_{(\text{com})} \end{bmatrix} = \begin{pmatrix} u^0 & u^1 \\ u^1 & u^0 \end{pmatrix} \begin{bmatrix} s^0_{(\text{com})} \\ s^1_{(\text{com})} \end{bmatrix} = \begin{bmatrix} u^1 \xi \\ u^0 \xi \end{bmatrix},$$

where $v \equiv u^1/u^0$ is the velocity of the stick in the laboratory system. The stick is attached to the observer moving along $x^\mu(\tau)$, so the proper coordinates (τ, ξ) of the

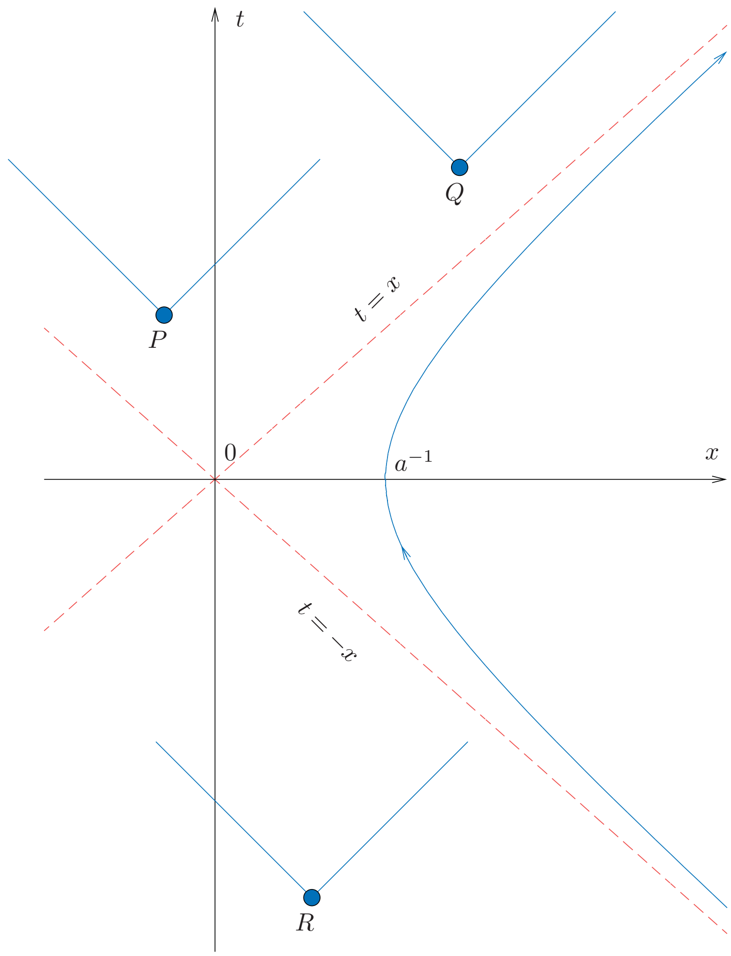

Figure 8.1: The worldline of a uniformly accelerated observer (proper acceleration $a \equiv |\mathbf{a}|$) in the Minkowski spacetime. The dashed lines show the lightcone. The observer cannot receive any signals from the events P, Q and cannot send signals to R.

far end of the stick correspond to the laboratory coordinates

$$t(\tau,\xi) = x^0(\tau) + s^0_{(\text{lab})} = x^0(\tau) + \frac{dx^1(\tau)}{d\tau}\xi, \tag{8.4}$$

$$x(\tau,\xi) = x^1(\tau) + s^1_{(\text{lab})} = x^1(\tau) + \frac{dx^0(\tau)}{d\tau}\xi. \tag{8.5}$$

Note that the relations (8.4)-(8.5) specify the proper frame for *any* trajectory $x^{0,1}(\tau)$ in the 1+1-dimensional Minkowski spacetime.

Now we can substitute Eq. (8.3) into the above relations to compute the proper coordinates for a uniformly accelerated observer. We choose the initial conditions $x^0(0) = 0$, $x^1(0) = a^{-1}$ for the observer's trajectory and obtain

$$t(\tau,\xi) = \frac{1+a\xi}{a}\sinh a\tau, \tag{8.6}$$

$$x(\tau,\xi) = \frac{1+a\xi}{a}\cosh a\tau. \tag{8.7}$$

The converse relations are

$$\tau(t,x) = \frac{1}{2a}\ln\frac{x+t}{x-t},$$
$$\xi(t,x) = -a^{-1} + \sqrt{x^2 - t^2}.$$

The horizon

From Eqs. (8.6)-(8.7) it can be seen that the coordinates (τ, ξ) vary in the intervals $-\infty < \tau < +\infty$ and $-a^{-1} < \xi < +\infty$. In particular, for $\xi < -a^{-1}$ we would find $\partial t/\partial\tau < 0$, i.e. the direction of time t would be opposite to that of τ. One can verify that an accelerated observer cannot measure distances longer than a^{-1} in the direction opposite to the acceleration, for instance, the distances to the events P and Q in Fig. 8.1. A measurement of the distance to a point requires to place a clock at that point and to synchronize that clock with the observer's clock. However, the observer cannot synchronize clocks with the events P and Q because no signals can be ever received from these events. One says that the accelerated observer perceives a *horizon* at proper distance a^{-1}.

The coordinate system (8.6)-(8.7) is incomplete and covers only a "quarter" of the Minkowski spacetime, more precisely, the subdomain $x > |t|$ (see Fig. 8.2). This is the subdomain of the Minkowski spacetime accessible to a uniformly accelerated observer. For instance, the events P, Q, R cannot be described by (real) values of τ and ξ. The past lightcone $x = -t$ corresponds to the proper coordinates $\tau = -\infty$ and $\xi = -a^{-1}$. The observer can see signals from the event R, however these signals appear to have originated not from R but from the horizon $\xi = -a^{-1}$ in the infinite past $\tau = -\infty$.

Another way to see that the line $\xi = -a^{-1}$ is a horizon is to consider a line of constant proper length $\xi = \xi_0 > -a^{-1}$. It follows from Eqs. (8.6)-(8.7) that the line

$\xi = \xi_0$ is a trajectory of the form $x^2 - t^2 =$ const with the proper acceleration

$$a_0 \equiv \frac{1}{\sqrt{x^2 - t^2}} = \left(\xi_0 + a^{-1}\right)^{-1}.$$

The observer cannot hold a rigid measuring stick longer than a^{-1} because the point $\xi = -a^{-1}$ of the stick would have to move with an infinite proper acceleration, which would require an infinitely large force and is thus impossible.

8.1.2 The Rindler spacetime

The Minkowski metric in the proper coordinates (τ, ξ) is

$$ds^2 = dt^2 - dx^2 = (1 + a\xi)^2 d\tau^2 - d\xi^2. \tag{8.8}$$

The spacetime with this metric is called the Rindler spacetime. The curvature of the Rindler spacetime is everywhere zero since it differs from the Minkowski spacetime merely by a change of coordinates.

> **Exercise 8.1**
> Derive the metric (8.8) from Eqs. (8.6)-(8.7).

To develop the quantum field theory in the Rindler spacetime, we first rewrite the metric (8.8) in a conformally flat form. This can be achieved by choosing the new spatial coordinate $\tilde{\xi}$ such that $d\xi = (1 + a\xi)d\tilde{\xi}$, because in that case both $d\tau^2$ and $d\tilde{\xi}^2$ will have a common factor $(1 + a\xi)^2$. The necessary replacement is therefore

$$\tilde{\xi} \equiv \frac{1}{a}\ln(1 + a\xi).$$

Since the proper distance ξ is constrained by $\xi > -a^{-1}$, the conformal distance $\tilde{\xi}$ varies in the interval $-\infty < \tilde{\xi} < +\infty$. The metric becomes

$$ds^2 = e^{2a\tilde{\xi}}(d\tau^2 - d\tilde{\xi}^2). \tag{8.9}$$

The relation between the laboratory coordinates and the conformal coordinates is

$$t(\tau, \tilde{\xi}) = a^{-1}e^{a\tilde{\xi}}\sinh a\tau, \quad x(\tau, \tilde{\xi}) = a^{-1}e^{a\tilde{\xi}}\cosh a\tau. \tag{8.10}$$

8.2 Quantum fields in the Rindler spacetime

The goal of this section is to quantize a scalar field in the proper reference frame of a uniformly accelerated observer. To simplify the problem, we consider a massless scalar field in the 1+1-dimensional spacetime. All physical conclusions will be the same as those drawn from a four-dimensional calculation.

The action for a massless scalar field $\phi(t, x)$ is

$$S[\phi] = \frac{1}{2}\int g^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta}\sqrt{-g}d^2x.$$

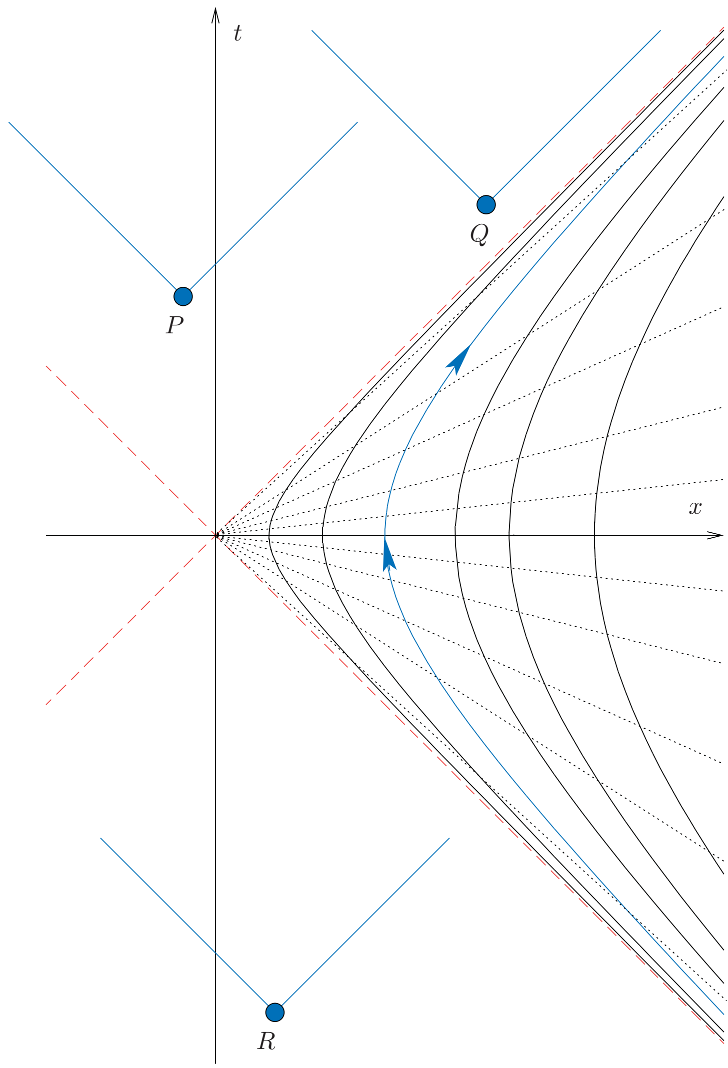

Figure 8.2: The proper coordinate system of a uniformly accelerated observer in the Minkowski spacetime. The solid hyperbolae are the lines of constant proper distance ξ; the hyperbola with arrows is $\xi = 0$, or $x^2 - t^2 = a^{-2}$. The lines of constant τ are dotted. The dashed lines show the lightcone which corresponds to $\xi = -a^{-1}$. The events P, Q, R are not covered by the proper coordinate system.

Here $x^\mu \equiv (t, x)$ is the two-dimensional coordinate. It is easy to see that this action is conformally invariant: indeed, if we replace

$$g_{\alpha\beta} \to \tilde{g}_{\alpha\beta} = \Omega^2(t, x) g_{\alpha\beta},$$

then the determinant $\sqrt{-g}$ and the contravariant metric are replaced by

$$\sqrt{-g} \to \Omega^2 \sqrt{-g}, \quad g^{\alpha\beta} \to \Omega^{-2} g^{\alpha\beta}, \tag{8.11}$$

so the factors Ω^2 cancel in the action. Therefore the minimally coupled massless scalar field in the 1+1-dimensional Minkowski spacetime is in fact *conformally* coupled. The conformal invariance causes a significant simplification of the theory in 1+1 dimensions. (Note that a minimally coupled massless scalar field in 3+1 dimensions is *not* conformally coupled!)

In the laboratory coordinates (t, x), the action is

$$S[\phi] = \frac{1}{2} \int \left[(\partial_t \phi)^2 - (\partial_x \phi)^2 \right] dt\, dx.$$

In the conformal coordinates, the metric (8.9) is equal to the flat Minkowski metric multiplied by a conformal factor $\Omega^2(\tau, \tilde{\xi}) \equiv \exp(2a\tilde{\xi})$. Therefore, due to the conformal invariance, the action has the same form in the coordinates $(\tau, \tilde{\xi})$:

$$S[\phi] = \frac{1}{2} \int \left[(\partial_\tau \phi)^2 - (\partial_{\tilde{\xi}} \phi)^2 \right] d\tau\, d\tilde{\xi}.$$

The classical equations of motion in the laboratory frame and in the accelerated frame are

$$\frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} = 0; \quad \frac{\partial^2 \phi}{\partial \tau^2} - \frac{\partial^2 \phi}{\partial \tilde{\xi}^2} = 0,$$

with the general solutions

$$\phi(t, x) = A(t - x) + B(t + x), \quad \phi(\tau, \tilde{\xi}) = P(\tau - \tilde{\xi}) + Q(\tau + \tilde{\xi}).$$

Here A, B, P, and Q are arbitrary smooth functions. Note that a solution $\phi(t, x)$ representing a certain state of the field will be a very different function of τ and $\tilde{\xi}$.

8.2.1 Quantization

We shall now quantize the field ϕ and compare the vacuum states in the laboratory frame and in the accelerated frame.

The procedure of quantization is formally the same in both coordinate systems (t, x) and $(\tau, \tilde{\xi})$. The mode expansion in the laboratory frame is found from Eq. (4.17) with the substitution $\omega_k = |k|$:

$$\hat{\phi}(t, x) = \int_{-\infty}^{+\infty} \frac{dk}{(2\pi)^{1/2}} \frac{1}{\sqrt{2\,|k|}} \left[e^{-i|k|t + ikx} \hat{a}_k^- + e^{i|k|t - ikx} \hat{a}_k^+ \right]. \tag{8.12}$$

The normalization factor $(2\pi)^{1/2}$ is used in 1+1 dimensions instead of the factor $(2\pi)^{3/2}$ used in 3+1 dimensions. The creation and annihilation operators $\hat{a}_k^{\pm}$ defined by Eq. (8.12) satisfy the usual commutation relations and describe particles moving with momentum k either in the positive x direction ($k > 0$) or in the negative x direction ($k < 0$).

> **Remark: the zero mode.** The mode expansion (8.12) ignores the $k = 0$ solution, $\phi(t, x) = c_0 + c_1 t$, called the *zero mode*. Quantization of the zero mode in the 1+1-dimensional spacetime is a somewhat complicated technical issue. However, the zero mode does not contribute to the four-dimensional theory and we ignore it here.

The vacuum state in the laboratory frame (the *Minkowski vacuum*), denoted by $|0_M\rangle$, is the zero eigenvector of all annihilation operators $\hat{a}_k^-$,

$$\hat{a}_k^- |0_M\rangle = 0 \text{ for all } k.$$

The mode expansion in the accelerated frame is quite similar to Eq. (8.12),

$$\hat{\phi}(\tau, \tilde{\xi}) = \int_{-\infty}^{+\infty} \frac{dk}{(2\pi)^{1/2}} \frac{1}{\sqrt{2|k|}} \left[e^{-i|k|\tau + ik\tilde{\xi}} \hat{b}_k^- + e^{i|k|\tau - ik\tilde{\xi}} \hat{b}_k^+ \right]. \tag{8.13}$$

Note that the mode expansions (8.12) and (8.13) are decompositions of the operator $\hat{\phi}(x, t)$ into linear combinations of two different sets of basis functions with operator-valued coefficients $\hat{a}_k^{\pm}$ and $\hat{b}_k^{\pm}$. So it is to be expected that the operators $\hat{a}_k^{\pm}$ and $\hat{b}_k^{\pm}$ are different, although they satisfy similar commutation relations.

The vacuum state in the accelerated frame $|0_R\rangle$ (the *Rindler vacuum*) is defined by

$$\hat{b}_k^- |0_R\rangle = 0 \text{ for all } k.$$

Since the operators $\hat{b}_k$ differ from $\hat{a}_k$, the Rindler vacuum $|0_R\rangle$ and the Minkowski vacuum $|0_M\rangle$ are two *different* quantum states of the field $\hat{\phi}$.

At this point, a natural question to ask is whether the state $|0_M\rangle$ or $|0_R\rangle$ is the "correct" vacuum. To answer this question, we need to consider the physical interpretation of the states $|0_M\rangle$ and $|0_R\rangle$ in a particular (perhaps imaginary) physical experiment. In Sec. 6.3.2 we discussed a hypothetical device for preparing the quantum field in the lowest-energy state. If mounted onto an accelerated spaceship, the device will prepare the field in the quantum state $|0_R\rangle$. All observers moving with the ship would agree that the field in the state $|0_R\rangle$ has the lowest possible energy, while the Minkowski state $|0_M\rangle$ has a higher energy. Thus a particle detector which remains at rest in the accelerated frame will register particles when the field is in the state $|0_M\rangle$. However, in the laboratory frame the state with the lowest energy is $|0_M\rangle$ and the state $|0_R\rangle$ has a higher energy. Therefore, if the field is in the Rindler state $|0_R\rangle$ (the vacuum prepared inside the spaceship), it will appear to be in an excited state when examined by observers in the laboratory frame.

Neither of the two vacuum states is "more correct" if considered by itself, without regard for realistic physical conditions in the universe. Ultimately the choice of vacuum is determined by experiment: the correct vacuum state must be such that the theoretical predictions agree with available experimental data. In an inertial reference frame in the Minkowski spacetime, we observe empty space that does not create any particles by itself. By virtue of this observation, we are justified to assume that fields in the empty Minkowski spacetime are in the vacuum state $|0_M\rangle$ and that any excitations in the field modes are always due to external sources. In particular, an accelerated observer moving through empty space will encounter fields in the state $|0_M\rangle$ and therefore will detect particles. This detection is the manifestation of the Unruh effect.

The rest of this chapter is devoted to a calculation relating the Minkowski frame operators $\hat{a}_k^{\pm}$ to the Rindler frame operators $\hat{b}_k^{\pm}$ through the appropriate Bogolyubov coefficients. This calculation will enable us to express the Minkowski vacuum as a superposition of excited states built on top of the Rindler vacuum and thus to compute the probability distribution for particle occupation numbers observed in the accelerated frame.

8.2.2 Lightcone mode expansions

It is convenient to introduce the lightcone coordinates[1]

$$\bar{u} \equiv t - x,\ \bar{v} \equiv t + x; \quad u \equiv \tau - \tilde{\xi},\ v \equiv \tau + \tilde{\xi}.$$

The relation between the laboratory frame and the accelerated frame has a simpler form in lightcone coordinates: from Eq. (8.10) we find

$$\bar{u} = -a^{-1}e^{-au}, \quad \bar{v} = a^{-1}e^{av}, \tag{8.14}$$

so the metric is

$$ds^2 = d\bar{u}\,d\bar{v} = e^{a(v-u)}du\,dv.$$

The field equations and their general solutions are also expressed more concisely in the lightcone coordinates:

$$\begin{aligned} \frac{\partial^2}{\partial\bar{u}\partial\bar{v}}\phi\left(\bar{u},\bar{v}\right) = 0, &\quad \phi\left(\bar{u},\bar{v}\right) = A\left(\bar{u}\right) + B\left(\bar{v}\right); \\ \frac{\partial^2}{\partial u\partial v}\phi(u,v) = 0, &\quad \phi(u,v) = P(u) + Q(v). \end{aligned} \tag{8.15}$$

The mode expansion (8.12) can be rewritten in the coordinates $\bar{u}, \bar{v}$ by first splitting the integration into the ranges of positive and negative k,

$$\begin{aligned} \hat{\phi}(t,x) = &\int_{-\infty}^{0} \frac{dk}{(2\pi)^{1/2}} \frac{1}{\sqrt{2\,|k|}} \left[e^{ikt+ikx}\hat{a}_k^- + e^{-ikt-ikx}\hat{a}_k^+ \right] \\ &+ \int_0^{+\infty} \frac{dk}{(2\pi)^{1/2}} \frac{1}{\sqrt{2k}} \left[e^{-ikt+ikx}\hat{a}_k^- + e^{ikt-ikx}\hat{a}_k^+ \right]. \end{aligned}$$

[1] The chosen notation (u,v) for the lightcone coordinates in a uniformly accelerated frame and $(\bar{u},\bar{v})$ for the freely falling (unaccelerated) frame will be used in Chapter 9 as well.

Then we introduce $\omega = |k|$ as the integration variable with the range $0 < \omega < +\infty$ and obtain the *lightcone mode expansion*

$$\hat{\phi}\left(\bar{u}, \bar{v}\right) = \int_0^{+\infty} \frac{d\omega}{(2\pi)^{1/2}} \frac{1}{\sqrt{2\omega}} \left[e^{-i\omega\bar{u}} \hat{a}_\omega^- + e^{i\omega\bar{u}} \hat{a}_\omega^+ + e^{-i\omega\bar{v}} \hat{a}_{-\omega}^- + e^{i\omega\bar{v}} \hat{a}_{-\omega}^+ \right]. \quad (8.16)$$

Lightcone mode expansions explicitly decompose the field $\hat{\phi}\left(\bar{u}, \bar{v}\right)$ into a sum of functions of $\bar{u}$ and functions of $\bar{v}$. This agrees with Eq. (8.15) from which we find that $A\left(\bar{u}\right)$ is a linear combination of the operators $\hat{a}_\omega^\pm$ with positive momenta ω, while $B\left(\bar{v}\right)$ is a linear combination of $\hat{a}_{-\omega}^\pm$ with negative momenta $-\omega$:

$$\begin{aligned} \hat{\phi}\left(\bar{u}, \bar{v}\right) &= \hat{A}\left(\bar{u}\right) + \hat{B}\left(\bar{v}\right); \\ \hat{A}\left(\bar{u}\right) &= \int_0^{+\infty} \frac{d\omega}{(2\pi)^{1/2}} \frac{1}{\sqrt{2\omega}} \left[e^{-i\omega\bar{u}} \hat{a}_\omega^- + e^{i\omega\bar{u}} \hat{a}_\omega^+ \right], \\ \hat{B}\left(\bar{v}\right) &= \int_0^{+\infty} \frac{d\omega}{(2\pi)^{1/2}} \frac{1}{\sqrt{2\omega}} \left[e^{-i\omega\bar{v}} \hat{a}_{-\omega}^- + e^{i\omega\bar{v}} \hat{a}_{-\omega}^+ \right]. \end{aligned}$$

The lightcone mode expansion in the Rindler frame has exactly the same form except for involving the coordinates (u, v) instead of $(\bar{u}, \bar{v})$. We use the integration variable Ω to distinguish the Rindler frame expansion from that of the Minkowski frame,

$$\begin{aligned} \hat{\phi}(u, v) &= \hat{P}(u) + \hat{Q}(v) \\ &= \int_0^{+\infty} \frac{d\Omega}{(2\pi)^{1/2}} \frac{1}{\sqrt{2\Omega}} \left[e^{-i\Omega u} \hat{b}_\Omega^- + e^{i\Omega u} \hat{b}_\Omega^+ + e^{-i\Omega v} \hat{b}_{-\Omega}^- + e^{i\Omega v} \hat{b}_{-\Omega}^+ \right]. \end{aligned} \quad (8.17)$$

As before, $\hat{P}(u)$ is expanded into operators $\hat{b}_\Omega^\pm$ with positive momenta Ω and $\hat{Q}(v)$ into the operators $\hat{b}_{-\Omega}^\pm$ with negative momenta $-\Omega$. (Note that in all lightcone mode expansions, the variables ω and Ω take only *positive* values.)

8.2.3 The Bogolyubov transformations

The relation between the operators $\hat{a}_{\pm\omega}^\pm$ and $\hat{b}_{\pm\Omega}^\pm$, which we shall presently derive, is a Bogolyubov transformation of a more general form than that considered in Sec. 6.2.2.

Since the coordinate transformation (8.14) does not mix u and v, the identity

$$\hat{\phi}(u, v) = \hat{A}\left(\bar{u}(u)\right) + \hat{B}\left(\bar{v}(v)\right) = \hat{P}(u) + \hat{Q}(v)$$

entails two separate relations for u and for v,

$$\hat{A}\left(\bar{u}(u)\right) = \hat{P}(u), \quad \hat{B}\left(\bar{v}(v)\right) = \hat{Q}(v).$$

Comparing the expansions (8.16) and (8.17), we find that the operators $\hat{a}_\omega^\pm$ with positive momenta ω are expressed through $\hat{b}_\Omega^\pm$ with positive momenta Ω, while the operators $\hat{a}_{-\omega}^\pm$ are expressed through negative-momentum operators $\hat{b}_{-\Omega}^\pm$. In other words, there is no mixing between operators of positive and negative momentum. The relation $\hat{A}(\bar{u}) = \hat{P}(u)$ is then rewritten as

$$\begin{aligned} \hat{A}(\bar{u}) &= \int_0^{+\infty} \frac{d\omega}{(2\pi)^{1/2}} \frac{1}{\sqrt{2\omega}} \left[e^{-i\omega\bar{u}} \hat{a}_\omega^- + e^{i\omega\bar{u}} \hat{a}_\omega^+ \right] \\ = \hat{P}(u) &= \int_0^{+\infty} \frac{d\Omega}{(2\pi)^{1/2}} \frac{1}{\sqrt{2\Omega}} \left[e^{-i\Omega u} \hat{b}_\Omega^- + e^{i\Omega u} \hat{b}_\Omega^+ \right]. \end{aligned} \tag{8.18}$$

Here $\bar{u}$ is understood to be the function of u given by Eq. (8.14); both sides of Eq. (8.18) are equal as functions of u.

We can now express the positive-momentum operators $\hat{a}_\omega^\pm$ as explicit linear combinations of $\hat{b}_\Omega^\pm$. To this end, we perform the Fourier transform of both sides of Eq. (8.18) in u. The RHS yields

$$\int_{-\infty}^{+\infty} \frac{du}{\sqrt{2\pi}} e^{i\Omega u} \hat{P}(u) = \frac{1}{\sqrt{2|\Omega|}} \begin{cases} \hat{b}_\Omega^-, & \Omega > 0; \\ \hat{b}_{|\Omega|}^+, & \Omega < 0. \end{cases} \tag{8.19}$$

(The Fourier transform variable is denoted also by Ω for convenience.) The Fourier transform of the LHS of Eq. (8.18) yields an expression involving all $\hat{a}_\omega^\pm$,

$$\begin{aligned} \int_{-\infty}^{+\infty} \frac{du}{\sqrt{2\pi}} e^{i\Omega u} \hat{A}(\bar{u}) &= \int_0^\infty \frac{d\omega}{\sqrt{2\omega}} \int_{-\infty}^{+\infty} \frac{du}{2\pi} \left[e^{i\Omega u - i\omega\bar{u}} \hat{a}_\omega^- + e^{i\Omega u + i\omega\bar{u}} \hat{a}_\omega^+ \right] \\ &\equiv \int_0^\infty \frac{d\omega}{\sqrt{2\omega}} \left[F(\omega,\Omega) \hat{a}_\omega^- + F(-\omega,\Omega) \hat{a}_\omega^+ \right], \end{aligned} \tag{8.20}$$

where we introduced the auxiliary function[2]

$$F(\omega,\Omega) \equiv \int_{-\infty}^{+\infty} \frac{du}{2\pi} e^{i\Omega u - i\omega\bar{u}} = \int_{-\infty}^{+\infty} \frac{du}{2\pi} \exp\left[i\Omega u + i\frac{\omega}{a} e^{-au} \right]. \tag{8.21}$$

Comparing Eqs. (8.19) and (8.20) restricted to positive Ω, we find that the relation between $\hat{a}_\omega^\pm$ and $\hat{b}_\Omega^-$ is of the form

$$\hat{b}_\Omega^- = \int_0^\infty d\omega \left[\alpha_{\omega\Omega} \hat{a}_\omega^- + \beta_{\omega\Omega} \hat{a}_\omega^+ \right], \tag{8.22}$$

where the coefficients $\alpha_{\omega\Omega}$ and $\beta_{\omega\Omega}$ are

$$\alpha_{\omega\Omega} = \sqrt{\frac{\Omega}{\omega}} F(\omega,\Omega), \quad \beta_{\omega\Omega} = \sqrt{\frac{\Omega}{\omega}} F(-\omega,\Omega); \quad \omega > 0, \Omega > 0. \tag{8.23}$$

[2] Because of the carelessly interchanged order of integration while deriving Eq. (8.20), the integral (8.21) diverges at $u \to +\infty$ and the definition of $F(\omega,\Omega)$ must be understood in the distributional sense. In Appendix A.3 it is shown how to express $F(\omega,\Omega)$ through Euler's gamma function, but we shall not need that representation.

The operators $\hat{b}^{+}_{\Omega}$ can be similarly expressed through $\hat{a}^{\pm}_{\omega}$ using the Hermitian conjugation of Eq. (8.22) and the identity

$$F^{*}(\omega,\Omega)=F(-\omega,-\Omega).$$

The relation (8.22) is a Bogolyubov transformation that mixes creation and annihilation operators with different momenta $\omega \neq \Omega$. In contrast, the Bogolyubov transformations considered in Sec. 6.2.2 are "diagonal," with $\alpha_{\omega\Omega}$ and $\beta_{\omega\Omega}$ proportional to $\delta(\omega-\Omega)$.

The relation between the operators $\hat{a}^{\pm}_{-\omega}$ and $\hat{b}^{\pm}_{-\Omega}$ is obtained from the equation $\hat{B}\left(\bar{v}\right)=\hat{Q}(v)$. We omit the corresponding straightforward calculations and concentrate on the positive-momentum modes; the results for negative momenta are completely analogous.

General Bogolyubov transformations

We need to briefly consider the properties of general Bogolyubov transformations,

$$\hat{b}^{-}_{\Omega}=\int_{-\infty}^{+\infty}d\omega\left[\alpha_{\omega\Omega}\hat{a}^{-}_{\omega}+\beta_{\omega\Omega}\hat{a}^{+}_{\omega}\right]. \tag{8.24}$$

The relation (8.22) is of this form except for the integration over $0<\omega<+\infty$ which is justified because the only nonzero Bogolyubov coefficients are those relating the momenta ω,Ω of equal sign, i.e. $\alpha_{-\omega,\Omega}=0$ and $\beta_{-\omega,\Omega}=0$. But for now we shall not limit ourselves to this case.

The relation for the operator $\hat{b}^{+}_{\Omega}$ is the Hermitian conjugate of Eq. (8.24).

> **Remark:** To avoid confusion in the notation, we always write the indices ω,Ω in the Bogolyubov coefficients in this order, i.e. $\alpha_{\omega\Omega}$, but never $\alpha_{\Omega\omega}$. In the calculations throughout this chapter, the integration is always over the first index ω corresponding to the momentum of a-particles.

Since the operators $\hat{a}^{\pm}_{\omega}$, $\hat{b}^{\pm}_{\Omega}$ satisfy the commutation relations

$$\left[\hat{a}^{-}_{\omega},\hat{a}^{+}_{\omega'}\right]=\delta(\omega-\omega'),\quad [\hat{b}^{-}_{\Omega},\hat{b}^{+}_{\Omega'}]=\delta(\Omega-\Omega'), \tag{8.25}$$

the Bogolyubov coefficients are constrained by

$$\int_{-\infty}^{+\infty}d\omega\left(\alpha_{\omega\Omega}\alpha^{*}_{\omega\Omega'}-\beta_{\omega\Omega}\beta^{*}_{\omega\Omega'}\right)=\delta(\Omega-\Omega'). \tag{8.26}$$

This is analogous to the normalization condition $\left|\alpha_{\mathbf{k}}\right|^{2}-\left|\beta_{\mathbf{k}}\right|^{2}=1$ we had earlier.

> **Exercise 8.2**
> Derive Eq. (8.26).

Note that the origin of the δ function in Eq. (8.25) is the infinite volume of the entire space. If the field were quantized in a finite box of volume V, the momenta ω and Ω would be discrete and the δ function would be replaced by the ordinary Kronecker symbol times the volume factor, i.e. $V\delta_{\Omega\Omega'}$. The δ function in Eq. (8.26) has the same origin. Below we shall use Eq. (8.26) with $\Omega = \Omega'$ and the divergent factor $\delta(0)$ will be interpreted as the infinite spatial volume.

Remark: inverse Bogolyubov transformations. The commutation relation $[\hat{b}^-_\Omega, \hat{b}^-_{\Omega'}] = 0$ yields another restriction on the Bogolyubov coefficients,

$$\int_{-\infty}^{+\infty} d\omega \left(\alpha_{\omega\Omega}\beta_{\omega\Omega'} - \alpha_{\omega\Omega'}\beta_{\omega\Omega}\right) = 0. \tag{8.27}$$

It follows from Eqs. (8.26), (8.27) that the inverse Bogolyubov transformation is

$$\hat{a}^-_\omega = \int_{-\infty}^{+\infty} d\Omega \left(\alpha^*_{\omega\Omega}\hat{b}^-_\Omega - \beta_{\omega\Omega}\hat{b}^+_\Omega\right).$$

This relation can be easily verified by substituting it into Eq. (8.24). One can also derive orthogonality relations similar to Eqs. (8.26), (8.27) but with the integration over Ω. We shall not need the inverse Bogolyubov transformations in this chapter.

8.2.4 Density of particles

Since the vacua $|0_M\rangle$ and $|0_R\rangle$ corresponding to the operators $\hat{a}^-_\omega$ and $\hat{b}^-_\Omega$ are different, the a-vacuum is a state with b-particles and vice versa. We now compute the density of b-particles in the a-vacuum state.

The b-particle number operator is $\hat{N}_\Omega \equiv \hat{b}^+_\Omega\hat{b}^-_\Omega$, so the average b-particle number in the a-vacuum $|0_M\rangle$ is equal to the expectation value of $\hat{N}_\Omega$,

$$\begin{aligned}
\langle\hat{N}_\Omega\rangle &\equiv \langle 0_M|\,\hat{b}^+_\Omega\hat{b}^-_\Omega\,|0_M\rangle \\
&= \langle 0_M|\int d\omega\left[\alpha^*_{\omega\Omega}\hat{a}^+_\omega + \beta^*_{\omega\Omega}\hat{a}^-_\omega\right]\int d\omega'\left[\alpha_{\omega'\Omega}\hat{a}^-_{\omega'} + \beta_{\omega'\Omega}\hat{a}^+_{\omega'}\right]|0_M\rangle \\
&= \int d\omega\,|\beta_{\omega\Omega}|^2\,.
\end{aligned} \tag{8.28}$$

This is the mean number of particles observed in the accelerated frame.

In principle one can explicitly compute the Bogolyubov coefficients $\beta_{\omega\Omega}$ defined by Eq. (8.23) in terms of the Γ function (see Appendix A.3). However, we only need to evaluate the RHS of Eq. (8.28) which involves an integral over ω, and we shall use a mathematical trick that allows us to compute just that integral and avoid cumbersome calculations.

We first note that the function $F(\omega,\Omega)$ satisfies the identity

$$F(\omega,\Omega) = F(-\omega,\Omega)\exp\left(\frac{\pi\Omega}{a}\right), \quad \text{for } \omega > 0,\ a > 0. \tag{8.29}$$

Exercise 8.3*
Derive the relation (8.29) from Eq. (8.21). *Hint:* deform the contour of integration in the complex plane.

We then substitute Eq. (8.23) into the normalization condition (8.26), use Eq. (8.29) and find

$$\delta(\Omega - \Omega') = \int_0^{+\infty} d\omega \frac{\sqrt{\Omega\Omega'}}{\omega} \left[F(\omega,\Omega)F^*(\omega,\Omega') - F(-\omega,\Omega)F^*(-\omega,\Omega')\right]$$
$$= \left[\exp\left(\frac{\pi\Omega + \pi\Omega'}{a}\right) - 1\right] \int_0^{+\infty} d\omega \frac{\Omega}{\omega} F^*(-\omega,\Omega)F(-\omega,\Omega).$$

The last line above yields the relation

$$\int_0^{+\infty} d\omega \frac{\Omega}{\omega} F(-\omega,\Omega)F^*(-\omega,\Omega') = \left[\exp\left(\frac{2\pi\Omega}{a}\right) - 1\right]^{-1} \delta(\Omega - \Omega'). \tag{8.30}$$

Setting $\Omega' = \Omega$ in Eq. (8.30), we directly compute the integral in the RHS of Eq. (8.28),

$$\langle \hat{N}_\Omega \rangle = \int_0^{+\infty} d\omega \left|\beta_{\omega\Omega}\right|^2 = \int_0^{+\infty} d\omega \frac{\Omega}{\omega} \left|F(-\omega,\Omega)\right|^2 = \left[\exp\left(\frac{2\pi\Omega}{a}\right) - 1\right]^{-1} \delta(0).$$

As usual, we expect $\langle \hat{N}_\Omega \rangle$ to be divergent since it is the total number of particles in the entire space. As discussed in Sec. 4.2, the divergent volume factor $\delta(0)$ represents the volume of space, and the remaining factor is the density n_Ω of b-particles with momentum Ω:

$$\int_0^{+\infty} d\omega \left|\beta_{\omega\Omega}\right|^2 \equiv n_\Omega \delta(0).$$

Therefore, the mean density of particles in the mode with momentum Ω is

$$n_\Omega = \left[\exp\left(\frac{2\pi\Omega}{a}\right) - 1\right]^{-1}. \tag{8.31}$$

This is the main result of this chapter.

So far we have computed n_Ω only for positive-momentum modes (with $\Omega > 0$). The result for negative-momentum modes is obtained by replacing Ω by $|\Omega|$ in Eq. (8.31).

8.2.5 The Unruh temperature

A massless particle with momentum Ω has energy $E = |\Omega|$, so the formula (8.31) is equivalent to the Bose distribution

$$n(E) = \left[\exp\left(\frac{E}{T}\right) - 1\right]^{-1}$$

where T is the *Unruh temperature*

$$T \equiv \frac{a}{2\pi}.$$

We found that an accelerated observer detects particles when the field ϕ is in the Minkowski vacuum state $|0_M\rangle$. The detected particles may have any momentum Ω, although the probability for registering a high-energy particle is very small. The particle distribution (8.31) is characteristic of the thermal blackbody radiation with the temperature $T = a/2\pi$, where a is the magnitude of the proper acceleration (in Planck units). An accelerated detector behaves as though it were placed in a thermal bath with temperature T. This is the Unruh effect.

> **Remark: conformal invariance.** Earlier we said that a conformally coupled field cannot exhibit particle production by gravity. This is not in contradiction with the detection of particles in accelerated frames. Conformal invariance means that identical initial conditions produce identical evolution in all conformally related frames. If the lowest-energy state is prepared in the accelerated frame (this is the Rindler vacuum $|0_R\rangle$) and later the number of particles is measured by a detector that remains accelerated in the same frame, then no particles will be registered after arbitrarily long times. This is exactly the same prediction as that obtained in the laboratory frame. Nevertheless, the vacuum state prepared in one frame of reference may be a state with particles in another frame.

A physical interpretation of the Unruh effect as seen in the laboratory frame is the following. The accelerated detector is coupled to the quantum fields and perturbs their quantum state around its trajectory. This perturbation is very small but as a result the detector registers particles, although the fields were previously in the vacuum state. The detected particles are real and the energy for these particles comes from the agent that accelerates the detector.

Finally, we note that the Unruh effect is impossible to use in practice because the acceleration required to produce a measurable temperature is enormous (see Exercise 1.6 on p. 12 for a numerical example). The energy spent by the accelerating agent is exponentially large compared with the energy in detected particles. The Unruh effect is an extremely inefficient way to produce particles.

> **Remark: more general motion.** Observers moving with a nonconstant acceleration will generally also detect particles but with a nonthermal spectrum. For a general trajectory $x^\mu(\tau)$ it is difficult to construct a proper reference frame; instead one considers a quantum-mechanical model of a detector coupled to the field $\phi(x)$ and computes the probability for observing an excited state of the detector. A calculation of this sort was first performed by W. G. Unruh; see the book by Birrell and Davies, §3.2.

9 The Hawking effect. Thermodynamics of black holes

Summary: Quantization of fields in a black hole spacetime. Choice of vacuum. Hawking radiation. Black hole evaporation. Thermodynamics of black holes.

In this chapter we consider a counter-intuitive effect: emission of particles by black holes.

9.1 The Hawking radiation

Classical general relativity describes black holes as massive objects with such a strong gravitational field that even light cannot escape their surface (the *black hole horizon*). However, quantum theory predicts that black holes emit particles moving away from the horizon. The particles are produced out of vacuum fluctuations of quantum fields present around the black hole. In effect, a black hole (BH) is not completely black but radiates a dim light as if it were an object with a low but nonzero temperature.

The theoretical prediction of radiation by black holes came as a complete surprise. It was thought that particles may be produced only by time-dependent gravitational fields. The first rigorous calculation of the rate of particle creation by a rotating BH was performed in 1974 by S. Hawking. He expected that in the limit of no rotation the particle production should disappear, but instead he found that nonrotating (static) black holes also create particles at a steady rate. This was so perplexing that Hawking thought he had made a mistake in calculations. It took some years before this theoretically derived effect (the *Hawking radiation*) was accepted by the scientific community.

An intuitive picture of the Hawking radiation involves a virtual particle-antiparticle pair at the BH horizon. It may happen that the first particle of the pair is inside the BH horizon while the second particle is outside. The first virtual particle always falls onto the BH center, but the second particle has a nonzero probability for moving away from the horizon and becoming a real radiated particle. The mass of the black hole is decreased in the process of radiation because the energy of the infalling virtual particle with respect to faraway observers is formally negative.

Another qualitative consideration is that a black hole of size R cannot capture radiation with wavelength much larger than R. It follows that particles (real or virtual) with sufficiently small energies $E \ll \hbar c/R$ might avoid falling into the BH horizon.

This argument indicates the correct order of magnitude for the energy of radiated particles, although it remains unclear whether and how the radiation is actually emitted.

The main focus of this section is to compute the density of particles emitted by a static black hole, as registered by observers far away from the BH horizon.

9.1.1 Scalar field in a BH spacetime

In quantum theory, particles are excitations of quantum fields, so we consider a scalar field in the presence of a single nonrotating black hole of mass M. The BH spacetime is described by the Schwarzschild metric,[1]

$$ds^2 = \left(1 - \frac{2M}{r}\right) dt^2 - \frac{dr^2}{1 - \frac{2M}{r}} - r^2 \left(d\theta^2 + d\varphi^2 \sin^2 \theta\right).$$

This metric is singular at $r = 2M$ which corresponds to the BH horizon, while for $r < 2M$ the coordinate t is spacelike and r is timelike. Therefore the coordinates (t, r) may be used with the normal interpretation of time and space only in the exterior region, $r > 2M$.

To simplify the calculations, we assume that the field ϕ is independent of the angular variables θ, φ and restrict our attention to a 1+1-dimensional section of the spacetime with the coordinates (t, r). The line element in 1+1 dimensions,

$$ds^2 = g_{ab} dx^a dx^b, \quad x^0 \equiv t,\ x^1 \equiv r,$$

involves the reduced metric

$$g_{ab} = \left[\begin{array}{cc} 1 - \frac{2M}{r} & 0 \\ 0 & -\left(1 - \frac{2M}{r}\right)^{-1} \end{array} \right].$$

The theory we are developing is a toy model (i.e. a drastically simplified version) of the full 3+1-dimensional QFT in the Schwarzschild spacetime. We expect that the main features of the full theory are preserved in the 1+1-dimensional model.

The action for a minimally coupled massless scalar field is

$$S[\phi] = \frac{1}{2} \int g^{ab} \phi_{,a} \phi_{,b} \sqrt{-g} d^2x.$$

As shown in Sec. 8.2, the field ϕ with this action is in fact conformally coupled. Because of the conformal invariance, a significant simplification occurs if the metric is brought to a conformally flat form. This is achieved by changing the coordinate $r \to r^*$, where the function $r^*(r)$ is chosen so that

$$dr = \left(1 - \frac{2M}{r}\right) dr^*.$$

[1] In our notation here and below, the asimuthal angle is φ while the scalar field is ϕ.

From this relation we find $r^*(r)$ up to an integration constant which we choose as $2M$ for convenience,

$$r^*(r) = r - 2M + 2M \ln\left(\frac{r}{2M} - 1\right). \tag{9.1}$$

The metric in the coordinates (t, r^*) is conformally flat,

$$ds^2 = \left(1 - \frac{2M}{r}\right)\left[dt^2 - dr^{*2}\right], \tag{9.2}$$

where r must be expressed through r^* using Eq. (9.1). We shall not need an explicit formula for the function $r(r^*)$.

The coordinate $r^*(r)$ is defined only for $r > 2M$ and varies in the range $-\infty < r^* < +\infty$. It is called the "tortoise coordinate" because an object approaching the horizon $r = 2M$ needs to cross an infinite coordinate distance in r^*. From Eq. (9.2) it is clear that the tortoise coordinates (t, r^*) are asymptotically the same as the Minkowski coordinates (t, r) when $r \to +\infty$, i.e. in regions far from the black hole where the spacetime is almost flat.

The action for the scalar field in the tortoise coordinates is

$$S[\phi] = \frac{1}{2}\int \left[(\partial_t \phi)^2 - (\partial_{r^*}\phi)^2\right] dt\, dr^*,$$

and the general solution of the equation of motion is of the form

$$\phi(t, r^*) = P(t - r^*) + Q(t + r^*),$$

where P and Q are arbitrary (but sufficiently smooth) functions.

In the lightcone coordinates (u, v) defined by

$$u \equiv t - r^*, \quad v \equiv t + r^*, \tag{9.3}$$

the metric is expressed as

$$ds^2 = \left(1 - \frac{2M}{r}\right) du\, dv. \tag{9.4}$$

Note that $r = 2M$ is a singularity where the metric becomes degenerate.

9.1.2 The Kruskal coordinates

The coordinate system (t, r^*) has the advantage that for $r^* \to +\infty$ it asymptotically coincides with the Minkowski coordinate system (t, r) naturally defined far away from the BH horizon. However, the coordinates (t, r^*) do not cover the black hole interior, $r < 2M$. To describe the entire spacetime, we need another coordinate system.

It is a standard result that the singularity in the Schwarzschild metric (9.4) which occurs at $r \to 2M$ is merely a *coordinate singularity* since a suitable change of coordinates yields a metric regular at the BH horizon. For instance, an observer freely falling into the black hole would see a normal, finitely curved space while crossing the horizon line $r = 2M$. Therefore one is motivated to consider the coordinate system $(\bar{t}, \bar{r})$ describing the proper time $\bar{t}$ and the proper distance $\bar{r}$ measured by a freely falling observer at the moment of horizon crossing. This coordinate system is called the Kruskal frame.

We omit the construction of the Kruskal frame[2] and write only the final formulae. The Kruskal lightcone coordinates

$$\bar{u} \equiv \bar{t} - \bar{r}, \quad \bar{v} \equiv \bar{t} + \bar{r}$$

are related to the tortoise lightcone coordinates (9.3) by

$$\bar{u} = -4M \exp\left(-\frac{u}{4M}\right), \quad \bar{v} = 4M \exp\left(\frac{v}{4M}\right). \tag{9.5}$$

The parameters $\bar{u}$, $\bar{v}$ vary in the intervals

$$-\infty < \bar{u} < 0, \quad 0 < \bar{v} < +\infty. \tag{9.6}$$

The inverse relation between $(\bar{u}, \bar{v})$ and the tortoise coordinates (t, r^*) is then found from Eqs. (9.1) and (9.5):

$$\begin{aligned} t &= 2M \ln\left(-\frac{\bar{v}}{\bar{u}}\right), \\ \exp\left(-\frac{r^*}{2M}\right) &= -\frac{\exp\left(1 - \frac{r}{2M}\right)}{1 - \frac{r}{2M}} = -\frac{16M^2}{\bar{u}\bar{v}}. \end{aligned} \tag{9.7}$$

The BH horizon $r = 2M$ corresponds to the lines $\bar{u} = 0$ and $\bar{v} = 0$. To examine the spacetime near the horizon, we need to rewrite the metric in the Kruskal coordinates. With the substitution

$$u = -4M \ln\left(-\frac{\bar{u}}{4M}\right), \quad v = 4M \ln\frac{\bar{v}}{4M},$$

the metric (9.4) becomes

$$ds^2 = -\frac{16M^2}{\bar{u}\bar{v}}\left(1 - \frac{2M}{r}\right) d\bar{u}\, d\bar{v}.$$

Using Eqs. (9.1) and (9.7), after some algebra we obtain

$$ds^2 = \frac{2M}{r} \exp\left(1 - \frac{r}{2M}\right) d\bar{u}\, d\bar{v}, \tag{9.8}$$

where it is implied that the Schwarzschild coordinate r is expressed through $\bar{u}$ and $\bar{v}$ using the relation (9.7).

It follows from Eq. (9.8) that at $r = 2M$ the metric is $ds^2 = d\bar{u}\, d\bar{v}$, the same as in the Minkowski spacetime. Although the coordinates $\bar{u}, \bar{v}$ were defined in the ranges (9.6),

[2] A detailed derivation can be found, for instance, in §31 of the book *Gravitation* by C.W. MISNER, K. THORNE, and J. WHEELER (W. H. Freeman, San Francisco, 1973).

there is no singularity at $\bar{u} = 0$ or at $\bar{v} = 0$ and therefore the coordinate system may be extended to $\bar{u} > 0$ and $\bar{v} < 0$. Thus the Kruskal coordinates cover a larger patch of the spacetime than the tortoise coordinates (t, r^*). For instance, Eq. (9.7) relates r to $\bar{u}, \bar{v}$ also for $0 < r < 2M$, even though r^* is undefined for these r.

The *Kruskal spacetime* is the extension of the Schwarzschild spacetime described by the Kruskal coordinates $\bar{t}, \bar{r}$.

> **Remark: the physical singularity.** The Kruskal metric (9.8) is undefined at $r = 0$. A calculation shows that the spacetime curvature grows without limit as $r \to 0$. Therefore $r = 0$ (the center of the black hole) is a real singularity where general relativity breaks down. From Eq. (9.7) one finds that $r = 0$ corresponds to the line $\bar{u}\bar{v} = \bar{t}^2 - \bar{r}^2 = 16e^{-1}M^2$. This line is a singular boundary of the Kruskal spacetime; the coordinates $\bar{t}$, $\bar{r}$ vary in the domain $|\bar{t}| < \sqrt{\bar{r}^2 + 16e^{-1}M^2}$.

Since the Kruskal metric (9.8) is conformally flat, the action and the classical field equations for a conformally coupled field in the Kruskal frame have the same form as in the tortoise coordinates. For instance, the general solution for the field ϕ is $\phi(\bar{u}, \bar{v}) = A(\bar{u}) + B(\bar{v})$.

We note that Eq. (9.5) is similar to the definition (8.14) of the proper frame for a uniformly accelerated observer. The formal analogy is exact if we set $a \equiv (4M)^{-1}$. Note that a freely falling observer (with the worldline $\bar{r} = \text{const}$) has zero proper acceleration. On the other hand, a spaceship remaining at a fixed position relative to the BH must keep its engine running at a constant thrust and thus has a constant proper acceleration. To make the analogy with the Unruh effect more apparent, we chose the notation in which the coordinates $(\bar{u}, \bar{v})$ always refer to freely falling observers while the coordinates (u, v) describe accelerated frames.

9.1.3 Field quantization

In the previous section we introduced two coordinate systems corresponding to a locally inertial observer (the Kruskal frame) and a locally accelerated observer (the tortoise frame). Now we quantize the field $\phi(x)$ in these two frames and compare the respective vacuum states. The considerations are formally quite similar to those in Chapter 8.

To quantize the field $\phi(x)$, it is convenient to employ the lightcone mode expansions (defined in Sec. 8.2.2) in the coordinates (u, v) and $(\bar{u}, \bar{v})$. Because of the intentionally chosen notation, the relations (8.16) and (8.17) can be directly used to describe the quantized field $\hat{\phi}$ in the BH spacetime.

The lightcone mode expansion in the tortoise coordinates is

$$\hat{\phi}(u, v) = \int_0^{+\infty} \frac{d\Omega}{\sqrt{2\pi}} \frac{1}{\sqrt{2\Omega}} \left[e^{-i\Omega u} \hat{b}_\Omega^- + H.c. + e^{-i\Omega v} \hat{b}_{-\Omega}^- + H.c. \right],$$

where the "$H.c.$" denotes the Hermitian conjugate terms. The operators $\hat{b}_{\pm\Omega}^{\pm}$ correspond to particles detected by a stationary observer at a constant distance from the BH. The role of this observer is completely analogous to that of the uniformly accelerated observer considered in Sec. 8.1.

The lightcone mode expansion in the Kruskal coordinates is

$$\hat{\phi}(\bar{u},\bar{v}) = \int_0^{+\infty} \frac{d\omega}{\sqrt{2\pi}} \frac{1}{\sqrt{2\omega}} \left[e^{-i\omega\bar{u}} \hat{a}_\omega^- + H.c. + e^{-i\omega\bar{v}} \hat{a}_{-\omega}^- + H.c. \right].$$

The operators $\hat{a}_{\pm\omega}^{\pm}$ are related to particles registered by an observer freely falling into the black hole.

It is clear that the two sets of creation and annihilation operators $\hat{a}_{\pm\omega}^{\pm}$, $\hat{b}_{\pm\Omega}^{\pm}$ specify two different vacuum states, $|0_K\rangle$ ("Kruskal") and $|0_T\rangle$ ("tortoise"),

$$\hat{a}_{\pm\omega}^{-} |0_K\rangle = 0; \quad \hat{b}_{\pm\Omega}^{-} |0_T\rangle = 0.$$

The state $|0_T\rangle$ is also called the *Boulware vacuum*.

Exactly as in the previous chapter, the operators $\hat{b}_{\pm\Omega}^{\pm}$ can be expressed through $\hat{a}_{\pm\omega}^{\pm}$ using the Bogolyubov transformation (8.22). The Bogolyubov coefficients are found from Eq. (8.23) if the acceleration a is replaced by $(4M)^{-1}$.

The correspondence between the Rindler and the Schwarzschild spacetimes is summarized in the following table. (We stress that this analogy is precise only for a conformally coupled field in 1+1 dimensions.)

Rindler	**Schwarzschild**
Inertial observers: vacuum $\|0_M\rangle$	Observers in free fall: vacuum $\|0_K\rangle$
Accelerated observers: $\|0_R\rangle$	Observers at $r =$ const: $\|0_T\rangle$
Proper acceleration a	Proper acceleration $(4M)^{-1}$
$\bar{u} = -a^{-1}\exp(-au)$	$\bar{u} = -4M \exp\left[-u/(4M)\right]$
$\bar{v} = a^{-1}\exp(av)$	$\bar{v} = 4M \exp\left[v/(4M)\right]$

9.1.4 Choice of vacuum

To find the expected number of particles measured by observers far outside of the black hole, we first need to make the correct choice of the quantum state of the field $\hat{\phi}$. In the present case, there are two candidate vacua, $|0_K\rangle$ and $|0_T\rangle$. We shall draw on the analogy with Sec. 8.2.1 to justify the choice of the Kruskal vacuum $|0_K\rangle$, which is the lowest-energy state for freely falling observers, as the quantum state of the field.

When considering a uniformly accelerated observer in the Minkowski spacetime, the correct choice of the vacuum state is $|0_M\rangle$ which is the lowest-energy state as measured by inertial observers. An accelerated observer registers this state as thermally excited. The other vacuum state, $|0_R\rangle$, can be physically realized by an accelerated vacuum preparation device occupying a very large volume of space. Consequently, the energy needed to prepare the field in the state $|0_R\rangle$ in the whole space is infinitely large. If one computes the mean energy density of the field $\hat{\phi}$ in the state $|0_R\rangle$, one finds (after subtracting the zero-point energy) that in the Minkowski frame the energy density diverges at the horizon.[3] On the other hand, the Minkowski vacuum

[3]This result can be qualitatively understood if we recall that the Rindler coordinate $\tilde{\xi}$ covers an infinite range when approaching the horizon ($\tilde{\xi} \to -\infty$ as $\xi \to -a^{-1}$). The zero-point energy density in the state $|0_R\rangle$ is constant in the Rindler frame and thus appears as an infinite concentration of energy density near the horizon in the Minkowski frame. We omit the detailed calculation.

state $|0_M\rangle$ has zero energy density everywhere.

It turns out that a very similar situation occurs in the BH spacetime. At first it may appear that the field $\hat{\phi}$ should be in the Boulware state $|0_T\rangle$ which is the vacuum measured by observers remaining at a constant distance from the black hole. However, the field $\hat{\phi}$ in the state $|0_T\rangle$ has an infinite energy density (after subtracting the zero-point energy) near the BH horizon.[4] Any energy density influences the metric via the Einstein equation. A divergent energy density indicates that the backreaction of the quantum fluctuations in the state $|0_T\rangle$ is so large near the BH horizon that the Schwarzschild metric is not a good approximation for the resulting spacetime. Thus the picture of a quantum field in the state $|0_T\rangle$ near an almost unperturbed black hole is inconsistent. On the other hand, the field $\hat{\phi}$ in the Kruskal state $|0_K\rangle$ has an everywhere finite and small energy density (when computed in the Schwarzschild frame after a subtraction of the zero-point energy). In this case, the backreaction of the quantum fluctuations on the metric is negligible. Therefore one has to employ the vacuum state $|0_K\rangle$ rather than the state $|0_T\rangle$ to describe quantum fields in the presence of a classical black hole.

Another argument for selecting the Kruskal vacuum $|0_K\rangle$ is the consideration of a star that turns into a black hole through the gravitational collapse. Before the collapse, the spacetime is almost flat and the initial state of quantum fields is the naturally defined Minkowski vacuum. It can be shown that the final quantum state of the field $\hat{\phi}$ after the collapse is the Kruskal vacuum.

9.1.5 The Hawking temperature

Observers remaining at $r = \text{const}$ far away from the black hole ($r \gg 2M$) are in an almost flat space where the natural vacuum is the Minkowski one. The Minkowski vacuum at $r \gg 2M$ is approximately the same as the Boulware vacuum $|0_T\rangle$. Since the field $\hat{\phi}$ is in the Kruskal vacuum state $|0_K\rangle$, these observers would register the presence of particles.

The calculations of Sec. 8.2.4 show that the temperature measured by an accelerated observer is $T = a/(2\pi)$, and we have seen that the correspondence between the Rindler and the Schwarzschild cases requires to set $a = (4M)^{-1}$. It follows that observers at a fixed distance $r \gg 2M$ from the black hole detect a thermal spectrum of particles with the temperature

$$T_H = \frac{1}{8\pi M}. \tag{9.9}$$

This temperature is known as the Hawking temperature. (Observers staying closer to the BH will see a higher temperature due to the inverse gravitational redshift.)

Similarly, we find that the density of observed particles with energy $E = k$ is

$$n_E = \left[\exp\left(\frac{E}{T_H}\right) - 1\right]^{-1}.$$

[4] This is analogous to the divergent energy density near the horizon in the Rindler vacuum state. We again omit the required calculations.

This formula remains valid for massive particles with mass m and momentum k, after the natural replacement $E = \sqrt{m^2 + k^2}$. One can see that the particle production is significant only for particles with very small masses $m \lesssim T_H$.

The Hawking effect is in principle measurable, although the Hawking temperature for plausible astrophysical black holes is extremely small.

> **Exercise 9.1**
> Rewrite Eq. (9.9) in the SI units and compute the Hawking temperature for black holes of masses $M_1 = M_\odot = 2\cdot 10^{30}$kg (one solar mass), $M_2 = 10^{15}$g, and $M_3 = 10^{-5}$g (of order of the Planck mass).

> **Exercise 9.2**
> **(a)** Estimate the typical wavelength of photons radiated by a black hole of mass M and compare it with the size of the black hole (the Schwarzschild radius $R = 2M$).
> **(b)** The temperature of a sufficiently small black hole can be high enough to efficiently produce baryons (e.g. protons) as components of the Hawking radiation. Estimate the required mass M of such black holes and compare their Schwarzschild radius with the size of the proton (its Compton length).

9.1.6 The Hawking effect in 3+1 dimensions

We have considered the 1+1-dimensional field $\hat{\phi}(t,r)$ that corresponds to spherically symmetric 3+1-dimensional field configurations. However, there is a difference between fields in 1+1 dimensions and spherically symmetric modes in 3+1 dimensions.

The field ϕ in 3+1 dimensions can be decomposed into spherical harmonics,

$$\phi(t,r,\theta,\varphi) = \sum_{l,m} \phi_{lm}(t,r) Y_{lm}(\theta,\varphi).$$

The mode $\phi_{00}(t,r)$ is spherically symmetric and independent of the angles θ, φ. However, the restriction of the 3+1-dimensional wave equation to the mode ϕ_{00} is not equivalent to the 1+1-dimensional problem. The four-dimensional wave equation ${}^{(4)}\Box\phi = 0$ for the spherically symmetric mode is

$$\left[{}^{(2)}\Box + \left(1 - \frac{2M}{r}\right) \frac{2M}{r^3} \right] \phi_{00}(t,r) = 0.$$

This equation represents a wave propagating in the potential

$$V(r) = \left(1 - \frac{2M}{r}\right) \frac{2M}{r^3}$$

instead of a free wave $\phi(t,r)$ considered above. The potential $V(r)$ has a barrier-like shape shown in Fig. 9.1, and a wave escaping the black hole needs to tunnel from $r \approx 2M$ to the potential-free region $r \gg 2M$. This decreases the intensity of the wave and changes the resulting distribution of produced particles by a *greybody factor* $\Gamma_{gb}(E) < 1$,

$$n_E = \Gamma_{gb}(E) \left[\exp\left(\frac{E}{T_H}\right) - 1 \right]^{-1}.$$

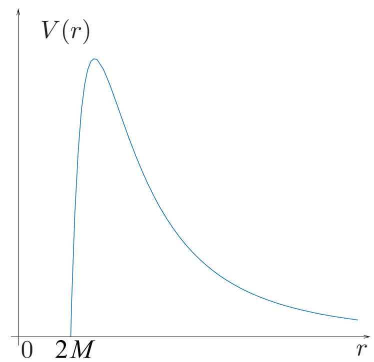

Figure 9.1: The potential $V(r)$ for the propagation of the spherically symmetric mode in 3+1 dimensions.

The computation of the greybody factor $\Gamma_{gb}(E)$ is beyond the scope of this book. This factor depends on the geometry of the radiated field mode and is different for fields of higher spin. (Of course, fermionic fields obey the Fermi instead of the Bose distribution.)

9.1.7 Remarks on other derivations

We derived the Hawking effect in one of the simplest possible cases, namely that of a conformally coupled field in a static BH spacetime restricted to 1+1 dimensions. This derivation cannot be straightforwardly generalized to the full 3+1-dimensional spacetime. For instance, a free massless scalar field is not conformally coupled in 3+1 dimensions, and spherically symmetric modes are not the only available ones. Realistic calculations must consider the production of photons or massive fermions instead of massless scalar particles. However, all such calculations yield the same temperature T_H of the black hole.

It is also important to consider a black hole formed by a gravitational collapse of matter (see Fig. 9.2). Hawking's original calculation involved wave packets of field modes that entered the collapsing region before the BH was formed (the dotted line in the figure). The BH horizon is a light-like surface, therefore massless and ultra-relativistic particles may remain near the horizon for a very long time before they escape to infinity. Since the spacetime is almost flat before the gravitational collapse, the "in" vacuum state of such modes is well-defined in the remote past. After the mode moves far away from the black hole, the "out" vacuum state is again the standard Minkowski ("tortoise") vacuum. A computation of the Bogolyubov coefficients between the "in" and the "out" vacuum states for this wave packet yields a thermal spectrum of particles with the temperature T_H.

This calculation implies that the radiation coming out of the black hole consists of particles that already existed at the time of BH formation but spent a long time near

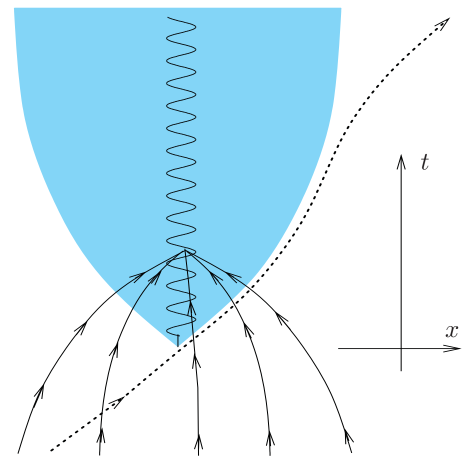

Figure 9.2: Black hole (shaded region) formed by gravitational collapse of matter (lines with arrows). The wavy line marks the singularity at the BH center. A light-like trajectory (dotted line) may linger near the horizon (the boundary of the shaded region) for a long time before escaping to infinity.

the horizon and only managed to escape at the present time. This explanation, however, contradicts the intuitive expectation that particles are created right at the present time by the gravitational field of the BH. The rate of particle creation should depend only on the present state of the black hole and not on the details of its formation in the distant past. One expects that an eternal black hole should radiate in the same way as a BH formed by gravitational collapse.

Another way to derive the Hawking radiation is to evaluate the energy-momentum tensor $T_{\mu\nu}$ of a quantum field in a BH spacetime and to verify that it corresponds to thermal excitations. However, a direct computation of the EMT is complicated and has been explicitly performed only for a 1+1-dimensional spacetime. The reason for the difficulty is that the EMT contains information about the quantum field at all points, not only the asymptotic properties at spatial infinity. This additional information is necessary to determine the backreaction of fields on the black hole during its evaporation. The detailed picture of the BH evaporation remains unknown.

There seems to be no unique physical explanation of the BH radiation. However, the resulting thermal spectrum of the created particles has been derived in many different ways and agrees with general thermodynamical arguments. There is little doubt that the Hawking radiation is a valid and in principle observable prediction of general relativity and quantum field theory.

9.2 Thermodynamics of black holes

9.2.1 Evaporation of black holes

In many situations, a static black hole of mass M behaves as a spherical body with radius $r = 2M$ and surface temperature T_H. According to the Stefan-Boltzmann law, a black body radiates the flux of energy

$$L = \gamma \sigma T_H^4 A,$$

where γ parametrizes the number of degrees of freedom available to the radiation, $\sigma = \pi^2/60$ is the Stefan-Boltzmann constant in Planck units, and

$$A = 4\pi R^2 = 16\pi M^2$$

is the surface area of the BH. The emitted flux determines the loss of energy due to radiation. The mass of the black hole decreases with time according to

$$\frac{dM}{dt} = -L = -\frac{\gamma}{15360\pi M^2}. \tag{9.10}$$

The solution with the initial condition $M|_{t=0} = M_0$ is

$$M(t) = M_0 \left(1 - \frac{t}{t_L}\right)^{1/3}, \quad t_L \equiv 5120\pi \frac{M_0^3}{\gamma}.$$

This calculation suggests that black holes are fundamentally unstable objects with the lifetime t_L during which the BH completely evaporates. Taking into account the greybody factor (see Sec. 9.1.6) would change only the numerical coefficient in the power law $t_L \sim M_0^3$.

Exercise 9.3
Estimate the lifetime of black holes with masses $M_1 = M_\odot = 2 \cdot 10^{30}$kg, $M_2 = 10^{15}$g, $M_3 = 10^{-5}$g.

It is almost certain that the final stage of the BH evaporation cannot be described by classical general relativity. The radius of the BH eventually reaches the Planck scale 10^{-33}cm and one expects unknown effects of quantum gravity to dominate in that regime. One possible outcome is that the BH is stabilized into a "remnant," a microscopic black hole that does not radiate, similarly to electrons in atoms that do not radiate on the lowest orbit. It is plausible that the horizon area is quantized to discrete levels and that a black hole becomes stable when its horizon reaches the minimum allowed area. In this case, quanta of Hawking radiation are emitted as a result of transitions between allowed horizon levels, so the spectrum of the Hawking radiation must consist of discrete lines. This prediction of the discreteness of the spectrum of the Hawking radiation may be one of the few testable effects of quantum gravity.

Remark: cosmological consequences of BH evaporation. Black holes formed by collapse of stars have extremely small Hawking temperatures. So the Hawking effect could be observed only if astronomers discovered a black hole near the end of its life, with a very high surface temperature. However, the lifetimes of astrophysically plausible black holes are much larger than the age of the Universe which is estimated as $\sim 10^{10}$ years. To evaporate within this time, a black hole must be lighter than $\sim 10^{15}$g (see Exercise 9.3). Such black holes could not have formed as a result of stellar collapse and must be *primordial*, i.e. created at very early times when the universe was extremely dense and hot. There is currently no direct observational evidence for the existence of primordial black holes.

9.2.2 Laws of BH thermodynamics

Prior to the discovery of the BH radiation it was already known that black holes require a thermodynamical description involving a nonzero intrinsic entropy.

The entropy of a system is defined as the logarithm of the number of internal microstates of the system that are indistinguishable on the basis of macroscopically available information. Since the gravitational field of a static black hole is completely determined (both inside and outside of the horizon) by the mass M of the BH, one might expect that a black hole has only one microstate and therefore its entropy is zero. However, this conclusion is inconsistent with the second law of thermodynamics. A black hole absorbs all energy that falls onto it. If the black hole always had zero entropy, it could absorb some thermal energy and decrease the entropy of the world. This would violate the second law unless one assumes that the black hole has an intrinsic entropy that grows in the process of absorption.

Similar *gedanken* experiments involving classical general relativity and thermodynamics lead J. Bekenstein to conjecture in 1971 that a static black hole must have an intrinsic entropy S_{BH} proportional to the surface area $A = 16\pi M^2$. However, the coefficient of proportionality between S_{BH} and A could not be computed until the discovery of the Hawking radiation. The precise relation between the BH entropy and the horizon area follows from the first law of thermodynamics,

$$dE \equiv dM = T_H dS_{BH}, \tag{9.11}$$

where T_H is the Hawking temperature for a black hole of mass M. A simple calculation using Eq. (9.9) shows that

$$S_{BH} = 4\pi M^2 = \frac{1}{4}A. \tag{9.12}$$

To date, there seems to be no completely satisfactory explanation of the BH entropy. Here is an illustration of the problem. A black hole of one solar mass has the entropy $S_{\odot} \sim 10^{76}$. A microscopic explanation of the BH entropy would require to demonstrate that a solar-mass BH actually has $\exp(10^{76})$ indistinguishable microstates. A large number of microstates implies many internal degrees of freedom not visible from the outside. Yet, a black hole is almost all empty space, with the exception of a Planck-sized region around its center where the classical general relativity does not apply. It is not clear how this microscopically small region could contain such a huge number of degrees of freedom. A fundamental explanation of the BH entropy probably requires a theory of quantum gravity which is not yet available.

The thermodynamical law (9.11) suggests that in certain circumstances black holes behave as objects in thermal contact with their environment. This description applies to black holes surrounded by thermal radiation and to adiabatic processes of emission and absorption of heat.

> **Remark: rotating black holes.** A static black hole has no degrees of freedom except its mass M. A more general situation is that of a rotating BH with an angular momentum J. In that case it is possible to perform work on the BH in a reversible way by making it rotate faster or slower. The first law (9.11) can be modified to include contributions to the energy in the form of work.

For a complete thermodynamical description of black holes, one needs an equation of state. This is provided by the relation

$$E(T) = M = \frac{1}{8\pi T}.$$

It follows that the heat capacity of the BH is negative,

$$C_{BH} = \frac{\partial E}{\partial T} = -\frac{1}{8\pi T^2} < 0.$$

In other words, black holes become *colder* when they absorb heat.

The second law of thermodynamics now states that the combined entropy of all existing black holes and of all ordinary thermal matter never decreases,

$$\delta S_{total} = \delta S_{matter} + \sum_k \delta S_{BH}^{(k)} \geq 0.$$

Here $S_{BH}^{(k)}$ is the entropy (9.12) of the k-th black hole.

In classical general relativity it has been established that the combined area of all BH horizons cannot decrease (this is Hawking's "area theorem"). This statement applies not only to adiabatic processes but also to strongly out-of-equilibrium situations, such as a collision of black holes with the resulting merger. It is mysterious that this theorem, derived from a purely classical theory, assumes the form of the second law of thermodynamics when one considers quantum thermal effects of black holes.

Moreover, there is a general connection between horizons and thermodynamics which has not yet been completely elucidated. The presence of a horizon in a spacetime means that a loss of information occurs, since one cannot observe events beyond the horizon. Intuitively, a loss of information entails a growth of entropy. It seems to be generally true in the theory of relativity that any event horizon behaves as a surface with a certain entropy and emits radiation with a certain temperature.[5] For instance, the Unruh effect considered in Chapter 8 can be interpreted as a thermodynamical consequence of the presence of a horizon in the Rindler spacetime.

[5] See e.g. the paper by T. Padmanabhan, *Classical and quantum thermodynamics of horizons in spherically symmetric spacetimes,* Class. Quant. Grav. **19** (2002), p. 5378.

9.2.3 Black holes in heat reservoirs

As an application of the thermodynamical description, we consider a black hole inside a reservoir of thermal energy. The simplest such reservoir is a reflecting cavity filled with radiation. Usual thermodynamical systems can be in a stable thermal equilibrium with an infinite heat reservoir. However, the behavior of black holes is different because of their negative heat capacity.

A black hole surrounded by an infinite heat bath at a lower temperature $T < T_{BH}$ would emit heat and become even hotter. The process of evaporation is not halted by the heat bath whose low temperature T remains constant. On the other hand, a black hole placed inside an infinite reservoir with a higher temperature $T > T_{BH}$ will tend to absorb radiation from the reservoir and become colder. The process of absorption will continue indefinitely. In either case, no stable equilibrium is possible. The following exercise demonstrates that a black hole can be stabilized with respect to absorption or emission of radiation only by a reservoir with a *finite* heat capacity.

> **Exercise 9.4**
> **(a)** Given the mass M of the black hole, find the range of heat capacities C_r of the reservoir for which the BH is in a stable equilibrium with the reservoir.
> **(b)** Assume that the reservoir is a completely reflecting cavity of volume V filled with thermal radiation (massless fields). The energy of the radiation is $E_r = \sigma V T^4$, where the constant σ characterizes the number of degrees of freedom in the radiation fields. Determine the largest volume V for which a black hole of mass M can remain in a stable equilibrium with the surrounding radiation.
> *Hint:* The stable equilibrium is the state with the largest total entropy.