Appendix B
Advanced field theoretic methods

B–1 The heat kernel

When using path integral techniques one must often evaluate quantities of the form

$$H(x, \tau) \equiv \left\langle x \left| e^{-\tau \mathcal{D}} \right| x \right\rangle \quad , \tag{1.1}$$

where $\mathcal{D}$ is a differential operator and τ is a parameter. In this section, we shall describe the *heat kernel* method by which $H(x, \tau)$ is expressed as a power series in τ. For example, if in d dimensions the differential operator $\mathcal{D}$ is of the form

$$\mathcal{D} = \Box + m^2 + V \,, \tag{1.2}$$

where V is some interaction, then the heat kernel expansion for $H(x, \tau)$ is

$$H(x, \tau) = \frac{i}{(4\pi)^{d/2}} \frac{e^{-\tau m^2}}{\tau^{d/2}} \left[a_0(x) + a_1(x)\tau + a_2(x)\tau^2 + \ldots \right] \,. \tag{1.3}$$

where $a_i(x)$ are coefficients which will be determined below.

Let us begin by citing the two most common occurrences of $H(x, \tau)$. One is in the evaluation of the functional determinant

$$\det \mathcal{D} = e^{\operatorname{tr} \ln \mathcal{D}} = e^{\int d^4 x \ \operatorname{Tr} \langle x | \ln \mathcal{D} | x \rangle} \quad , \tag{1.4}$$

where 'Tr' is a trace over internal variables like isospin, Dirac matrices, *etc.*, and 'tr' is a trace over these plus spacetime. The (generally singular) matrix element $\langle x | \ln \mathcal{D} | x \rangle$ appearing in Eq. (1.4) can be expressed in a variety of ways. For example, in dimensional regularization one can use the identity

$$\ln \frac{b}{a} = \int_0^\infty \frac{dx}{x} \left(e^{-ax} - e^{-bx} \right) \tag{1.5}$$

to write

$$\langle x| \ln \mathcal{D} |x\rangle = -\int_0^\infty \frac{d\tau}{\tau} \left\langle x \left| e^{-\tau \mathcal{D}} \right| x \right\rangle + C \ , \qquad (1.6)$$

where C is a divergent constant having no physical consequences. Substituting Eq. (1.3) into the above yields

$$\langle x| \ln \mathcal{D} |x\rangle - C = -\frac{i}{(4\pi)^{d/2}} \sum_{n=0}^\infty m^{d-2n} \ \Gamma\left(n - \frac{d}{2}\right) a_n(x) \ . \qquad (1.7)$$

The divergences in the series representation arise from the Γ-function and are restricted in four dimensions to the terms $a_0(x), a_1(x), a_2(x)$.

The heat kernel can likewise be used to analyze the functional determinant in alternative regularization procedures, such as zeta-function regularization. Here, one expresses the matrix element $\langle x| \ln \mathcal{D} |x\rangle$ as

$$\langle x| \ln \mathcal{D} |x\rangle = -\left\langle x \left| \left[\frac{d}{ds} e^{-s \ln \mathcal{D}}\right]_{s=0} \right| x \right\rangle$$
$$= -\left[\frac{d}{ds} \left\langle x \left| \frac{1}{\mathcal{D}^s} \right| x \right\rangle\right]_{s=0} = -\frac{d}{ds} \xi_{\mathcal{D}}(x,s)\bigg|_{s=0} \ , \qquad (1.8)$$
$$\xi_{\mathcal{D}}(x,s) \equiv \frac{1}{\Gamma(s)} \int_0^\infty d\tau \, \tau^{s-1} H(x,\tau) \ .$$

The penultimate equality in Eq. (1.8) is obtained from repeated formal differentiation of Eq. (1.6) with respect to $\mathcal{D}$. Upon expanding the $H(x,\tau)$ term in $\xi_{\mathcal{D}}(x,s)$, one arrives at the desired power series expansion of $\langle x| \ln \mathcal{D} |x\rangle$. This usage is applied in the next section.

The other main use of the heat kernel is in the regularization of anomalies. Often one is faced with making sense of Tr $\langle x\,|O(x)|\,x\rangle$, where O is a local operator. Although such quantities are generally singular, they can be defined in a gauge-invariant manner by damping out the contributions of large eigenvalues,

$$\text{Tr} \ \langle x\,|O(x)|\,x\rangle = \lim_{\epsilon \to 0} \text{Tr} \ \left\langle x \left| O(x) e^{-\epsilon \mathcal{D}} \right| x \right\rangle \ , \qquad (1.9)$$

where $\mathcal{D}$ is a gauge-invariant differential operator. Again it is only the low-order coefficients, generally those up to $a_2(x)$, which contribute in the $\epsilon \to 0$ limit. We employ this technique in Sects. III–3,4.

As an example of heat kernel techniques, let us consider the following operator defined in d dimensions:

$$\mathcal{D} = d_\mu d^\mu + m^2 + \sigma(x) \qquad \left(d_\mu \equiv \frac{\partial}{\partial x^\mu} + \Gamma_\mu(x)\right) \ , \qquad (1.10)$$

where $\Gamma_\mu(x)$ and $\sigma(x)$ are functions and/or matrices defined in some internal symmetry space. In particular, neither Γ_μ nor σ contains derivative operators. Employing a complete set of momentum eigenstates $\{|p\rangle\}$

allows us to express the heat kernel as

$$H(x,\tau) = \int \frac{d^d p}{(2\pi)^d}\, e^{-ip\cdot x} e^{-\tau\mathcal{D}} e^{ip\cdot x} \quad, \tag{1.11}$$

where in d dimensions, use is made of the relations

$$\langle p|x\rangle = \frac{1}{(2\pi)^{d/2}} e^{ip\cdot x}\ ,$$

$$\langle x|x'\rangle = \int \frac{d^d p}{(2\pi)^d} e^{ip\cdot(x'-x)} = \delta^{(d)}(x-x')\ , \tag{1.12}$$

$$\langle p'|p\rangle = \int \frac{d^d x}{(2\pi)^d} e^{i(p'-p)\cdot x} = \delta^{(d)}(p'-p)\ \ .$$

From the identities

$$\begin{aligned}
d_\mu e^{ip\cdot x} &= e^{ip\cdot x}(ip_\mu + d_\mu)\ , \\
d_\mu d^\mu e^{ip\cdot x} &= e^{ip\cdot x}(ip_\mu + d_\mu)(ip^\mu + d^\mu)\ ,
\end{aligned} \tag{1.13}$$

we can then write

$$\begin{aligned}
H(x,\tau) &= \int \frac{d^d p}{(2\pi)^d}\, e^{-\tau\left[(ip_\mu + d_\mu)^2 + m^2 + \sigma\right]} \\
&= \int \frac{d^d p}{(2\pi)^d}\, e^{\tau[p^2 - m^2]} e^{-\tau[d\cdot d + \sigma + 2ip\cdot d]} \quad.
\end{aligned} \tag{1.14}$$

The first exponential factor is simply the free field result, while all the interesting physics is in the second exponential. The latter can be Taylor expanded in powers of τ, keeping those terms which contribute up to order τ^2 after integration over momentum. Note that each power of p^2 contributes a factor of $1/\tau$. Thus we obtain the expansion

$$\begin{aligned}
H(x,\tau) = \int \frac{d^d p}{(2\pi)^d}\, e^{\tau(p^2 - m^2)} &\Big[1 - \tau[d\cdot d + \sigma] \\
&+ \frac{\tau^2}{2}\left[(d\cdot d + \sigma)(d\cdot d + \sigma) - 4p\cdot d\, p\cdot d\right] \\
&+ \frac{4\tau^3}{3!}[p\cdot d\, p\cdot d(d\cdot d + \sigma) + p\cdot d(d\cdot d + \sigma)p\cdot d \\
&\quad + (d\cdot d + \sigma)p\cdot d\, p\cdot d] \\
&+ \frac{16\tau^4}{4!} p\cdot d\, p\cdot d\, p\cdot d\, p\cdot d + \dots \Big]\ ,
\end{aligned} \tag{1.15}$$

where terms odd in p have been dropped and we have displayed only those $\mathcal{O}(\tau^3)$ and $\mathcal{O}(\tau^4)$ terms which contribute to H at order τ^2 after p is integrated over. To perform the integral, it is convenient to continue to euclidean momentum $p_E = \{p_1, p_2, p_3, p_4 = -ip_0\}$. Then with

the replacement $p_\mu p^\mu \to -|p_E^\mu p_E^\mu| = -p_E^2$, we obtain

$$\int \frac{d^d p_E}{(2\pi)^d} e^{-(p_E^2+m^2)\tau} = \int \frac{d\Omega_d}{(2\pi)^d} \int dp_E \; p_E^{d-1} e^{-(p_E^2+m^2)\tau}$$

$$= \frac{2\pi^{d/2}}{\Gamma(d/2)} \frac{1}{(2\pi)^d} \frac{e^{-m^2\tau}\Gamma(d/2)}{2\tau^{d/2}}$$

$$= \frac{1}{(4\pi)^{d/2}} \frac{e^{-m^2\tau}}{\tau^{d/2}} \; ,$$

$$\int \frac{d^d p_E}{(2\pi)^d} e^{-(p_E^2+m^2)\tau} p_E^\mu p_E^\nu = \frac{\delta^{\mu\nu}}{d} \frac{1}{(4\pi)^{d/2}} \frac{e^{-m^2\tau}}{\tau^{d/2+1}} \frac{\Gamma(d/2+1)}{\Gamma(d/2)} \quad (1.16)$$

$$= \frac{\delta^{\mu\nu}}{2} \frac{e^{-m^2\tau}}{(4\pi)^{d/2}\tau^{d/2+1}} \; ,$$

$$\int \frac{d^d p_E}{(2\pi)^d} e^{-(p_E^2+m^2)\tau} p_E^\mu p_E^\nu p_E^\lambda p_E^\sigma = \frac{e^{-m^2\tau}}{(4\pi)^{d/2}\tau^{d/2+2}}$$

$$\times \frac{\left(\delta^{\mu\nu}\delta^{\lambda\sigma} + \delta^{\mu\lambda}\delta^{\nu\sigma} + \delta^{\mu\sigma}\delta^{\lambda\nu}\right)}{4} \; .$$

Employing these relations to evaluate Eq. (1.14) gives (to second order in τ),

$$H(x,\tau) = \frac{ie^{-m^2\tau}}{(4\pi)^{d/2}\tau^{d/2}}$$

$$\times \left[1 - \tau\sigma + \tau^2 \left(\frac{1}{2}\sigma^2 + \frac{1}{12}[d_\mu, d_\nu][d^\mu, d^\nu] + \frac{1}{6}[d_\mu, [d^\mu, \sigma]]\right)\right] \; , \quad (1.17)$$

or in the notation of Eq. (1.3),

$$a_0(x) = 1 \; , \qquad a_1(x) = -\sigma \; ,$$

$$a_2(x) = \frac{1}{2}\sigma^2 + \frac{1}{12}[d_\mu, d_\nu][d^\mu, d^\nu] + \frac{1}{6}[d_\mu, [d^\mu, \sigma]] \; . \quad (1.18)$$

Fermions are treated in a similar manner. For example, the identity

$$\ln\!\not{D} = \frac{1}{2}\ln(\not{D}\,\not{D}) \quad (1.19)$$

allows the same technique to be used for the operator $\not{D}\,\not{D}$. In particular let us consider the case where

$$\not{D} = \not{\partial} + i\,\not{V} + i\,\not{A}\gamma_5 \; . \quad (1.20)$$

With some work, one can cast this into the form of Eq. (1.9) with the identifications

$$
\begin{aligned}
\not{D}\,\not{D} &\equiv \mathcal{D} = d_\mu d^\mu + \sigma \ , \\
d_\mu &= \partial_\mu + iV_\mu + \sigma_{\mu\nu}A^\nu\gamma_5 \equiv \partial_\mu + \Gamma_\mu \ , \\
\sigma &= \frac{1}{2}\sigma_{\mu\nu}V^{\mu\nu} - 2A_\mu A^\mu + \left(i\partial_\mu A^\mu - [V_\mu, A^\mu]\right)\gamma_5 \ , \\
V_{\mu\nu} &= \partial_\mu V_\nu - \partial_\nu V_\mu + i[V_\mu, V_\nu] + i[A_\mu, A_\nu] \ .
\end{aligned}
\tag{1.21}
$$

The values of $a_i(x)$ appearing in Eq. (1.18) can also be used in this case. The heat kernel coefficients have been worked out for more general situations [Gi 75].

B–2 Chiral renormalization and background fields

In this section, we illustrate the method described above while also proving an important result for the theory of chiral symmetry. The goal is to demonstrate that all the divergences encountered at one loop can be absorbed into a renormalization of the coefficients of the $\mathcal{O}(E^4)$ chiral lagrangian and to identify the renormalization constants. The technique used here, the *background field method*, is of considerable interest in its own right [Sc 51, De 67, Ab 82, Bal 89] and is applicable to areas such as general relativity [BiD 82].

The basic idea of the background field method is to calculate quantum corrections about some nonvanishing field configuration $\overline{\varphi}$,

$$
\varphi(x) = \bar{\varphi}(x) + \delta\varphi(x) \ ,
\tag{2.1}
$$

rather than about the zero field,* and to then compute the path integral over the fluctuation $\delta\varphi(x)$. The result is an effective action for $\bar{\varphi}$. This effective action can be expanded in powers of $\overline{\varphi}$ and applied to matrix elements at tree level, resulting in a description of scattering processes at one-loop order. In the case of the chiral lagrangian, one expands the full chiral matrix

$$
U = \bar{U} + \delta U \ ,
\tag{2.2}
$$

where $\bar{U}$ satisfies the classical equation of motion. Upon integration over δU, one obtains the one-loop effective action for $\bar{U}$. This contains a great deal of information. In particular, $\bar{U}$ can be expanded in the usual way in terms of a set of external meson fields

$$
\bar{U} = \exp(i\lambda^a\bar{\varphi}^a/F) \ \ (a = 1, \ldots, 8) \ .
\tag{2.3}
$$

---

* See the discussion in Appendix A–4.

Contained in $S_{\text{eff}}(\bar{U})$ is the effective one-loop action for arbitrary numbers of meson fields. Upon identification of renormalization constants, all processes become renormalized at the same time.

Our starting point is, in the notation of Sect. IV–6, the $\mathcal{O}(E^2)$ lagrangian

$$\mathcal{L}_2 = \frac{F_0^2}{4} \operatorname{Tr}\left(D_\mu U D^\mu U^\dagger\right) + \frac{F_0^2}{4} \operatorname{Tr}\left(\chi^\dagger U + U^\dagger \chi\right) \quad . \qquad (2.4)$$

The procedure to follow is rather technical, so let us first quote the end result of the calculation. Upon performing the one-loop quantum corrections, the effective action will have the form

$$S_{\text{eff}} = S_2^{\text{ren}} + S_4^{\text{ren}} + S_4^{\text{finite}} + \ldots \quad .$$

Here the lagrangians in S_2^{ren}, S_4^{ren} are the ones quoted in Sect. VII–2, but now with renormalized coefficients. In particular S_4^{ren} is the sum $S_4^{\text{ren}} = S_4^{\text{bare}} + S_4^{\text{div}}$ where, in chiral $SU(3)$ and employing dimensional regularization, S_4^{div} is given by

$$
\begin{aligned}
S_4^{\text{div}} = -\lambda \int d^4x \Bigg[&\frac{3}{32} \left[\operatorname{Tr}\left(D_\mu U D^\mu U^\dagger\right) \right]^2 \\
&+ \frac{3}{16} \operatorname{Tr}\left(D_\mu U D_\nu U^\dagger\right) \operatorname{Tr}\left(D^\mu U D^\nu U^\dagger\right) \\
&+ \frac{1}{8} \operatorname{Tr}\left(D_\mu U D^\mu U^\dagger\right) \operatorname{Tr}\left(\chi^\dagger U + U^\dagger \chi\right) \\
&+ \frac{3}{8} \operatorname{Tr}\left[D_\mu U D^\mu U^\dagger \left(\chi U^\dagger + U \chi^\dagger\right)\right] \\
&+ \frac{11}{144} \left[\operatorname{Tr}\left(\chi U^\dagger + U \chi^\dagger\right) \right]^2 + \frac{5}{48} \operatorname{Tr}\left(\chi U^\dagger \chi U^\dagger + U \chi^\dagger U \chi^\dagger\right) \\
&+ \frac{i}{4} \operatorname{Tr}\left(L_{\mu\nu} D^\mu U D^\nu U^\dagger + R_{\mu\nu} D^\mu U^\dagger D^\nu U\right) - \frac{1}{4} \operatorname{Tr}\left(L_{\mu\nu} U R^{\mu\nu} U^\dagger\right) \Bigg]
\end{aligned}
\qquad (2.5)
$$

with

$$\lambda \equiv \frac{1}{32\pi^2} \left\{ \frac{2}{d-4} - \ln 4\pi - 1 + \gamma \right\} \quad . \qquad (2.6)$$

The terms in S_4^{div} are all of the same form as the terms in the bare lagrangian at order E^4. Therefore, all the divergences can be absorbed into renormalized values of these constants. The finite remainder, S_4^{finite}, cannot be simply expressed as a local lagrangian, but can be worked out for any given transition. When S_4^{div} is added to the $\mathcal{O}(E^4)$ tree-level lagrangian of Eq. (VII–2.7), the result has the same form but with coefficients

$$\alpha_i^r = \alpha_i - \gamma_i \lambda \quad , \qquad (2.7)$$

where the $\{\gamma_i\}$ are numbers which are given in Table B–1 for both the case of chiral $SU(2)$ and $SU(3)$. Thus the divergences can all be absorbed into the redefined parameters and these in turn can be determined from experiment. Let us now turn to the task of obtaining this result.

In applying the background field method, there are a variety of ways to parameterize δU, and several different ones are used in the literature. The prime consideration is to maintain the unitarity property $U^\dagger U = 1 = \left(\bar{U}^\dagger + \delta U^\dagger\right)\left(\bar{U} + \delta U\right)$ along with $\bar{U}^\dagger \bar{U} = 1$. We shall take

$$U = \bar{U}e^{i\Delta} , \tag{2.8}$$

with $\Delta \equiv \lambda^a \Delta^a$ representing the quantum fluctuations. This choice is made to simplify the algebra in the heat kernel renormalization approach, which we shall describe shortly. Another possible choice is

$$U = \xi e^{i\eta}\xi \tag{2.9}$$

with $\eta = \lambda^a \eta^a$ and $\xi\xi \equiv \bar{U}$. These two forms are related by $\eta = \xi\Delta\xi^\dagger$. Since in the path integral, we integrate over all values of Δ (or η) at each point of spacetime, these two choices are equivalent.

The expansion of the lagrangian in terms of $\bar{U}$ and Δ is straightforward, and we find

$$
\begin{aligned}
\mathrm{Tr}\left(D_\mu U D^\mu U^\dagger\right) &= \mathrm{Tr}\left(D_\mu \bar{U} D^\mu \bar{U}^\dagger\right) - 2i\,\mathrm{Tr}\left(\bar{U}^\dagger D_\mu \bar{U}\tilde{D}^\mu\Delta\right) \\
&\quad + \mathrm{Tr}\left[\tilde{D}_\mu\Delta\tilde{D}^\mu\Delta + \bar{U}^\dagger D_\mu\bar{U}\left(\Delta\tilde{D}^\mu\Delta - \tilde{D}^\mu\Delta\Delta\right)\right] , \\
\mathrm{Tr}\left(\chi^\dagger U + U^\dagger\chi\right) &= \mathrm{Tr}\left(\chi^\dagger\bar{U} + \bar{U}^\dagger\chi\right) + i\,\mathrm{Tr}\left(\Delta\left(\chi^\dagger\bar{U} - \bar{U}^\dagger\chi\right)\right) \\
&\quad - \frac{1}{2}\,\mathrm{Tr}\left[\Delta^2\left(\chi^\dagger\bar{U} + \bar{U}^\dagger\chi\right)\right] ,
\end{aligned}
\tag{2.10}
$$

where

$$\tilde{D}_\mu\Delta \equiv \partial_\mu\Delta + i\left[r_\mu, \Delta\right] . \tag{2.11}$$

Table B–1. Renormalization coefficients

	i	1	2	3	4	5	6	7	8	9	10
$SU(2)$	γ_i	$\frac{1}{12}$	$\frac{1}{6}$	0	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{3}{32}$	0	0	$\frac{1}{6}$	$-\frac{1}{6}$
$SU(3)$	γ_i	$\frac{3}{32}$	$\frac{3}{16}$	0	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{11}{144}$	0	$\frac{5}{48}$	$\frac{1}{4}$	$-\frac{1}{4}$

Since $\bar{U}$ satisfies the equation of motion, there is no term linear in Δ. One may integrate various terms in the action by parts to obtain

$$S_2^{(0)} = \int d^4x \left\{ \mathcal{L}_2(\bar{U}) - \frac{F_0^2}{2} \Delta^a \left(d_\mu d^\mu + \sigma \right)^{ab} \Delta^b + \dots \right\} \tag{2.12}$$

where

$$d_\mu^{ab} = \delta^{ab} \partial_\mu + \Gamma_\mu^{ab} ,$$

$$\Gamma_\mu^{ab} = -\frac{1}{4} \operatorname{Tr} \left(\left[\lambda^a, \lambda^b \right] \left(\bar{U}^\dagger \partial_\mu \bar{U} + i \bar{U}^\dagger \ell_\mu \bar{U} + i r_\mu \right) \right) ,$$

$$\sigma^{ab} = \frac{1}{8} \operatorname{Tr} \left(\left\{ \lambda^a, \lambda^b \right\} \left(\chi^\dagger \bar{U} + \bar{U}^\dagger \chi \right) + \left[\lambda^a, \bar{U}^\dagger D_\mu \bar{U} \right] \left[\lambda^b, \bar{U}^\dagger D^\mu \bar{U} \right] \right) . \tag{2.13}$$

The action is now a simple quadratic form, and the path integral may be performed. The only potential complication is the question of interpreting the integration variables. This is referred to as the 'question of the path integral measure'. The integration over all the unitary matrices U can be accomplished by an integration over the parameters in the exponential

$$\int [dU] = N \int [d\Delta^a] , \tag{2.14}$$

where N is a constant which plays no dynamical role. With this identification one obtains

$$e^{iW_{\text{loop}}} = \int [d\Delta^a] e^{i \int d^4x \frac{F^2}{2} \Delta^a (d_\mu d^\mu + \sigma)^{ab} \Delta^b}$$

$$= \left(\det \left[d_\mu d^\mu + \sigma \right] \right)^{-1/2} = \exp \left\{ -\frac{1}{2} \operatorname{tr} \ln \left(d_\mu d^\mu + \sigma \right) \right\} . \tag{2.15}$$

Here 'tr' indicates a trace over the spacetime indices as well as over the $SU(N)$ indices a, b.

The identification of divergences is most conveniently done by using the heat kernel expansion derived earlier in this appendix, where it is shown that all the ultraviolet divergences are contained in the first few expansion coefficients. The relevant terms are

$$W_{\text{loop}} = \frac{i}{2} \operatorname{tr} \ln \left(d_\mu d^\mu + \sigma \right)$$

$$= \frac{1}{2(4\pi)^{d/2}} \int d^4x \lim_{m \to 0} \left\{ \Gamma \left(1 - \frac{d}{2} \right) m^{d-2} \operatorname{Tr} \sigma \right. \tag{2.16}$$

$$\left. + m^{d-4} \Gamma \left(2 - \frac{d}{2} \right) \operatorname{Tr} \left(\frac{1}{12} \Gamma_{\mu\nu} \Gamma^{\mu\nu} + \frac{1}{2} \sigma^2 \right) + \dots \right\} ,$$

where

$$\Gamma_{\mu\nu}^{ab} = \partial_\mu \Gamma_\nu^{ab} - \partial_\nu \Gamma_\mu^{ab} + \Gamma_\mu^{ac} \Gamma_\nu^{cb} - \Gamma_\nu^{ac} \Gamma_\mu^{cb} = [d_\mu, d_\nu]^{ab} . \tag{2.17}$$

For N_f flavors, the operator part of the first term in Eq. (2.16) is

$$\text{Tr}\,\sigma = \frac{N_f}{2}\,\text{Tr}\,\left(D_\mu \bar{U} D^\mu \bar{U}^\dagger\right) + \frac{N_f^2 - 1}{2N_f}\,\text{Tr}\,\left(\chi^\dagger \bar{U} + \bar{U}^\dagger \chi\right)\ . \tag{2.18}$$

The above two traces are just those which appear in $\mathcal{L}_2^{(0)}$, so that they can only modify F_π and m_π^2. The remaining terms can be worked out with a bit more algebra. Using the identity

$$\partial_\mu \left(\bar{U}^\dagger \partial_\nu \bar{U}\right) - \partial_\nu \left(\bar{U}^\dagger \partial_\mu \bar{U}\right) = -\left[\bar{U}^\dagger \partial_\mu U, \bar{U}^\dagger \partial_\nu U\right]\ , \tag{2.19}$$

we find for the field strength,

$$\Gamma_{\mu\nu}^{ab} = \frac{1}{8}\,\text{Tr}\,\left\{\left[\lambda^a, \lambda^b\right]\left(\left[\bar{U}^\dagger D_\mu \bar{U}, \bar{U}^\dagger D_\nu \bar{U}\right] + i\bar{U}^\dagger L_{\mu\nu}\bar{U} + iR_{\mu\nu}\right)\right\}\ . \tag{2.20}$$

This produces, for N_f flavors in chiral $SU(N_f)$,

$$\begin{aligned}
\text{Tr}\,\left(\Gamma_{\mu\nu}\Gamma^{\mu\nu}\right) &= \frac{N_f}{8}\,\text{Tr}\,\left(\left[\bar{U}^\dagger D_\mu \bar{U}, \bar{U}^\dagger D_\nu \bar{U}\right]\left[\bar{U}^\dagger D^\mu \bar{U}, \bar{U}^\dagger D^\nu \bar{U}\right]\right) \\
&\quad + iN_f\,\text{Tr}\,\left(R_{\mu\nu}\partial^\mu \bar{U}^\dagger \partial^\nu \bar{U} + L_{\mu\nu}\partial^\mu \bar{U}\partial^\nu \bar{U}^\dagger\right) \\
&\quad - N_f\,\text{Tr}\,\left(L_{\mu\nu}\bar{U}R^{\mu\nu}\bar{U}^\dagger\right) - \frac{N_f}{2}\,\text{Tr}\,\left(L_{\mu\nu}L^{\mu\nu} + R_{\mu\nu}R^{\mu\nu}\right)\ ,
\end{aligned}$$

$$\begin{aligned}
\text{Tr}\,\sigma^2 &= \frac{1}{8}\left[\text{Tr}\,\left(D_\mu \bar{U}D^\mu \bar{U}^\dagger\right)\right]^2 + \frac{1}{4}\,\text{Tr}\,\left(D_\mu \bar{U}D_\nu \bar{U}^\dagger\right)\,\text{Tr}\,\left(D^\mu \bar{U}D^\nu \bar{U}^\dagger\right) \\
&\quad + \frac{N_f}{8}\,\text{Tr}\,\left(D_\mu \bar{U}D^\mu \bar{U}^\dagger D_\nu \bar{U}D^\nu \bar{U}^\dagger\right) + \frac{2 + N_f^2}{8N_f^2}\left[\text{Tr}\,\left(\chi\bar{U}^\dagger + \bar{U}^\dagger \chi\right)\right]^2 \\
&\quad + \frac{1}{4}\,\text{Tr}\,\left(D_\mu \bar{U}D^\mu \bar{U}^\dagger\right)\,\text{Tr}\,\left(\chi\bar{U}^\dagger + \bar{U}\chi^\dagger\right) \\
&\quad + \frac{N_f}{4}\,\text{Tr}\,\left(D_\mu \bar{U}D^\mu \bar{U}^\dagger \left(\chi\bar{U}^\dagger + \bar{U}\chi^\dagger\right)\right) \\
&\quad + \frac{N_f^2 - 4}{8N_f}\,\text{Tr}\,\left(\left(\chi\bar{U}^\dagger + \bar{U}\chi^\dagger\right)\left(\chi\bar{U}^\dagger + \bar{U}\chi^\dagger\right)\right)\ .
\end{aligned}$$

$$\tag{2.21}$$

The only operator which is not of the same form as the basic $\mathcal{O}(E^4)$ lagrangian occurs in the first term of $\text{Tr}\,\Gamma^2$. However, by use of Eq. (VII–2.3) for $SU(3)$, it can be written as a linear combination of our standard forms. For $N_f = 3$, these add up to the result previously quoted in Eq. (2.5). Here the divergence is in the parameter λ. For convenience in applications, we have added some finite terms to the definitions of λ. The results for $N_f = 2$ are also quoted in Table B–1, although some of the operators are redundant for that case.

The reader who has understood the above development as well as the standard perturbative methods presented in the main text will be prepared for the use of the background field method in the full calculation of transition amplitudes. This procedure consists of writing

$$
\begin{aligned}
d_\mu d^\mu + \sigma &= \mathcal{D}_0 + V \\
\mathcal{D}_0 &= \square + m^2 \\
V &= \{\partial_\mu, \Gamma^\mu\} + \Gamma_\mu \Gamma^\mu + \sigma - m^2 \quad,
\end{aligned}
\tag{2.22}
$$

where m^2 is the meson mass-squared matrix. The one-loop action is then expanded in powers of the interaction V

$$
\begin{aligned}
W_{loop} &= \frac{i}{2} \operatorname{tr} \ln(d_\mu d^\mu + \sigma) = \frac{i}{2} \operatorname{tr} \left[\ln \mathcal{D}_0 + \ln(1 + \mathcal{D}_0^{-1} V) \right] \\
&= \frac{i}{2} \operatorname{tr} \left[\ln \mathcal{D}_0 + \mathcal{D}_0^{-1} V - \frac{1}{2} \mathcal{D}_0^{-1} V \mathcal{D}_0^{-1} V + \ldots \right] \quad.
\end{aligned}
\tag{2.23}
$$

The first term is an uninteresting constant which may be dropped, and the remainder has the coordinate space form

$$
\begin{aligned}
W_{loop} = &-\frac{i}{2} \int d^4x \operatorname{Tr} \left[\Delta_F(x - x) V(x) \right] \\
&- \frac{i}{4} \int d^4x d^4y \operatorname{Tr} \left[\Delta_F(x - y) V(y) \Delta_F(y - x) V(x) \right] + \ldots \quad.
\end{aligned}
\tag{2.24}
$$

When the matrix elements of this action are taken, the result contains not only the divergent terms calculated above, but also the finite components of the one-loop amplitudes. The resulting expressions are presented fully in [GaL 84,GaL 85a]. This method allows one to calculate the one-loop corrections to many processes at the same time, and in practice is a much simpler procedure for some of the more difficult calculations.

B–3 PCAC and the soft-pion theorem

We have emphasized the use of effective lagrangians to elucidate the symmetry predictions of a theory. For a dynamically broken chiral symmetry such as QCD these predictions will relate processes which have different numbers of Goldstone bosons. The machinery of effective lagrangians will correctly yield such predictions, but it is often useful to have an alternative technique for understanding or calculating these results. In the case of chiral symmetry, this is provided by the *soft-pion theorem* which explicitly relates a process with a pion to one with that pion removed from the amplitude. Calculations performed this way uses current algebra methods which go by the name of *Partial Conservation*

of the Axial Current or *PCAC* [AdD 68]. While these techniques are often more cumbersome, they often are useful. This section describes these methods.

We can again turn to the sigma model to introduce this subject. We return to the effective lagrangian treatment of Chap.–IV, with a pion mass included and the S-field integrated out. The lagrangian of Eq. (IV-6.12) gives rise to the vector and axial-vector currents

$$V_\mu^k = -i\frac{v^2}{4}\,\mathrm{Tr}\,\left[\tau^k\left(U^\dagger\partial_\mu U + U\partial_\mu U^\dagger\right)\right] \; ,$$
$$A_\mu^k = i\frac{v^2}{4}\,\mathrm{Tr}\,\left[\tau^k\left(U^\dagger\partial_\mu U - U\partial_\mu U^\dagger\right)\right] \quad , \tag{3.1}$$

with $k = 1, 2, 3$. The equation of motion is found to be

$$\partial^\mu\left(U^\dagger\partial_\mu U\right) + \frac{m_\pi^2}{2}\left(U - U^\dagger\right) = 0 \;\; . \tag{3.2}$$

and two important matrix elements are

$$\left\langle 0\left|A_\mu^k\right|\pi^j(\mathbf{p})\right\rangle = ivp_\mu\delta^{kj} \;\; , \quad \left\langle 0\left|\partial^\mu A_\mu^k\right|\pi^j(\mathbf{p})\right\rangle = vm_\pi^2\delta^{kj} \;\; . \tag{3.3}$$

The former allows the identification $v = F_\pi$, where F_π is the pion decay constant $F_\pi \simeq 92$ MeV, while the latter follows either from Eq. (3.1) directly, or by use of the equation of motion for A_μ^k,

$$\partial^\mu A_\mu^k = -i\frac{v^2 m_\pi^2}{4}\,\mathrm{Tr}\,\left[\tau^k\left(U - U^\dagger\right)\right] = F_\pi m_\pi^2\pi^k + \dots \;\; . \tag{3.4}$$

This last equation forms the heart of the *PCAC* method. It describes a situation covered by Haag's theorem (recall Sect. IV–1), and says that we may use either π^k or $\partial^\mu A_\mu^k$ (properly normalized) as the pion field. It is more general than the sigma model which we used to motivate it. This, plus certain smoothness assumptions, gives rise to a *soft-pion theorem* for the following matrix element of a local operator O,

$$\lim_{q^\mu\to 0}\left\langle\pi^k(\mathbf{q})\beta|O|\alpha\right\rangle = -\frac{i}{F_\pi}\left\langle\beta\left|\left[Q_5^k, O\right]\right|\alpha\right\rangle \;\; , \tag{3.5}$$

where β, α are arbitrary states and $Q_5^k = \int d^3x\, A_0^k(x)$ is an axial charge.

The proof of this starts with the *LSZ* reduction formula. We consider the matrix element for the process $\alpha \to \beta + \pi^k(q)$ as the pion four-momentum q is taken off the mass-shell,

$$\left\langle\pi^k(q)\beta|O(0)|\alpha\right\rangle = i\int d^4x\, e^{iq\cdot x}\left(\Box + m_\pi^2\right)\left\langle\beta|T\left(\pi^k(x)O(0)\right)|\alpha\right\rangle$$
$$= i\int d^4x\, e^{iq\cdot x}(-q^2 + m_\pi^2)\left\langle\beta|T\left(\pi^k(x)\mathcal{O}(0)\right)|\alpha\right\rangle \; , \tag{3.6}$$

The pion field can be replaced by using the $PCAC$ relation (valid in the sense of the Haag theorem),

$$\pi^k = \frac{1}{F_\pi m_\pi^2} \partial^\mu A_\mu^k \ , \tag{3.7}$$

leading to

$$\left\langle \pi^k(q)\beta|O(0)|\alpha \right\rangle = i\frac{(m_\pi^2 - q^2)}{F_\pi m_\pi^2} \int d^4x \, e^{iq\cdot x} \left\langle \beta|T\left(\partial^\mu A_\mu^k(x)O(0)\right)|\alpha \right\rangle \ . \tag{3.8}$$

The derivative can be extracted from the time-ordered product by using

$$\partial^\mu \left\langle \beta|T\left(A_\mu^k(x)O(0)\right)|\alpha \right\rangle$$
$$= \left\langle \beta|T\left(\partial^\mu A_\mu^k(x)O(0)\right)|\alpha \right\rangle + \delta(x_0)\left\langle \beta|\left[A_0^k(x),O(0)\right]|\alpha \right\rangle \ , \tag{3.9}$$

where the last term arises from differentiating the functions $\theta(\pm x_0)$ which occur in the time-ordering prescription. Upon integrating by parts, we find

$$\left\langle \pi^k(q)\beta|O(0)|\alpha \right\rangle = i\frac{(m_\pi^2 - q^2)}{F_\pi m_\pi^2} \int d^4x \, e^{iq\cdot x}$$
$$\times \left[-\left\langle \beta|\left[A_0^k(x),O(0)\right]|\alpha \right\rangle \delta(x_0) - iq^\mu \left\langle \beta|T\left(A_\mu^k(x)\mathcal{O}(0)\right)|\alpha \right\rangle \right] \ . \tag{3.10}$$

Up to this stage all the formulae are exact for physical processes, even if appearing rather senseless, since $\partial^\mu A_\mu^k$ has the same singularity for $q^2 \to m_\pi^2$ as does the field π^k. However, to obtain the soft-pion theorem one assumes that the matrix element does not vary much between its on-shell value and the point where the pion's four-momentum vanishes. In that circumstance, we have [NaL 62, AdD 68]

$$\lim_{q^\mu \to 0}\left\langle \pi^k(q)\beta|O|\alpha \right\rangle = -\frac{i}{F_\pi}\left\langle \beta|\left[Q_5^k,O(0)\right]|\alpha \right\rangle + \lim_{q^\mu \to 0}iq^\mu R_\mu^k \ , \tag{3.11}$$

where

$$R_\mu^k = -\frac{i}{F_\pi}\int d^4x \, e^{iq\cdot x}\left\langle \beta|T\left(A_\mu^k(x)O(0)\right)|\alpha \right\rangle \ . \tag{3.12}$$

The remainder term of Eq. (3.11) vanishes unless R_μ^k has a singularity as $q^\mu \to 0$. Such a singularity can occur if there are intermediate states in R_μ^k which are degenerate in mass with either α or β. This last statement can be proven by inserting a complete set of intermediate states in the time-ordered product in R_μ^k, and taking the $q^\mu \to 0$ limit. This caveat should be kept in mind as it is sometimes relevant.

The soft-pion theorem relates to the intuitive picture for dynamically broken symmetries mentioned in Sect. I–6. Since a chiral transformation

corresponds in the symmetry limit to the addition of a zero energy Gold-stone boson, we expect the states $\langle\beta|$ and $\langle\pi^k_{q_\mu=0}\beta|$ to be related by the symmetry, and indeed, the soft-pion theorem expresses this. Although the soft-pion theorem is exact in the symmetry limit, a smoothness assumption is needed in the real world to pass from $q_\mu = 0$ to $q^2 = m^2_\pi$, implying that corrections of order q_μ or of order m^2_π can be expected.

In the Standard Model, the charge commutation rules are commonly abstracted from those of the quark model. Upon expressing charge operators in terms of quark fields,

$$Q^k = \int d^3x \ \bar{\psi}\gamma_0\frac{\lambda^k}{2}\psi \ , \quad Q^k_5 = \int d^3x \ \bar{\psi}\gamma_0\gamma_5\frac{\lambda^k}{2}\psi \ , \tag{3.13}$$

one obtains the algebra

$$\begin{aligned}
\left[Q^i, V^j_\mu\right] &= if^{ijk}V^k_\mu \ , & \left[Q^i_5, V^j_\mu\right] &= if^{ijk}A^k_\mu \ , \\
\left[Q^i, A^j_\mu\right] &= if^{ijk}A^k_\mu \ , & \left[Q^i_5, A^j_\mu\right] &= if^{ijk}V^k_\mu \ .
\end{aligned} \tag{3.14}$$

These commutation rules can be extended to equal-time commutators which contain a charge density, *e.g.*

$$\left[V^i_0(x), A^j_\mu(y)\right]_{x^0=y^0} = if^{ijk}A^k_\mu\delta^{(3)}(\mathbf{x}-\mathbf{y}) \ . \tag{3.15}$$

However, commutators which involve two spatial components can be more problematic [AdD 68].

Sometimes, in the *PCAC* approach, if the matrix element is assumed to be strictly constant, the various soft-pion limits turn out to be contradictory. If so, the amplitude must be extended to include momentum dependence, as happens in nonleptonic kaon decay. By contrast, the effective lagrangian approach automatically gives the appropriate momentum dependence, and its predictions follow in a straightforward manner. Moreover, effective lagrangians are especially useful in identifying and parameterizing corrections to the lowest order results. They allow a systematic expansion in terms of energy and mass.

B–4 Matching fields with different symmetry transformation properties

In Chapter IV, we described the construction of an effective lagrangian for pion fields with chiral transformation properties. However, most particles do not transform in the same way as the pions of that Chapter. In a broader context, we require a procedure for combining fields with different symmetry properties. For example, in the case of hadronic physics one often needs to consider particles such as nucleons, $\rho(770)$, *etc.* interacting with pions. A general approach for this was presented in a set of

classic papers on the subject [We 68, CoWZ 69, CaCWZ 69]. We shall introduce this framework by again referring to the sigma model, and then we shall extend the results.

Heavy particles do not themselves exist in chiral multiplets. For example, the chiral partner of $\rho(770)$ would be the $J^{\mathrm{PC}} = 1^{++}$ state $a(1260)$. The $a(1260)$–$\rho(770)$ mass difference is considerable, and attempts to pair these particles in a chiral multiplet would clearly be a matter of speculation. However, since each falls into vectorial flavor ($SU(2)$ or $SU(3)$) multiplets, it makes sense to build in only vectorial flavor invariance without invoking assumptions about chiral properties.

We shall proceed by first working out an example, the fermionic sector of the linear sigma model,

$$
\begin{aligned}
\mathcal{L}_f &= \bar{\psi} \left[i\slashed{\partial} - g \left(\sigma - i\boldsymbol{\tau} \cdot \boldsymbol{\pi} \gamma_5 \right) \right] \psi \\
&= \bar{\psi}_L i\slashed{\partial} \psi_L + \bar{\psi}_R i\slashed{\partial} \psi_R - gv \left(1 + \frac{S}{v} \right) \left(\bar{\psi}_L U \psi_R + \bar{\psi}_R U^\dagger \psi_L \right) \quad .
\end{aligned}
\tag{4.1}
$$

We shall drop reference to the scalar field S in the following. The above lagrangian is invariant under the chiral transformations

$$
\psi_L \to L\psi_L, \qquad \psi_R \to R\psi_R, \qquad U \to L\,U\,R^\dagger \quad ,
\tag{4.2}
$$

with L in $SU(2)_L$ and R in $SU(2)_R$. As always, we are free to change variables via contact transformations. In this instance, a useful choice of field redefinitions turns out to be

$$
N_L \equiv \xi^\dagger \psi_L, \qquad N_R \equiv \xi \psi_R, \qquad U = \xi\,\xi \quad ,
\tag{4.3}
$$

where $\xi = \exp(i\boldsymbol{\tau} \cdot \boldsymbol{\pi}/2F_\pi)$. This is seen, after some algebra, to convert the fermion lagrangian to

$$
\begin{aligned}
\mathcal{L}'_f &= \overline{N} \left(i\slashed{\mathcal{D}} - \overline{\slashed{A}}\gamma_5 - \mathbf{M} \right) N \ , \qquad \mathcal{D}_\mu = \partial_\mu + i\overline{V}_\mu \ , \\
\overline{V}_\mu &= -\frac{i}{2} \left(\xi^\dagger \partial_\mu \xi + \xi \partial_\mu \xi^\dagger \right) \ , \qquad \overline{A}_\mu = -\frac{i}{2} \left(\xi^\dagger \partial_\mu \xi - \xi \partial_\mu \xi^\dagger \right) \ ,
\end{aligned}
\tag{4.4}
$$

which is a theory of fermions of mass $M = gv$ having pseudovector coupling. The new fields transform as

$$
\begin{aligned}
\xi &\to L\xi V^\dagger \equiv V\,\xi\,R^\dagger \ , & N_L &\to V\,N_L \ , & N_R &\to V\,N_R \ , \\
\overline{V}_\mu &\to V \left(\overline{V}_\mu - i\partial_\mu V^\dagger \cdot V \right) V^\dagger \ , & \overline{A}_\mu &\to V\,\overline{A}_\mu V^\dagger \ , & \mathcal{D}_\mu N &\to V\,\mathcal{D}_\mu N \ .
\end{aligned}
\tag{4.5}
$$

For purely vector transformations we have $L = R = V$. For $L \neq R$, the property of V is more complicated, and Eq. (4.5) implies that it cannot be a simple global transformation, but must be a function of $\boldsymbol{\pi}(x)$ and hence a function of x. At first sight, the need to express an $SU(2)$ transformation matrix like V as a function of $\boldsymbol{\pi}(x)$ appears unnatural. However, it is in fact consistent with physical expectations. Recall from

the general discussion of dynamical symmetry breaking in Sect. I–6 that in the symmetry limit, axial transformations mix the proton not with the neutron (as in isospin transformations), but rather with states consisting of nucleons plus zero momentum pions. Mathematically, the important point is that N_L and N_R transform in an identical fashion. This corresponds to the fact that heavy fields do not transform chirally, but have a common vectorial $SU(2)$ transformation. It can be directly verified that Eq. (3.5) is a symmetry of the lagrangian. Thus we have obtained the expected result that the baryons can have a vectorial $SU(2)$ invariance, while maintaining a chiral invariance for pion couplings.

We see in the above example the ingredients of a general procedure for adding heavy fields to effective chiral lagrangians. The heavy fields are assumed to have an $SU(2)$ (or $SU(n)$, if desired) transformation described by the matrix V. A derivative ∂_μ acting on a heavy field must be incorporated as part of a covariant derivative $\mathcal{D}_\mu$ in order to maintain this invariance. Couplings to pions are described by the matrices ξ and U, with ξ having the same transformation as in Eq. (4.5). It is usually straightforward to combine factors of ξ and U in such a way that the overall lagrangian is invariant. In the general case, each invariant term will have an unknown coefficient which must be determined phenomenologically. For example, the $\bar{N}\,\slashed{A}\gamma_5\,N$ term in Eq. (4.4) would be expected to have a coefficient different from unity; the unit coefficient is a prediction specific to the linear sigma model. Effects which break the symmetry in an explicit fashion, like mass terms or electroweak interactions, can be added by using appropriate external sources. To date, heavy field lagrangians have been used in applications primarily at tree level. The feature which is essential for their application is that the *pion* momenta are small, and hence the heavy fields are essentially static.