

# EPFL Lectures - Effective Field Theory

Note Title

0/27/2016

What is Effective Field Theory?

full  $\xrightarrow{\text{QFT}}$  with clear thinking of scales

We "do physics" at some range of energies  
- particles + interactions

Physics from much higher energy scale only slightly relevant  $\Rightarrow$  small  $\# \alpha$

EFT helps separate relevant from irrelevant - Simplifies theory

EFT teaches us how to handle "non-renormalizable" theories

My treatment here focuses on  
"non-linear, nonrenormalizable"

$\Rightarrow$  most like GR

Other flavors of EFT exist

All of our theories are effective field theories

Homework:

- you decide
- something

Lecture 1 : ✓

Why do quantum calculations work?

$$\sum_I \frac{\langle f | V | I \rangle \langle I | V | i \rangle}{E - E_i}$$

dip  $\nearrow$   $\nwarrow$  all state  $\rightarrow$  new unknown states

Loops  $\int_{\text{on}}$   $\leftarrow$  all p

All of physics is sensitive to high energy at some order

First organizing principle:

Uncertainty principle and locality

$\Delta x \Delta p \sim h$

High E  $\rightarrow$  Local,  $\Delta x, \Delta t$  small

Looks like some term in a Lagrangian  
parameters  $\epsilon, m, g$   
measure

Unknown stuff  $\Rightarrow$  renorm of parameters

Example - Heavy particle in vacuum polarization,

$$\overbrace{\text{---}}^{\sim} + \overbrace{\text{---}}^{\sim} + \overbrace{\text{---}}^{\sim} + \dots = \frac{1}{g^2} \frac{\ell_s}{1 - \Pi(g^2)}$$

MS renormalization

$$\frac{\ell_s^2}{\ell_{\text{MS}}^2} = \frac{\ell_s^2}{1 - \frac{e^2}{12\pi^2} \left[ \frac{1}{\epsilon} - \gamma + \ln \frac{m^2}{\mu^2} \right]}$$

But there is also a  $\ln M_H^2$  in result

$$\Rightarrow \frac{1}{g^2} \frac{\ell_s}{1 + \frac{e^2}{12\pi^2} \ln M_H^2} \underbrace{\ln M_H^2}_{\sim}$$

$$\begin{aligned} \Pi(q) &= \frac{e_0^2}{12\pi^2} \left[ \frac{1}{\epsilon} + \ln(4\pi) - \gamma \right. \\ &\quad \left. - 6 \int_0^1 dx x(1-x) \ln \left( \frac{m^2 - q^2 x(1-x)}{\mu^2} \right) + \mathcal{O}(\epsilon) \right] \\ &= \frac{e_0^2}{12\pi^2} \begin{cases} \frac{1}{\epsilon} + \ln(4\pi) - \gamma + \frac{5}{3} - \ln \frac{-q^2}{\mu^2} + \dots & (|q^2| \gg m^2), \\ \frac{1}{\epsilon} + \ln(4\pi) - \gamma - \ln \frac{m^2}{\mu^2} + \frac{q^2}{5m^2} + \dots & (m^2 \gg |q^2|). \end{cases} \end{aligned} \quad (1.26)$$

But when you measure  $e^2$ ,  $\ln M_H^2$  disappears

$$\ln \frac{1}{g^2} \frac{\ell_d^2}{1 - \pi(g^2)} \rightarrow \frac{1}{g^2} \underbrace{\ell_{ph}^2}_{M_H^2 \text{ disappear}} = \frac{\ell_p^2}{4\pi} = \frac{1}{187}$$

Residual effect suppressed:

$$\frac{1}{g^2} \frac{\ell^2}{1 - \pi(g^2)} = \frac{\ell_{ph}^2}{g^2} + \frac{\ell^2}{12\pi^2} \frac{g^2}{5m^2} \frac{1}{g^2}$$

$\overbrace{\qquad\qquad\qquad}^{\Rightarrow F_T} \overbrace{\frac{S^3(k)}{M_H^2}}^{\text{Suppressed}}$

## Appelquist-Carrazonne Theorem

Effects of a heavy mass particle appear either as

a renormalization of a coupling constant      or

a power suppressed correction

1) EFT "heavy particle"  $\rightarrow$  "high energy"  $\xrightarrow{\gamma}$   $\frac{m_t}{\epsilon^2} \gamma^2 \rightarrow \delta M$

2) Caveat - Not true if limit  $m_t \rightarrow$  large violates a symmetry of the theory

$\rightarrow \begin{pmatrix} t \\ b \end{pmatrix}$ ,  $M_t \rightarrow \infty$  leads to  $M_t^2$  effects  $\leftarrow$  left over  
 $M_b$  fixed

but  $M_t \rightarrow \infty$  leads to only  $1/M_t^2$  effects  
 $M_b \rightarrow \infty$

Integrating out a heavy field - Tree level

$$\mathcal{L} = \frac{1}{2} [\partial^\mu \phi \partial^\nu \phi - m_H^2 \phi^2] + \phi F(\psi) + \mathcal{L}(\psi)$$

Then the PI is

$$\begin{aligned} Z &= S d\phi d\psi \exp \left\{ i \int d^4x [ \mathcal{L}(\phi) + \mathcal{L}(\phi, \psi) + \mathcal{L}(\psi) ] \right\} \\ &= S d\psi e^{i \mathcal{L}(\psi)} \left[ S d\phi e^{i \int d^4x (\mathcal{L}(\phi) + \mathcal{L}(\phi, \psi))} \right] \end{aligned}$$

Complete the square

$$\mathcal{L}(\phi) + \phi F(\psi) = -\frac{1}{2} \phi^2 (\Box + m^2) + \phi F(\psi)$$

First  $\Rightarrow Z_1$   
 ↓ int by part

$$\text{Defin } \tilde{\phi}(x) = \phi(x) + \int d^4y D_i(x-y) F(y)$$

$$\text{with } (\bar{D} + m^2) \bar{D}_F(x-y) = -\delta^4(x-y)$$

Then

$$-\frac{1}{2} \tilde{\phi} (\bar{D} + m^2) \tilde{\phi} = -\frac{1}{2} \phi (\bar{D} + m^2) \phi + \frac{1}{2} \times 2 \phi F \phi$$

$$+ \frac{1}{2} \int d^4y F(\psi(x)) D_F(x-y) F(\psi(y))$$

$$Z_1 = \int [d\tilde{\phi}] e^{-i \int d^4x \frac{1}{2} \tilde{\phi} (\bar{D} + m^2) \tilde{\phi}} e^{-i \int d^4x d^4y F_{\alpha\beta} D_F(x-y) F_{\beta\alpha}}$$

$d\tilde{\phi} = d\tilde{\phi}$

const

$$Z = \int [d\psi] e^{i \int d^4x \mathcal{L}(\psi)} \langle F D F \rangle$$

Now locality

$$D_F(x-y) = \int \frac{d^4 g}{(2\pi)^4} \frac{e^{-ig \cdot (x-y)}}{g^2 - m_H^2} = \int \frac{d^4 g}{(2\pi)^4} e^{-ig \cdot (x-y)} \left[ \frac{-1}{m_H^2} - \frac{g^2}{m_H^4} \dots \right]$$
$$= \left( -\frac{1}{m_H^2} + \frac{\square}{m_H^4} + \dots \right) \underbrace{\int \frac{d^4 g}{(2\pi)^4} e^{-ig \cdot (x-y)}}_{\delta^4(x-y) \text{ locality}}$$

$$\mathcal{L}_{\text{eff}} = \mathcal{L}(\psi) + \frac{1}{2} F(\psi) \frac{1}{m_H^2} F(\psi) - \frac{1}{2m_H^4} F(\psi) \square F(\psi) + \dots$$

Light particle are not local

$$D_F(x-y) \Big|_{m=0} = \frac{1}{(2\pi)^2} \frac{1}{(x-y)^2 - i\varepsilon} \quad (m=0)$$

↑ deriv  
expansion

## Second Organizing Principle - The Energy Expansion

Powers of  $\frac{E^2}{M_H^2} \sim \frac{\Box}{M_H^2}$

$$\mathcal{L} = \mathcal{L}_{\text{ren}} + \mathcal{L}_{d=6} + \mathcal{L}_{d=8} \dots$$

↖ perhaps

$$\Box \sim \left( \frac{E^2}{M_H^2} \right)^n$$

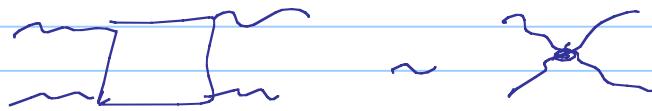
See in BSM

$$\text{New physics} \Rightarrow \mathcal{L}_{d=5} + \mathcal{L}_{d=6} \dots$$

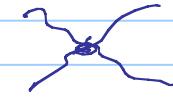
Much of EFT usage is just Effective Lagrangians

- local operators ordered in an energy expansion

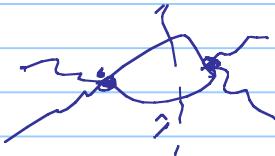
\* \* But EFT is really full QFT



heavy



light



Cuts, on shell --  
full field theory

Here + GR

Third organizing principle: Loops, renormalization, matching/measuring

A) Divergence from high Energy

- not reliable

- local

$\Rightarrow$  something in  $L$  to renormalize

B) When you renormalize, you measure

i) Known HE theory  $\Rightarrow$  match (express parameter of  $L_{\text{eff}}$   
in terms of this theory)

ii) Unknown - measure

c) Separate HE part from low E - non local

$\curvearrowleft$  predictions live here

## Linear sigma model

- construct EFT by hand
- EFT closest in style to General Relativity

Full theory

$$\mathcal{L}(\sigma, \vec{\pi}, \psi) = \frac{1}{2} \left[ (\partial_\mu \sigma)^2 + (\partial_\mu \vec{\pi})^2 \right] + \frac{m^2}{2} (\sigma^2 + \vec{\pi}^2) - \frac{\lambda}{4} (\sigma^2 + \vec{\pi}^2)^2 + \bar{\psi}_i \gamma^\mu \psi_i$$

$$+ g \bar{\psi} (\sigma + i \vec{\tau} \cdot \vec{\pi}) \gamma_5 \psi$$

↑ Pauli SU(2)

Effective Theory

$$\mathcal{L}_{\text{eff}} = \frac{v^2}{4} \text{Tr} (\partial_\mu U \delta^\mu U^\dagger) \quad U = \exp \left( i \frac{\vec{\sigma} \cdot \vec{\pi}}{v} \right)$$

+ - - -

$$v = \sqrt{\frac{m^2}{\lambda}}$$

Alternate form of full theory

$$\Sigma = \left( \sigma + i \vec{e} \cdot \vec{\pi} \right) \quad \text{then} \quad \frac{1}{2} \text{Tr}(\Sigma^\dagger \Sigma) = \frac{1}{2} \text{Tr} \left( \sigma^2 + \frac{(\vec{e} \cdot \vec{\pi})^2}{\pi^2} \right) = \sigma^2 + \frac{\vec{e}^2}{\pi^2}$$

$$\mathcal{L} = \frac{1}{4} \text{Tr} \left( \partial_\mu \Sigma^\dagger \partial^\mu \Sigma \right) + \frac{m^2}{4} \text{Tr} (\Sigma^\dagger \Sigma) - \frac{g^2}{16} \left[ \text{Tr} (\Sigma^\dagger \Sigma)^2 + \bar{\psi}_L \not{D} \psi_L + \bar{\psi}_R \not{D} \psi_R + g \bar{\psi}_L \Sigma^\dagger \psi_R + g \bar{\psi}_R \Sigma^\dagger \psi_L \right]$$

$\uparrow$        $\nwarrow$        $\underbrace{(n+s)^2 u^+ u^-}_{\text{or}}$

Symmetry of the theory

$$\psi_L \rightarrow L \psi_L$$

$SU(2)$  matrix

$$\psi_R \rightarrow R \psi_R$$

$\left. \begin{array}{l} \\ \end{array} \right\} \psi_L \in \psi_R \text{ invariant}$

$$\varepsilon \rightarrow L \varepsilon R^+$$

and  $\text{Tr}(\varepsilon^+ \varepsilon) \Rightarrow \text{Tr}(R \varepsilon^+ \underbrace{\varepsilon^+ L}_{L} \varepsilon R^+) = \text{Tr} \varepsilon^+ \varepsilon$

$SU(2)_L \times SU(2)_R$

Spontaneous Symmetry Breaking  
- physical fields

$$\langle \vec{\sigma} \rangle = \sqrt{\frac{u^2}{\lambda}}, \quad \langle \vec{\pi} \rangle = 0$$

$\propto v$

$$\frac{\vec{\pi}}{|\vec{\sigma}|}$$

$$\sigma = v + \tilde{\sigma}$$

$$\begin{aligned} \mathcal{L} = & \frac{1}{2} [(2\omega \vec{\sigma})^2 - 2m^2 \vec{\pi}^2] - \frac{1}{2} \partial_\mu \vec{\pi}^\mu - \lambda v \tilde{\sigma} (\tilde{\sigma}^2 + \vec{\pi}^2) \\ & - \frac{\lambda}{4} (\tilde{\sigma}^2 + \vec{\pi}^2)^2 + \bar{\psi} (i \not{D} - g \sigma) \psi \\ & + g \bar{\psi} (\sigma + i \vec{\pi} \cdot \vec{\gamma}) \psi \end{aligned}$$

We have lost track of the symmetry of the theory — but it is still there

## Transitions to EFT

$$\Sigma = (\nu + i \tilde{\tau} \tilde{\pi}) = (\nu + \tilde{\tau} + i \tilde{\pi} \tilde{\pi}) = (\nu + S) U$$

$$= (\nu + S + i \tilde{\tau} \tilde{\pi}' + \dots)$$

$$\Rightarrow U = \exp(i \tilde{\tau} \tilde{\pi}')$$

$$\tilde{\tau} = S + \dots$$

$$\tilde{\pi} = \tilde{\tau}' + \dots$$

equal at KE

Rewrite as

$$L = \frac{1}{2} \left[ (\partial_\mu S)^2 - 2 m^2 S^2 \right] + \frac{(\nu + S)^2}{4} \text{Tr} (\partial_\mu U \partial^\mu U^\dagger) - \lambda \nu S^3 - \frac{\lambda}{4} S^4$$

pro approx

$$\text{Symmetry } \Sigma \rightarrow L \Sigma R^+ \rightarrow U \rightarrow L U R^+$$

no  $\tilde{\pi}$ 's

$$X + \overline{Ts}$$

First approximation Drop heavy field

(Appelquist Carrasco)

$$I_{\text{eff}} = \frac{N^2}{4} \text{Tr}(\partial_\mu U \partial^\mu U^\dagger)$$

Tree level  $\frac{1}{\kappa (\partial_\mu U)^2} \sim \partial^4$

Test of equivalence  $\boxed{\pi^+ \pi^0 \rightarrow \pi^+ \pi^0}$

Full theory  $\vec{\sigma}, \vec{\pi}$  rep  $\frac{\pi^+}{\pi^0} \frac{\pi^+}{\pi^0}$

$$\cancel{X} + \frac{|\vec{\sigma}|}{\pi^0 \pi^0}$$

$$I = -\frac{\lambda}{c!} \left( \frac{\vec{\sigma} \cdot \vec{\pi}}{\pi^0 \pi^0} \right)^2 - \lambda N \vec{\sigma} \vec{\pi}^2$$

$$\begin{aligned} -\mathcal{W} &= -2i\lambda + \frac{(-2i\lambda N)^2}{g^2 - \frac{m_0^2}{2\mu^2}} = -2i\lambda \left[ 1 + \frac{2\lambda N^2}{g^2 - \frac{m_0^2}{2\mu^2}} \right] \\ &= i \frac{g^2}{\mu^2} + \mathcal{O}(g^4) \end{aligned}$$

Second form of the full theory  $\cdot S_{\pi^+ \pi^-} - S_{\pi^0 \pi^0}$

$$\text{X} + \frac{1}{\pi^0 \pi^0} \left[ \frac{\partial u_{\pi^+}}{\partial u_{\pi^+}} \frac{\partial u_{\pi^-}}{\partial u_{\pi^-}} \right] \rightarrow \mathcal{L}(g^4)$$

$$\frac{n^2}{4} \text{Tr}(\partial_u u \partial^u u^*) \rightarrow \frac{1}{6n^2} \left[ \left( \frac{\partial u^*}{\partial u} \partial_u \pi^+ \right)^2 - \pi^+ \left( \partial_u \pi^+ \partial^u \pi^+ \right) \right]$$

$$\text{X} - n M = \frac{g^4}{N} >$$

Now Effective Lagrangian  $\frac{N^2}{4} \text{Tr}(\bar{J} \partial^1 J^m U^+) \leftarrow$

$-i M = i g^3 \frac{N^2}{N^2}$  ✓

Haag's theorem

"Names do not matter" - matrix

with  $\tilde{\sigma} = S + \dots$       } K-E. unchanged  
 $\tilde{\pi} = \tilde{\pi}' + \dots$

$\psi \rightarrow F(\phi)$  with  $F(\phi) = \phi + \dots$

## Exercises:

$$\phi \rightarrow \phi' = e^{i\theta} \phi \quad U(1)$$

### - 1) U(1) effective lagrangian

Consider a theory with a complex scalar field  $\varphi$  with a  $U(1)$  global symmetry  $\varphi \rightarrow \varphi' = \exp(i\theta) \varphi$ . The lagrangian will be

$$\mathcal{L} = \partial_\mu \varphi^* \partial^\mu \varphi + \mu^2 \varphi^* \varphi - \lambda (\varphi^* \varphi)^2$$

- a) Minimize the potential to find the ground state and write out the lagrangian in the basis

$$\varphi = \frac{1}{\sqrt{2}} (v + \varphi_1(x) + i\varphi_2(x))$$

Show that  $\varphi_2$  is the Goldstone boson.

- b) Use this lagrangian to calculate the low-energy scattering of  $\varphi_2 + \varphi_2 \rightarrow \varphi_2 + \varphi_2$ . Show that despite the non-derivative interactions of the lagrangian, cancellations occur such that leading scattering amplitude starts at order  $p^4$ .
- c) Instead of the basis above express the lagrangian using an exponential basis,

$$\varphi = \frac{1}{\sqrt{2}} (v + \Phi(x)) e^{i\chi(x)/v} .$$

Show that in this basis a 'shift symmetry'  $\chi \rightarrow \chi + c$  is manifest.

- d) Calculate the same scattering amplitude using this basis and show that the results agree. Note that the fact that the amplitude is of order  $p^4$  is more readily apparent in this basis.

### 2) Why is the sky blue?

Write some examples of gauge invariant effective Lagrangians for the scattering of light off of a neutral object. Use these to calculate the frequency dependence of light scattered from molecules in the sky – showing that the sky is blue.

For more detail about this calculation, see Barry R. Holstein, *Blue skies and effective interactions*, American Journal of Physics, **67**, 422 (1999)

$$\phi = \frac{1}{\sqrt{2}} (n + \phi_1 + i\phi_2)$$

$$\cancel{\phi_2} \times \cancel{\phi_1} + \cancel{i\phi_2} \sim g^4 \quad \leftarrow G.B$$

$$\leftarrow \phi = \frac{1}{\sqrt{2}} (n + \overline{\phi}) e^{i\chi(x)/v}$$

$$\Rightarrow \mathcal{L}_{eff} \rightarrow \text{Compare}$$

Pause: Why are we doing this?

$$\text{Low } E \quad D.O.F = \overline{T} \bar{I}$$

$\widehat{T}, S$  not very relevant

Simplify theory  $L(\pi)$

Some symmetry  $\Rightarrow$  consequences

$$\text{Amp} \sim g^2$$

We can recover all of  $\pi$  physics at low  $E$  using EFT  $\Rightarrow$  Full QFT

Formal correspondence

$$\begin{aligned} Z[\vec{J}] &= S[dS] d\vec{\pi}] e^{i \int d^4x [L_{\text{Full}}(S, \vec{\pi}) + \vec{J} \cdot \vec{\pi}]} \\ &= S[d\vec{\pi}] e^{i \int d^4x [L_{\text{eff}}(\vec{\pi}) + \vec{J} \cdot \vec{\pi}]} \end{aligned}$$

$\uparrow$  only light D.O.F  
 $\uparrow$  full QFT  
L<sub>eff</sub> local  
S heavy

Recall

$$\cancel{X} + \overline{T\tilde{\alpha}}$$

$$-iM = -2i\lambda \left[ 1 - \underbrace{\frac{m_0^2}{g^2 - m_0^2}}_{\sim} \right] = i\frac{g^2}{N^2} - \frac{g^2 \times g^3}{m_0^2} +$$

*expand*

$$= i\frac{g^2}{N} \left[ 1 - \frac{g^2}{m_0^2} \right] \quad \begin{matrix} \curvearrowleft \\ \curvearrowleft \end{matrix}$$

↑  
Now lets go beyond the lowest order      -Tree level      T  
S

First tree level       $\tau, S$

$$\mathcal{L} = \frac{N^2}{4} \text{Tr} (\partial_\mu U \partial^\mu U^\dagger) + \frac{NS}{2} \text{Tr} (\partial_\mu U \partial^\mu U^\dagger) + \frac{1}{2} [(S)^2 - M_0^2 S^2] + \frac{S^2}{2} (\text{Tr} \partial_\mu U \partial^\mu U^\dagger)$$

Integrates

$$\mathcal{L}_{\text{eff}} = \frac{N^2}{4} \text{Tr} (\partial_\mu U \partial^\mu U^\dagger) + \frac{N^2}{8M_0^2} [\text{Tr} (\partial_\mu U \partial^\mu U^\dagger)]^2$$

← →

$\times \frac{m g^2}{N}$       ↑  $\frac{g^4}{N^2 M_0^2}$

⇒ same result to  $O(E^4)$

Now lets tackle loop effects:

Overview:

- 1) Linear  $\sigma$  model is renormalizable theory
  - loop, renormalize, calculate, renorm param
- 2) Renormalize, then look at low energy predictions
- 3) EFT is a nonrenormalizable theory
  - loops, renormalize
- 4) Renormalize, then look at low energy predictions
- 5) Match ✓

$$g = \lambda_1, \lambda_2$$

Why will this work?

All HE effects are local  $\Rightarrow$  some terms in  $L$

HE part of EFT is wrong, but can adjust by local term

Low Energy parts of both theories are the same

$\Rightarrow$  Low E can be predicted

Most general Lagrangian to next order in the energy expansion

Require symmetry

$$U \rightarrow L U R^+$$

$$\rightarrow \text{Tr}(U U^+)$$

General form:  $SU(n)$

$$\mathcal{L} = \frac{F^2}{4} \text{Tr}(\partial_\mu U \partial^\mu U^+) + L_1 \left[ \text{Tr}(\partial_\mu U \partial^\mu U^+) \right]^2 + L_2 \text{Tr}(\partial_\mu U_{\nu} U^{\mu\nu}) \text{Tr}(\partial_\mu U^{\nu} U_{\nu}^{\mu}) \\ + L_3 \text{Tr}(\partial_\mu U \partial_\nu U^+ \partial^\mu Y \partial^\nu U^+)$$

For  $SU(2)$ , only two terms are independent

$$\mathcal{L} = \frac{F^2}{4} \text{Tr}(\partial_\mu U \partial^\mu U^+) + L_1 \left[ \text{Tr}( ) \right]^2 + L_2 \left[ \text{Tr}(\partial_\mu U_{\nu} U^{\mu\nu}) \right]^2$$

- Notation Gasser, Leutwyler

## Renormalization of EFT

Background field method:  $U = \bar{U} e^{i\Delta} \leftarrow \Delta = \vec{\tau} \cdot \vec{\Delta}$   $\checkmark$  quantus

$$\left. \begin{array}{l} \text{App B of DSY} \\ \text{on web site} \end{array} \right\} \quad \left( \begin{array}{l} U = \xi^{i\delta'} \bar{U} \\ \text{or } U = \xi^i e^{i\eta} \end{array} \right) \quad \xi^2 = \bar{U}$$

Expand in  $\Delta$

$$\begin{aligned} \text{Tr} \left( D_\mu U D^\mu U^\dagger \right) &= \text{Tr} \left( D_\mu \bar{U} D^\mu \bar{U}^\dagger \right) - 2i \text{Tr} \left( \bar{U}^\dagger D_\mu \bar{U} \tilde{D}^\mu \Delta \right) \\ &\quad + \text{Tr} \left[ \tilde{D}_\mu \Delta \tilde{D}^\mu \Delta + \bar{U}^\dagger D_\mu \bar{U} \left( \Delta \tilde{D}^\mu \Delta - \tilde{D}^\mu \Delta \Delta \right) \right] \end{aligned}$$

vanish by E of M

$$\begin{array}{l} D_m \rightarrow \partial_m \\ X \rightarrow 0 \end{array}$$

$$\left[ d_\mu d^\mu + \sigma \right]^{ab}$$

$$S_2^{(0)} = \int d^4x \left\{ \mathcal{L}_2(\bar{U}) - \frac{F_0^2}{2} \Delta^a (d_\mu d^\mu + \sigma)^{ab} \Delta^b + \dots \right\}$$

where

$$d_\mu^{ab} = \delta^{ab} \partial_\mu + \Gamma_\mu^{ab},$$

$$\Gamma_\mu^{ab} = -\frac{1}{4} \text{Tr} \left( [\lambda^a, \lambda^b] \left( \bar{U}^\dagger \partial_\mu \bar{U} + \cancel{\bar{U}^\dagger \partial_\mu \bar{U}} \right) \right),$$

$$\sigma^{ab} = \frac{1}{8} \text{Tr} \left( \{ \lambda^a, \lambda^b \} \left( \bar{\chi}^\dagger \bar{U} + \bar{U}^\dagger \chi \right) + [\lambda^a, \bar{U}^\dagger D_\mu \bar{U}] [\lambda^b, \bar{U}^\dagger D^\mu \bar{U}] \right).$$

$$d^{ab} = \partial_\mu \delta^{ab} + \Gamma^{ab}$$
(2.12)

## Heat Kernel:

Recall heat kernel expansion

$$a_0(x) = 1 \ , \quad \quad \quad a_1(x) = -\sigma \ ,$$

$$a_2(x) = \frac{1}{2}\sigma^2 + \frac{1}{12}[d_\mu, d_\nu][d^\mu, d^\nu] + \frac{1}{6}[d_\mu, [d^\mu, \sigma]]$$

$$\begin{aligned}
 W_{\text{loop}} &= \frac{i}{2} \text{tr} \ln (d_\mu d^\mu + \sigma) \\
 &= \frac{1}{2(4\pi)^{d/2}} \int d^4x \lim_{m \rightarrow 0} \left\{ \Gamma \left( 1 - \frac{d}{2} \right) m^{d-2} \text{Tr} \sigma \right. \\
 &\quad \left. + m^{d-4} \Gamma \left( 2 - \frac{d}{2} \right) \text{Tr} \left( \frac{1}{12} \Gamma_{\mu\nu} \Gamma^{\mu\nu} + \frac{1}{2} \sigma^2 \right) + \dots \right\}
 \end{aligned}$$

$$L_{\mu\nu}^{(a)} = \frac{1}{8} \text{Tr} \left[ [\lambda^a, \lambda^b] \left[ \bar{u}^\dagger \partial_\mu u, \bar{d}^\dagger \partial_\mu d \right] \right]$$

$$\text{Tr} (\Gamma_{\mu\nu} \Gamma^{\mu\nu}) = \frac{N_f}{8} \text{Tr} \left( \left[ \bar{U}^\dagger D_\mu \bar{U}, \bar{U}^\dagger D_\nu \bar{U} \right] \left[ \bar{U}^\dagger D^\mu \bar{U}, \bar{U}^\dagger D^\nu \bar{U} \right] \right)$$

$$\begin{aligned} \text{Tr} \sigma^2 &= \frac{1}{8} \left[ \text{Tr} (D_\mu \bar{U} D^\mu \bar{U}^\dagger) \right]^2 + \frac{1}{4} \text{Tr} (D_\mu \bar{U} D_\nu \bar{U}^\dagger) \text{Tr} (D^\mu \bar{U} D^\nu \bar{U}^\dagger) \\ &\quad + \frac{N_f}{8} \text{Tr} (D_\mu \bar{U} D^\mu \bar{U}^\dagger D_\nu \bar{U} D^\nu \bar{U}^\dagger) + \frac{2+N_f^2}{8N_f^2} \left[ \text{Tr} (\chi \bar{U}^\dagger + \bar{U}^\dagger \chi) \right]^2 \end{aligned}$$

Renormalize:

$$\mathcal{L} = \frac{N^2}{4} \text{Tr} (\partial_\mu u \partial^\mu u^\dagger) + \ell_1 \left[ \text{Tr} (\partial_\mu u \partial^\mu u^\dagger) \right]^2 + \ell_2 \underset{\uparrow}{\text{Tr}} (\partial_\mu u \partial^\mu u^\dagger) \text{Tr} (\partial^\mu u \partial^\nu u^\dagger)$$

$$\ell_1' = \ell_1 + \frac{1}{384\pi^2} \left[ \frac{1}{\varepsilon} - \gamma + \ln 4\pi \right]$$

$$\ell_2' = \ell_2 + \frac{1}{192\pi^2} \left[ \frac{1}{\varepsilon} - \gamma + \ln 4\pi \right]$$

} renormalized

Now lets go for finite effects: Two ways

1) Direct Background Field calculation

$$\Delta_F^2(x-y) = \frac{1}{16\pi^2} \left\{ \frac{1}{\epsilon} \delta^4(x-y) + L(x-y) \right\}$$

↑  
renorm

$\sim \frac{\int d^4q}{(2\pi)^4} e^{iq(x-y)} \frac{\ln q^2/\mu^2}{\mu^2}$

$$(S_{\text{loop}} \frac{1}{d-1} \frac{1}{\epsilon} = \frac{1}{3+d-4} \frac{1}{\epsilon} \Rightarrow \left[ \frac{1}{3} + \frac{1}{3} \left( \frac{1}{3} (d-4) \right) \right] \frac{1}{\epsilon})$$

$\frac{1}{9}$  const

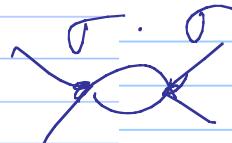
Recall from last lecture:

Generalizes

$$\mathcal{L} = \phi^* [d_\mu d^\mu + \sigma h] \phi$$

$$d_\mu = \partial_\mu + \Gamma_\mu^\nu \quad \text{matrix}$$

$$\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \sigma \end{pmatrix}$$



$$\Delta S = \int d^4x dy \operatorname{Tr} \left[ \Gamma_{\mu\nu}(y) \frac{\Delta_f^2(x-y)}{4(d-1)} \Gamma^{\mu\nu} + \frac{1}{2} \delta(y) \Delta_f^2(x-y) \delta(y) \right]$$

$$\text{with } \Gamma_{\mu\nu} = \partial_\mu \Gamma_\nu - \partial_\nu \Gamma_\mu + [\Gamma_\mu, \Gamma_\nu]$$

Divergence  $\nearrow$  local:

$$\Delta S_{loc} = \int d^4x \frac{1}{16\pi} \left[ \frac{1}{\epsilon} \dots \right] \operatorname{Tr} \left[ \frac{1}{12} \Gamma_{\mu\nu} \Gamma^{\mu\nu} + \frac{1}{2} \sigma^2(x) \right] \quad \checkmark$$

This yields:

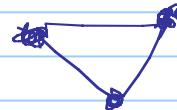
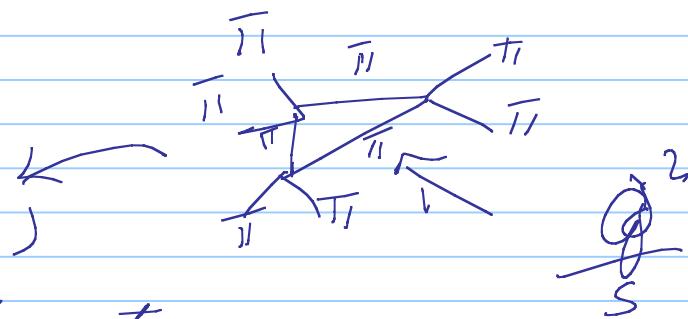
$$\Delta S_{\text{parts}} = \int d^4x \downarrow d^4y \text{Tr} \left\{ \frac{1}{12} \Gamma_{\mu\nu}(x) L(x-y) \Gamma^{\mu\nu}(y) + \frac{1}{2} \sigma(x) L(x-y) \sigma(y) \right\}$$



$G+L$   
NP B250  
(1984)

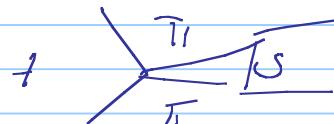
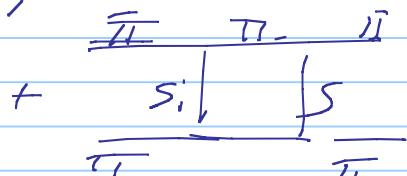
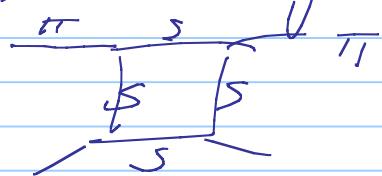
Includes all processes (up to  $\pi^6$ )

Simply expand and take matrix elements

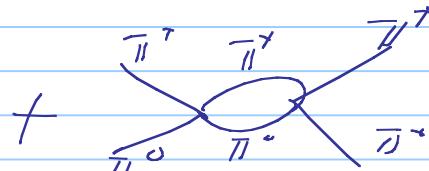
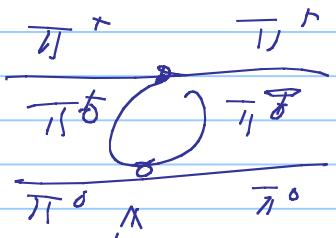


Direct calculation  $\pi^+ \pi^- \rightarrow \pi^+ \pi^0$  to one loop

True theory  $\tilde{F}, \tilde{\pi}$



+ --



Linear O model to one loop  
- expanded at low energy

Mandar + Matean

Terrible in general

Expand at low momentum

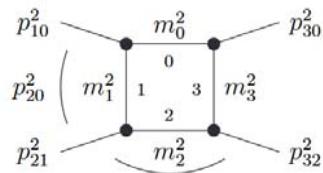
$$\begin{aligned} \mathcal{M}_{full} = & \frac{t}{v^2} + \left[ \frac{1}{m_\sigma^2 v^2} - \frac{11}{96\pi^2 v^4} \right] t^2 \\ & - \frac{1}{144\pi^2 v^4} [s(s-u) + u(u-s)] \\ & - \frac{1}{96\pi^2 v^4} \left[ 3t^2 \ln \frac{-t}{m_\sigma^2} + s(s-u) \ln \frac{-s}{m_\sigma^2} + u(u-s) \ln \frac{-u}{m_\sigma^2} \right] \end{aligned}$$

↑

In general, this calculation is very complex

Scalar one-loop 4-point integrals

A. DENNER<sup>1</sup> AND S. DITTMAYER<sup>2</sup>



$$\begin{aligned} &\equiv D_0(p_1, p_2, p_3, m_0^2, m_1^2, m_2^2, m_3^2) \\ &\equiv D_0(p_{10}^2, p_{21}^2, p_{32}^2, p_{30}^2, p_{20}^2, p_{31}^2, m_0^2, m_1^2, m_2^2, m_3^2) \end{aligned}$$

$$= \frac{(2\pi\mu)^{4-D}}{i\pi^2} \int d^D q \frac{1}{(q^2 - m_0^2)[(q+p_1)^2 - m_1^2][(q+p_2)^2 - m_2^2][(q+p_3)^2 - m_3^2]}$$

$$\begin{aligned} D_0 = & \int_0^{-\beta_1} dz \frac{1}{C_1(z)} \left\{ \ln A_1\left(-\frac{z}{\beta_1}, 0\right) - \ln \left[ \left(1 + \frac{z}{\beta_1}\right) A_1\left(-\frac{z}{\beta_1}, 0\right) + B_1\left(-\frac{z}{\beta_1}, 0\right) \right] \right. \\ & + \int_0^{-\beta_2} dz \frac{1}{C_2(z)} \left\{ \ln A_2\left(-\frac{z}{\beta_2}, 0\right) - \ln B_2\left(-\frac{z}{\beta_2}, 0\right) \right\} \\ & + \int_0^{-\beta_3} dz \frac{1}{C_3(z)} \left\{ \ln A_3\left(-\frac{z}{\beta_3}, 0\right) - \ln B_3\left(-\frac{z}{\beta_3}, 0\right) \right\} \\ & + \int_{-\beta_1}^1 dz \frac{1}{C_1(z)} \left\{ \ln A_1\left(\frac{1-z}{1+\beta_1}, \frac{z+\beta_1}{1+\beta_1}\right) - \ln B_1\left(\frac{1-z}{1+\beta_1}, \frac{z+\beta_1}{1+\beta_1}\right) \right\} \\ & \int_{-\beta_2}^1 dz \frac{1}{C_2(z)} \left\{ \ln A_2\left(\frac{1-z}{1+\beta_2}, \frac{z+\beta_2}{1+\beta_2}\right) \right. \\ & \left. - \ln \left[ \frac{z+\beta_2}{1+\beta_2} A_2\left(\frac{1-z}{1+\beta_2}, \frac{z+\beta_2}{1+\beta_2}\right) + B_2\left(\frac{1-z}{1+\beta_2}, \frac{z+\beta_2}{1+\beta_2}\right) \right] \right\} \\ & \int_{-\beta_3}^1 dz \frac{1}{C_3(z)} \left\{ \ln A_3\left(\frac{1-z}{1+\beta_3}, \frac{z+\beta_3}{1+\beta_3}\right) - \ln B_3\left(\frac{1-z}{1+\beta_3}, \frac{z+\beta_3}{1+\beta_3}\right) \right\} \\ (2.1) \quad & \int_0^1 dz \frac{1}{C_1(z)} \left\{ \ln A_1(0, z) - \ln \left[ (1-z) A_1(0, z) + B_1(0, z) \right] \right\} \\ & - \int_0^1 dz \frac{1}{C_2(z)} \left\{ \ln A_2(0, z) - \ln \left[ z A_2(0, z) + B_2(0, z) \right] \right\} \end{aligned}$$

Matching

$$\mathcal{M}_{eff} = \frac{t}{v^2} + \left[ 8\ell_1^r + 2\ell_2^r + \frac{5}{192\pi^2} \right] \frac{t^2}{v^4}$$

$$+ \left[ 2\ell_2^r + \frac{7}{576\pi^2} \right] [s(s-u) + u(u-s)]/v^4$$

$$- \frac{1}{96\pi^2 v^4} \left[ 3t^2 \ln \frac{-t}{\mu^2} + s(s-u) \ln \frac{-s}{\mu^2} + u(u-s) \ln \frac{-u}{\mu^2} \right]$$

Eff. Theory

$\cancel{\lambda}$

$$\ell_1^r, \ell_2^r$$

]

Match  $\ell_1^r = \frac{v^2}{8M_0^2} + \frac{1}{384\pi^2} \left[ \ln \frac{m_0^2}{\mu^2} - \frac{35}{6} \right]$

$\ell_2^r = \frac{1}{192\pi^2} \left[ \ln \frac{m_0^2}{\mu^2} - \frac{11}{6} \right]$

$\ell_2^r(\mu)$

no  $\mu$  in answer

In more detail - Why does this work

Recall our first tree level matching:

Now full theory:

$$\cancel{I} + \cancel{II} + \cancel{III} + \cancel{IV}$$

$$M_{\text{full}} = \int \frac{d^4 k}{(2\pi)^4} \left[ -2i\lambda + (-2i\lambda N)^2 \right] \frac{i}{(k+p_T - M_S^2)} \frac{i}{k^2 (k+p_+ + p_\perp)^2} \left[ -2i\lambda + (2i\lambda N)^2 \right] \frac{i}{(k+p_+)^2 - m^2}$$

EFT

$$\cancel{I}$$

$$M_{\text{eff}} \int \frac{d^4 k}{(2\pi)^4} \left[ i \frac{(k+p_+)^2}{N^2} \right]$$

EFT  $\cancel{X}$

at low momenta

$$\left[ \frac{(k+p_+)^2}{N^2} \right] \cancel{X}$$

Low momentum agrees ✓ by construction

High momentum different  $\Rightarrow$  but local  
correct by local counter terms ] ✗ +

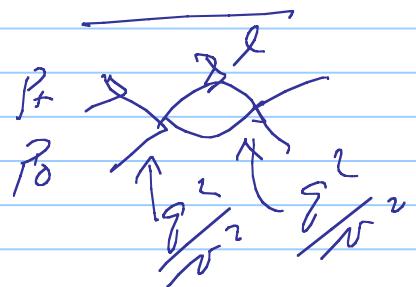
Power counting:

- we have seen 1 loop effect at order  $E^4 \sim \mathcal{J}^4$

$$X = \frac{g^2}{\mu^2}$$

$$\mathcal{O}(E^2)$$

$$\mathcal{O}(E^4)$$



$$\Rightarrow i \mathcal{M} \sim \frac{1}{\mu^4} I(P_+, P_0) \sim \text{dimensionless}$$

$P_+^2 = 0 \quad P_0^2 = 0 \quad I P_+ \cdot P_0 = S$

$$\approx \frac{g^2}{\mu^4}, \quad \frac{1}{\mu^4}, \quad \dots$$

Weinberg Theorem

$$\mathcal{L} = \mathcal{L}_2 + \mathcal{L}_4 + \mathcal{L}_6 + \dots$$

$\mathcal{O}(E^2)$  = tree level  $\mathcal{L}_2$

$\mathcal{O}(E^4)$  = one loop with  $\mathcal{L}_2$  + tree level with  $\mathcal{L}_4$

$\mathcal{O}(E^6) \Rightarrow$  2 loops with  $\mathcal{L}_2$  + one loop with one  $\mathcal{L}_4 + \mathcal{L}_2$   
+ tree level with  $\mathcal{L}_6$

## The Effective Field Theory of QCD

- Chiral perturbation theory

$$\mathcal{L}_{QCD} = \bar{\psi}_L D^\mu \psi_L + \bar{\psi}_R D^\mu \psi_R - \bar{\psi}_L m \psi_R - \bar{\psi}_R m \psi_L$$

Approximately the same symmetry as sigma model

not invariant

$$\psi_L \rightarrow L \psi_L$$

$$\psi_R \rightarrow R \psi_R$$

## Background Field Constructors of Effective Lagrangians

(partial treatment)

- if  $m_i = 0$ , must start with same  $\mathcal{L}_{\text{eff}}$

$$\mathcal{L}_{\text{eff}} = \frac{E^2}{4} \text{Tr}(\partial_\mu U \partial^\mu U^\dagger)$$

But, mass terms is new.

BF construction, consider

$$\mathcal{L} = \bar{\psi}_L iD\psi_L + \bar{\psi}_R iD\psi_R - \bar{\psi}_L^i (s + ip) \psi_R - \bar{\psi}_R (s - ip) \psi_L$$

This equals QCD if  $s \rightarrow m$ ,  $p \rightarrow 0$

But, in general can make this fully symmetric if we assume

$$s + ip \rightarrow \underline{s + ip} R$$

Now  $L_{eff}$  with symmetry must be :

Let  $\chi = 2B_0(s + ip)$   
-  $\rightarrow$  const

$$U = e^{i \frac{\chi \cdot \vec{r}}{F}}$$

Then

$$L_{eff} = \frac{F^2}{4} \text{Tr}(\partial_\mu U \partial^\mu U^\dagger) + \frac{F^2}{4} \text{Tr}(U^\dagger U + U^\dagger U) \quad \uparrow \chi = 2B_0 m$$

$$= \partial_\mu \bar{\psi} \partial^\mu \psi - B_0(m_u + m_d) \bar{\psi} \psi$$

$$m_\pi^2 = B_0(m_u + m_d)$$

$$\langle \bar{\psi} \psi \rangle = -F_\pi^2 B_0$$

The full EFT program can be carried out

$$\begin{aligned} \mathcal{L}_A &= \sum_{i=1}^{10} L_i O_i \\ &= l_1 \left[ \text{Tr} \left( D_\mu U D^\mu U^\dagger \right) \right]^2 + L_2 \text{Tr} \left( D_\mu U D_\nu U^\dagger \right) \cdot \text{Tr} \left( D^\mu U D^\nu U^\dagger \right) \\ &\quad + L_3 \text{Tr} \left( D_\mu U D^\mu U^\dagger D_\nu U D^\nu U^\dagger \right) \\ &\quad + L_4 \text{Tr} \left( D_\mu U D^\mu U^\dagger \right) \text{Tr} \left( \chi U^\dagger + U \chi^\dagger \right) \\ &\quad + L_5 \text{Tr} \left( D_\mu U D^\mu U^\dagger \left( \chi U^\dagger + U \chi^\dagger \right) \right) + L_6 \left[ \text{Tr} \left( \chi U^\dagger + U \chi^\dagger \right) \right]^2 \\ &\quad + L_7 \left[ \text{Tr} \left( \chi^\dagger U - U \chi^\dagger \right) \right]^2 + L_8 \text{Tr} \left( \chi U^\dagger \chi U^\dagger + U \chi^\dagger U \chi^\dagger \right) \\ &\quad + i L_9 \text{Tr} \left( L_{\mu\nu} D^\mu U D^\nu U^\dagger + R_{\mu\nu} D^\mu U^\dagger D^\nu U \right) + L_{10} \text{Tr} \left( L_{\mu\nu} U R^{\mu\nu} U^\dagger \right) \end{aligned}$$

Gasser + Leutwyler

In QCD it is hard to do matching

$\Rightarrow$  Measure

Table VII-1. Renormalized coefficients in the chiral lagrangian  $\mathcal{L}_4$  given in units of  $10^{-3}$  and evaluated at renormalization point  $\mu = m_\rho$  [BiJ 12].

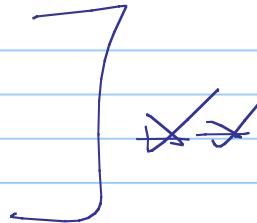
Coefficient	Value	Origin
$L_1^r$	$1.12 \pm 0.20$	$\pi\pi$ scattering
$L_2^r$	$2.23 \pm 0.40$	and
$L_3^r$	$-3.98 \pm 0.50$	$K_{\ell 4}$ decay
$L_4^r$	$1.50 \pm 1.01$	$F_K/F_\pi$
$L_5^r$	$1.21 \pm 0.08$	$F_K/F_\pi$
$L_6^r$	$1.17 \pm 0.95$	$F_K/F_\pi$
$L_7^r$	$-0.36 \pm 0.18$	meson masses
$L_8^r$	$0.62 \pm 0.16$	$F_K/F_\pi$
$L_9^r$	$7.0 \pm 0.2$	rare pion
$L_{10}^r$	$-5.6 \pm 0.2$	decays

<sup>a</sup>Determined only with the additional assumption of  $\eta\text{-}\eta'$  mixing.

<sup>b</sup>Vanishes in the  $N_c \rightarrow \infty$  limit.

## QCD as the Hydrogen atoms of EFT's

- all aspects are displayed
- experimental tests
- now working to 2 loops



## Some predictions

Table VII-3. Chiral predictions and data in the radiative complex of transitions.

Reaction	Quantity	Theory	Experiment
$\gamma \rightarrow \pi^+ \pi^-$	$\langle r_\pi^2 \rangle$ (fm <sup>2</sup> )	0.45 <sup>a</sup>	$0.45 \pm 0.01$
$\gamma \rightarrow K^+ K^-$	$\langle r_K^2 \rangle$ (fm <sup>2</sup> )	0.45	$0.31 \pm 0.03$
$\pi^+ \rightarrow e^+ \nu_e \gamma$	$h_V(m_\pi^{-1})$	0.027	$0.0254 \pm 0.0017$
	$h_A/h_V$	0.441 <sup>a</sup>	$0.441 \pm 0.004$
$K^+ \rightarrow e^+ \nu_e \gamma$	$(h_V + h_A)(m_K^{-1})$	0.136	$0.133 \pm 0.008$
$\pi^+ \rightarrow e^+ \nu_e e^+ e^-$	$r_A/h_V$	2.6	$2.2 \pm 0.3$
$\gamma \pi^+ \rightarrow \gamma \pi^+$	$(\alpha_E + \beta_M)(10^{-4}\text{fm})$	0	$0.17 \pm 0.02$
	$(\alpha_E - \beta_M)(10^{-4}\text{fm})$	5.6	$13.6 \pm 2.8$
$K \rightarrow \pi e^+ \nu_e$	$\xi = f_-(0)/f_+(0)$	-0.13	$-0.17 \pm 0.02$
	$\lambda_+$ (fm <sup>2</sup> )	0.067	$0.0605 \pm 0.001$
	$\lambda_0$ (fm <sup>2</sup> )	0.040	$0.0400 \pm 0.002$

<sup>a</sup>Used as input.

Table VII-4. The pion scattering lengths and slopes.

	Experimental	Lowest Order <sup>a</sup>	First Two Orders <sup>a</sup>
$a_0^0$	$0.220 \pm 0.005$	0.16	0.20
$b_0^0$	$0.25 \pm 0.03$	0.18	0.26
$a_0^2$	$-0.044 \pm 0.001$	-0.045	-0.041
$b_2^2$	$-0.082 \pm 0.008$	-0.089	-0.070
$a_1^1$	$0.038 \pm 0.002$	0.030	0.036
$b_1^1$	—	0	0.043
$a_2^0$	$(17 \pm 3) \times 10^{-4}$	0	$20 \times 10^{-4}$
$a_2^2$	$(1.3 \pm 3) \times 10^{-4}$	0	$3.5 \times 10^{-4}$

<sup>a</sup>Predictions of chiral symmetry.

## Summary of EFT principle :

- 1) Identify low energy degrees of freedom and symmetry ←
- 2) Write most general  $\mathcal{L}_{\text{eff}}$  ←
- 3) Order it in the energy expansion
- 4) Starting with lowest order, start calculations
- 5) Renormalize →
- 6) Match or measure
- 7) Residual low energy effects are predictions )