

General Relativity as a Quantum Effective Field Theory

9/22/16

Note Title

8/3/2016

Overall goal: GR is a valid QFT like our other fundamental theories

Topic 1: Constructing GR as a gauge theory
- QFT point of view

Topic 2: Quantization of GR
- Feynman rules, background field method, ghosts, heat kernel, loops

Topic 3: Effective Field Theory
- both in general and for GR

Topic 4: Calculating in Quantum GR
- long distance quantum effects
- anomalies
- non-local effective actions
⋮

John Donoghue.

Bookkeeping

- course web site - blogs.umass.edu/grft/
 - Google: (~~GRQFT UMass~~) or (Donoghue UMass Teaching)
- notes + references
class Donoghue UMass general relativity
- email: donoghue@physics.umass.edu
- email me a selfie Assign #1
- research interests

metric = (+ - - -)

conventions - Dynamics of S.M.
- Gasperini

"Trust but verify"

Assessment/grade for those who need grade for credit

- Each week document for me your active engagement in the course

You decide what + how much

Background material not covered

Problems

Working through details of material

Reading classic papers (notes)

Your choice

Theory for gravity

Want $V(r) = -G \frac{m_1 m_2}{r}$

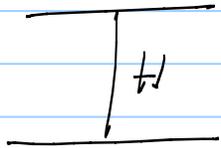
$$G = \frac{1}{M_p^2} = (1.22 \times 10^{19} \text{ GeV})^{-2}$$

- QFT Logic
- not spin 1
 - spin 0 \Rightarrow always attractive
 - want coupling \propto mass

Try "Higgs"

$$\mathcal{L} = -m_i \left(1 + \frac{H}{\nu} \right) \bar{\psi}_i \psi_i$$

$$\frac{1}{\{t\}} = -i \frac{m_1}{\nu}$$



$$-iM = -i \frac{m_1}{\nu} \frac{1}{q^2 - M_H^2} \left(-i \frac{m_2}{\nu} \right) \Rightarrow V(r) = \underbrace{-\frac{1}{4\pi r^2}}_G m_1 m_2 \frac{e^{-M_H r}}{r}$$

>

What goes wrong?

→ more to mass than constituent mass ($m_u \sim 4 \text{ MeV}$, $m_d \sim 7 \text{ MeV}$)

$$m_p = \langle P | T^4_4 | P \rangle = \langle P | \underbrace{\beta \vec{F}^2}_{900 \text{ MeV}} + \underbrace{m_u \bar{u} u + m_d \bar{d} d}_{40 \text{ MeV}} | P \rangle$$

- Binding energy $\sim 10 \text{ MeV}$ / nucleon
- Photon would not couple
- $\Delta E \sim \hbar \omega \sim \frac{1}{\Omega}$

⇒ Really should use total energy

EP #10) No combination of ϕF^2 , $\phi \vec{F} \cdot \vec{D} \phi$... gives $E|P$ for all nuclei

⇒ focus on E, \vec{p}

[Damour
- Donoghue]

Role of equivalence principle

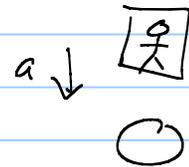
Eotvos grav mass = inertial mass

EP #1) Focus on energy

Conceptually



also free fall



$F = 0$
locally

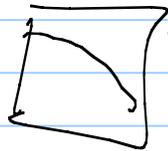
Should then couple to light



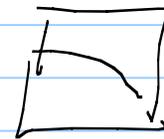
$v = 0$



$v \neq 0$
 $a = 0$



$a \neq 0$



Scalar coupling $\phi F_{\mu\nu} F^{\mu\nu} \propto \phi (E^2 - B^2) = 0$

EP #2 Gravity like non-inertial coordinate systems \leftarrow
#3 There exists coordinate system where no local gravity effects

Energy + Momentum

Tied together in relativistic theory - $T_{\mu\nu} =$ energy momentum tensor
stress tensor

$$H = \int d^3x T_{00} \quad , \quad P_i = \int d^3x T_{0i}$$

Conservation $\partial^\mu T_{\mu\nu} = 0$

Noether construction - time + space translation invariance

$$T_{\mu\nu} = \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \partial_\nu \phi - g_{\mu\nu} \mathcal{L}$$

Example:

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \phi \partial^\mu \phi - m^2 \phi^2)$$

$$T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g_{\mu\nu} (\partial_\lambda \phi \partial^\lambda \phi - m^2 \phi^2)$$

Reasonable to use $T_{\mu\nu}$ as the source

$$\underline{\underline{\{g\}}} = \frac{1}{2} \frac{\kappa}{2} T_{\mu\nu} \frac{1}{4\pi\Omega} \frac{\kappa}{2} T_{\mu\nu} \rightarrow \frac{\kappa^2}{32\pi} \frac{m_{\mu\nu} m_{\nu\mu}}{\Omega}$$

Note on normalization

$$\langle P | P' \rangle = 2E \delta^3(\vec{p} - \vec{p}')$$

In this norm

$$\langle P | T_{\mu\nu} | P' \rangle = \frac{1}{\sqrt{2E} \sqrt{2E'}} \left[(P_{\mu} P'_{\nu} + P'_{\mu} P_{\nu}) - g_{\mu\nu} (p \cdot p' - m^2) \right]$$

Question: How to get $T_{\mu\nu}$ as source?

- answer - gauge a symmetry

Reminders / notation

Global invariance $\psi \rightarrow e^{-i\theta} \psi$
corresponding charge $J^\mu = \frac{q}{2} \bar{\psi} \gamma^\mu \psi \leftarrow \text{QED}$
invariance $\partial_\mu J^\mu = 0$
conserved quantities $Q = \int d^3x J^0$

Get this as a source by gauging the symmetry

$$\psi \rightarrow e^{-i\theta(x)} \psi$$

Covariant Derivative

$$L = \bar{\psi} \not{\partial} \psi \rightarrow \underline{\underline{L = \bar{\psi} i \not{D} \psi}} \Rightarrow$$

\Rightarrow

$$D_\mu = \partial_\mu + ie A_\mu$$

$$D_\mu \psi \rightarrow e^{-ie(x)} D_\mu \psi$$

Field strength

$$[D_\mu, D_\nu] = +ie(\partial_\mu A_\nu - \partial_\nu A_\mu) = +ie F_{\mu\nu}$$

Then

$$L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi} (i \not{D} - m) \psi$$

$$\underbrace{\hspace{10em}}_{\text{J}_\mu \text{ as source}}$$

Then To get $T_{\mu\nu}$ as source

- gauge space-time translations

Non abelian rotations, \swarrow matrix

eg $SU(N) \quad \psi \rightarrow U \psi$

$$\psi = \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_N \end{pmatrix}$$

$$U = \exp \left\{ -i \frac{\alpha^a}{2} \lambda^a \right\}$$

$$\left[\frac{\lambda^a}{2}, \frac{\lambda^b}{2} \right] = i f^{abc} \frac{\lambda^c}{2}$$

$$; \text{Tr} \left[\frac{\lambda^a}{2} \frac{\lambda^b}{2} \right] = \frac{1}{2} \delta^{ab}$$

Gauging:

$$\psi \rightarrow \psi' \rightarrow U(x) \psi$$

$$D_\mu \psi \rightarrow D'_\mu \psi' = U(x) D_\mu \psi$$

$$D_\mu = \partial_\mu + g \frac{\lambda^a}{2} A_\mu^a = \partial_\mu + \underline{g A_\mu}$$

Then $\underline{A_\mu} \rightarrow A'_\mu = U A_\mu U^{-1} + \frac{i}{g} (D_\mu U) U^{-1}$

We have $D'_\mu = U D_\mu U^{-1}$

Field strength $[D_\mu, D_\nu] \psi = i g \underline{\underline{F_{\mu\nu}}} \psi = i g \frac{1}{2} \underline{\underline{F_{\mu\nu}^a}} \tau^a \psi$

$$\underline{\underline{F_{\mu\nu}}} = \partial_\mu \underline{\underline{A_\nu}} - \partial_\nu \underline{\underline{A_\mu}} + g \underline{\underline{[A_\mu, A_\nu]}}$$

$$\underline{\underline{F_{\mu\nu}'}} = U \underline{\underline{F_{\mu\nu}}} U^{-1}$$

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a - g f^{abc} A_\mu^b A_\nu^c$$

✓



Gauging Spacetime invariance

Global symmetry $x'^{\mu} = \Lambda^{\mu}_{\nu} x^{\nu} + a^{\mu}$

Gauged symmetry $x'^{\mu} = \Lambda^{\mu}_{\nu}(x) x^{\nu} + \underline{a^{\mu}(x)} = x'^{\mu}(x)$

\Rightarrow new field $g_{\mu\nu}(x)$

Covariant derivative $D_{\mu} = \partial_{\mu} + \dots$

Invariant matter action $\mathcal{L}(\phi, g)$ gives source $\frac{\delta \mathcal{L}}{\delta g^{\mu\nu}} = \frac{1}{2} T_{\mu\nu}$

Field strengths from $\{D_{\mu}, D_{\nu}\} \sim \mathcal{F}$

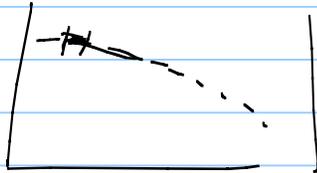
Invariant action $S_g \sim R$

Varying action $\delta(S_g + S_m) \Rightarrow \underline{G_{\mu\nu}} = \frac{1}{2} T_{\mu\nu}$

Local Coordinate systems - EP at work

- rulers + light beams

Gravity -



← quadratic

Locally flat - infinitesimal dy^{μ}

$$\underline{ds^2 = \eta_{ab} dy^a dy^b}$$

$$\eta = \text{diag}(1, -1, -1, -1)$$

But these may not be useful everywhere

Can use other coordinates also locally

$$dy^a = e_\mu^a(x) dx^\mu$$

Then

↖ vierbein or tetrad

$$ds^2 = \eta_{ab} dy^a dy^b = \eta_{ab} e_\mu^a(x) e_\nu^b dx^\mu dx^\nu$$

$$\equiv g_{\mu\nu}(x) dx^\mu dx^\nu$$

metre

General Coordinate Invariance

$$x^m \rightarrow x'^m = x'^m(x)$$

Look locally $dx'^m = J^m_{\nu} dx^\nu$

$$J^m_{\nu} = \frac{\partial x'^m}{\partial x^\nu}(x)$$

Inverse $(J^{-1})^{\mu}_{\nu} = \frac{\partial x^\mu}{\partial x'^\nu}$

To have invariance:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = g'_{\mu\nu} dx'^\mu dx'^\nu$$
$$\Rightarrow g'_{\alpha\beta} = (J^{-1})^{\mu}_{\alpha} g_{\mu\nu} (J^{-1})^{\nu}_{\beta}$$

$$\text{or } \begin{aligned} x' &= J x, \\ g' &= J^{-1} g J^{-1} \end{aligned}$$

\Rightarrow New field + transformation $g_{\mu\nu}(x) \rightarrow g' = (J^{-1})^T g J$

Vierbein / Tetrad

From EP: $g_{\mu\nu}(x) = e_{\mu}^a(x) e_{\nu}^b(x) \eta_{ab}$

Transformation is now $e \rightarrow J^{-1} e$ $x \rightarrow Jx = x'$

$$e_{\mu}^{a'} = (J^{-1})_{\mu}^{\alpha} e_{\alpha}^a$$

Note: extra local Lorentz invariance

(not $\Lambda_{\nu}^{\mu}(x)$
from before)

Lorentz

Here $e_{\mu}^a \rightarrow e_{\mu}^{a'} = \Lambda^a_c(x) e_{\mu}^c$ with $\eta_{ab} \Lambda^a_c(x) \Lambda^b_d(x) = \eta_{cd}$

is symmetry

Boring technical details

Fundamental def. $x^m \leftarrow \boxed{\text{up}}$

then $\boxed{\text{down}} \rightarrow \partial_\mu \equiv \frac{\partial}{\partial x^\mu}$

$$\text{so } \partial_\mu x^\nu = \delta_\mu^\nu \quad \&$$

$g_{\mu\nu} \leftarrow \text{down}$

Definitions: $g^{\mu\nu} = [g_{\mu\nu}]^{-1} \rightarrow g^{\mu\alpha} g_{\alpha\nu} = \delta_\nu^\mu$

$$x_\mu \equiv g_{\mu\nu} x^\nu, \quad \partial^\mu \equiv g^{\mu\nu} \partial_\nu$$

"Ups" transform with J , "Downs" transform with J^{-1}

$$g'^{\mu\nu} = J^\mu_\alpha J^\nu_\beta g^{\alpha\beta} = \frac{\partial x'^\mu}{\partial x^\alpha} \frac{\partial x'^\nu}{\partial x^\beta} g^{\alpha\beta}$$

Invariant Volume

Using flat coord $dV = d^4 y$

In other coord $dy^a = e^a_\mu dx^\mu$

$$d^4 y = d^4 x \left| \frac{\partial y^a}{\partial x^\mu} \right| = d^4 x \underbrace{\det e^a_\mu}_{\sqrt{-g_{\mu\nu}}}$$

$$\begin{aligned} \text{Then } g &\equiv \det g_{\mu\nu} = \det (e^a_\mu e^b_\nu \eta_{ab}) = (\det e)^2 \underbrace{\det \eta}_{=-1} \\ &= -(\det e)^2 \end{aligned}$$

Then $d^4 x \sqrt{-g}$ is invariant volume

—

An invariant action

Scalar field $\phi(x)$

$$\phi'(x') = \phi(x)$$

Construct using $d^4x \sqrt{g}$, $g_{\mu\nu}$, $g^{\mu\nu}$, ∂_μ

Then

$$S = \int d^4x \sqrt{-g} \frac{1}{2} \left[\underset{\equiv}{\underset{\neq}{g^{\mu\nu}}} \partial_\mu \phi \partial_\nu \phi - m^2 \phi^2 \right]$$

Immediate success!
- vary action

$T_{\mu\nu}$ as source *mission accomplished*

$$\frac{\delta S}{\delta g^{\mu\nu}} = \frac{1}{2} \sqrt{-g} \left[\partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g_{\mu\nu} \left[\partial_\alpha \phi \partial^\alpha \phi - m^2 \phi^2 \right] \right]$$

recall $T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g_{\mu\nu} \left[\partial_\alpha \phi \partial^\alpha \phi - m^2 \phi^2 \right]$

$$\frac{2}{\sqrt{-g}} \frac{\delta S_m}{\delta g^{\mu\nu}} = T_{\mu\nu}$$

Technology here:

$$\delta(g_{\mu\nu} g^{\nu\sigma}) = \delta(\delta_{\mu}^{\sigma}) = 0$$

$$\Rightarrow \delta g_{\mu\nu} = -g_{\mu\rho} g_{\nu\sigma} \delta g^{\rho\sigma}$$

$$\begin{aligned} \text{And for } \delta \det M &= \det(M + \delta M) - \det M = \\ &= e^{\text{Tr} \ln(M + \delta M)} - e^{\text{Tr} \ln M} = \text{Tr}(M^{-1} \delta M) \end{aligned}$$

$$\Rightarrow \frac{\delta \sqrt{-g}}{\delta g^{\mu\nu}} = -\sqrt{-g} \frac{1}{2} g_{\mu\nu}$$

Matter eq of motion

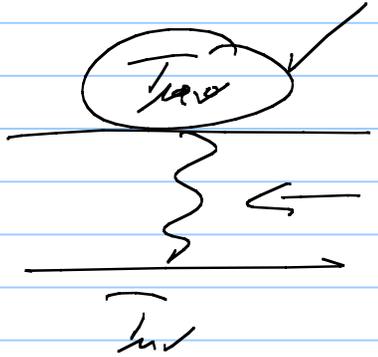
$$S_m = \int d^4x \sqrt{-g} \frac{1}{2} \left[g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - m^2 \phi^2 \right]$$

$$\left[\frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu) - m^2 \right] \phi = 0$$

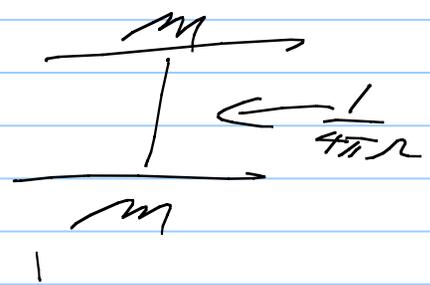
Sch. Eq in grav field

FP?

$$T_{uv} \rightarrow T_{00} = \mathcal{M}_{TOT} \leftarrow \begin{matrix} \text{Bending } E \\ \text{const.} \end{matrix}$$



NR \rightarrow



Now gravity itself

Pathway - follow $[D_\mu, D_\nu] = \frac{1}{2} g \lambda^a F_{\mu\nu}^a$

I do simplest path first then return for fermions

Vector field V^μ - transforms like x^μ

$$V^\mu \rightarrow V'^\mu(x') = J^\mu_{\sigma(x)} V^\sigma(x)$$

Covariant derivative

$$D_\mu V^\lambda \rightarrow D'_\mu V'^\lambda = J^\lambda_{\rho} J^{-1\sigma}_{\mu} D_\sigma V^\rho$$

Most general form

$$D_\mu V^\lambda = \partial_\mu V^\lambda + \Gamma_{\mu\lambda}^\lambda V^\lambda$$

Γ = "connection"

To get covariance need to cancel \rightarrow

$$\partial_\mu \underbrace{J^\lambda_p}_{= V^\lambda} = J^\lambda_p \partial_\mu V^\lambda + (\partial_\mu J^\lambda_p) V^\lambda$$

Let

$$\Gamma_{\mu\nu}^\lambda = \underbrace{J_\mu^{-1}{}^{\lambda'} J_\nu^{-1}{}^{\lambda'} J^{\lambda'}_{\lambda'}}_{\text{usual trans.}} \Gamma_{\mu\nu}^{\lambda'} + \underbrace{\partial_\mu J_\nu^\lambda}_{\text{gauge trans}}$$

This condition is

$$\partial_{\mu} A^{\mu} + J_{\mu}^{-1\nu} J_{\nu}^{-1\rho} \partial_{\mu} J^{\rho}_{\nu} = 0$$

It looks like a standard gauge condition

$$\partial_{\mu} A^{\mu} = J_{\mu}^{-1\nu} J_{\nu}^{-1\rho} J^{\rho}_{\lambda} \left[\partial_{\mu} J^{\lambda}_{\nu} + J^{-1\lambda'}_{\nu''} \partial_{\mu} J^{\nu''}_{\lambda'} \right]$$

compare with

$$A'_{\mu} = U A_{\mu} U^{-1} + \frac{i}{g} (\partial_{\mu} U) U^{-1}$$

This is solved by

$$\Gamma_{\mu\nu}^{\lambda} = \frac{1}{2} g^{\lambda\sigma} [\partial_{\mu} g_{\nu\sigma} + \partial_{\nu} g_{\mu\sigma} - \partial_{\sigma} g_{\mu\nu}] = \{ \mu\nu \}^{\lambda}$$

(Plug away!)

Christoffel
symbol

Unique form composed only of ∂g

But literature has discussions of extra DOFs ("torsion" \leftarrow \star "non-metricity" \leftarrow)

$$T' = \underbrace{\text{JJJ}}_{\leftarrow} [\underbrace{T'}_{\leftarrow} + \underbrace{\partial J}_{\leftarrow}] + \underbrace{\text{JJJ}}_{\leftarrow} \underbrace{\Gamma_{\mu\nu}^{\lambda}}_{\leftarrow} \neq \neq$$

Easiest derivation - "Metricity" condition

- from EP - for some coordinate - locally flat $g_{\mu\nu} = \eta_{\mu\nu}$
- no grav force $\Gamma = 0$
- deviations can be made quadratic

$\Rightarrow \partial_\alpha g_{\mu\nu} = 0$ in this coordinates

$\Rightarrow D_\alpha g_{\mu\nu} = 0$ in general
↙ metricity

Then

- ① $D_\alpha g_{\mu\nu} = 0 = \partial_\alpha g_{\mu\nu} - \Gamma_{\alpha\mu}^\sigma g_{\sigma\nu} - \Gamma_{\alpha\nu}^\sigma g_{\mu\sigma}$
- ② $D_\mu g_{\nu\sigma} = 0 = \dots$
- ③ $D_\nu g_{\sigma\mu} = 0 = \dots$

Choose $\Gamma_{\mu\nu}^\lambda = \Gamma_{\nu\mu}^\lambda \leftarrow$ symmetric $\nabla \leftarrow$

Then $\frac{1}{2}[\text{①} - \text{②} - \text{③}]$ isolates $\Gamma_{\mu\nu}^\lambda$

Formalism

"Lower" vectors

$$D_\mu V_\nu = \partial_\mu V_\nu - \Gamma_{\mu\nu}^\lambda V_\lambda$$

Tensors $T^{\alpha\beta} \rightarrow T'^{\alpha\beta} = J^\alpha_{\alpha'} J^\beta_{\beta'} T^{\alpha'\beta'}$ --- etc transform covariantly

$$D_\mu T^{\alpha\beta\dots}_{\rho\sigma\dots} = \partial_\mu T^{\alpha\beta\dots}_{\rho\sigma\dots} + \Gamma_{\mu\nu}^\alpha T^{\nu\beta\dots}_{\rho\sigma\dots} + \Gamma_{\mu\nu}^\beta T^{\alpha\nu\dots}_{\rho\sigma\dots} - \Gamma_{\mu\rho}^\lambda T^{\alpha\beta\dots}_{\lambda\sigma\dots} - \dots$$

Note despite indices

$\Gamma_{\mu\nu}^\lambda$ is not a "tensor" (the gauge trans term spoils it)

Riemann tensor

Recall in YM $[D_\mu, D_\nu] \psi = i \frac{g}{2} \lambda^a F_{\mu\nu}^a \psi = -i \frac{g}{2} \underline{F}_{\mu\nu} \psi$

Do the same here:

$$[D_\mu, D_\nu] V^\beta = R_{\mu\nu\alpha}{}^\beta V^\alpha$$

With no effort we know this transforms covariantly, since LHS does

$$R_{\mu\nu\alpha}{}^\beta = \partial_\mu \Gamma_{\nu\alpha}^\beta - \partial_\nu \Gamma_{\mu\alpha}^\beta + \Gamma_{\mu\rho}^\beta \Gamma_{\nu\alpha}^\rho - \Gamma_{\nu\rho}^\beta \Gamma_{\mu\alpha}^\rho$$

\uparrow
covariant

$$\left(\text{like } \underline{F}_{\mu\nu} = \partial_\mu \underline{A}_\nu - \partial_\nu \underline{A}_\mu - g [\underline{A}_\mu, \underline{A}_\nu] \right)$$

From Riemann

$$R_{\nu\alpha} = R_{\mu\nu\alpha}{}^{\mu} \quad (\text{note conventions vary here})$$

↖ Ricci tensor

And Ricci scalar

$$R = g^{\nu\alpha} R_{\nu\alpha}$$

By construction R is an invariant

Seeing past the formalism here?

We have made $g_{\mu\nu}$ a field $g_{\mu\nu}(x)$

No gravity effect $g_{\mu\nu} = \eta_{\mu\nu}$ \hookrightarrow

Weak field $g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$

$$\Gamma_{\mu\nu}^\lambda = \frac{1}{2} g^{\lambda\sigma} [\partial_\mu g_{\sigma\nu} + \dots] \sim \partial_\mu h$$

$$R_{\mu\nu} \approx \partial^2 + \mathcal{O}^2 \sim \underbrace{\partial^2 h + h \partial h \partial h}_{\mathcal{O}' \text{ terms}} \neq \partial h \partial h$$

$$R_{\mu\nu}, R \sim \partial^2 h + \partial h \partial h$$

$$R^2 \sim \quad \quad \quad |, \quad \quad \quad |,$$

Here is a difference with gauge theory

$$\rightarrow F_{\mu\nu} \sim \partial_\mu A_\nu - \partial_\nu A_\mu \sim \partial A \quad \leftarrow \partial x^\alpha A^\alpha$$

$$R_{\mu\nu\alpha}{}^\beta \sim \partial_\mu \overset{\uparrow \text{gauge}}{\Pi_{\nu\alpha}{}^\beta} - \partial_\nu \overset{\uparrow \text{gauge}}{\Pi_{\mu\alpha}{}^\beta} \sim \partial^2 h$$

Connection seems more natural gauge variable

But it is a composed field

Why? The gauged symmetry is that of spacetime translations

The generators of translations are $\underline{P}_\mu \sim \underline{\partial}_\mu$

The Gravitational action

— need to form invariants

YM analogy $F_{\mu\nu}^A F^{A\mu\nu}$ suggests $R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta}$

but also $R_{\mu\nu} R^{\mu\nu}$, R^2

' And R ← invariants

Think like a particle physicist

$$R^2 \sim \partial h \partial^2 h + \dots$$

$$R \sim \partial h \partial h + \dots$$

low E R wins!

Einstein Hilbert Action

$$S = S_{EH} + \underline{S_m} \quad \leftarrow \text{before}$$

convention $R_{\nu\alpha} = R_{\mu\nu\alpha}{}^\mu$

$$S_{EH} = \int d^4x \sqrt{-g} \left(\frac{-2}{\kappa^2} R \right) + \int \Sigma^m K_m$$

$$\uparrow \kappa^2 = 32\pi G$$

Surface
York Hawking
Gibbons

Einstein Eq: $g^{\mu\nu} \rightarrow g^{\mu\nu} + \delta g^{\mu\nu}$

$$\delta S_{EH} = \int d^4x \sqrt{-g} \left(-\frac{2}{\kappa^2} \right) [R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R] \delta g^{\mu\nu}$$

$$\delta S_m = \int d^4x \sqrt{-g} \frac{1}{2} T_{\mu\nu} \delta g^{\mu\nu}$$

$$\text{from } T_{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta S_m}{\delta g^{\mu\nu}}$$

$$\delta (S_{EH} + S_m) = 0$$

$$\Rightarrow \left[R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{\kappa^2}{4} T_{\mu\nu} = 8\pi G T_{\mu\nu} \right]$$

Parse summary

Constructed GR by gauging spacetime translations

S_m invariant produced source

S_{EH} produced gravity eq