

Overall Plan:

Overall goal: GR is a valid QFT like our other fundamental theories

Topic 1: Constructing GR as a gauge theory
- QFT point of view

Topic 2: Quantization of GR
- Feynman rules, background field method, ghosts, heat kernel, loops

Topic 3 Effective Field Theory
- both in general and for GR

Topic 4: Calculating in Quantum GR
- long distance quantum effects
- anomalies
- non-local effective action

What we have accomplished:

Constructing GR as a QFT:

- identify source of interactions $T_{\mu\nu}$
- gauge symmetry to obtain the interaction $x_\mu' = x_\mu + a_\mu(x)$
- new field and covariant derivative to allow symmetry $g_{\mu\nu}(x)$, D_μ
- Field strength via $[D_\mu, D_\nu] \sim R_{\mu\nu\rho}{}^\sigma$
- invariant action $\int d^4x \sqrt{g} R$
- simple solution
- Schrödinger eq in gravitational field ✓

$$\boxed{\frac{\partial^u}{\partial x^\mu} \frac{\partial^v}{\partial x^\nu} R_{\mu\nu}^{ab}(A)}$$

Today's topics

Quantization

1) Some basic facts (weak field limit + Schrodinger eq.)

2) Gravitons — Feynman rules

3) Simple quantizations ←

4) Background field method

5) Gauge fixing & ghost

6) Heat kernel methods

7) One loop effect

Gravitons

if $T_{\mu\nu} = 0$ then $\underbrace{[R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R]}_{{\mathcal O}(h^2)} = 0 \Rightarrow \Box h_{\mu\nu} = 0$

But let's work to $\mathcal{O}(h^2)$

harmonic gauge

to first order
in h

$$[R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R]^{(1)} + \underbrace{[R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R]^{(2)}}_{\mathcal{O}(h^2)} = 8\pi G T_{\mu\nu}$$

$$\text{Define } -\frac{1}{8\pi G} [R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R]^{(2)} = t_{\mu\nu} \quad (\mathcal{O}(h^2))$$

then

$$\Box h_{\mu\nu} = 8\pi G (T_{\mu\nu} + t_{\mu\nu})$$

$\underbrace{\{ \}_{\text{It}}}$ gravity acts as a source
of gravity

'not a' true tensor'

Calculate this



$$h = h^\lambda_\lambda$$

total
deriv.

$$\begin{aligned} t_{\mu\nu} = & -\frac{1}{4}h_{\alpha\beta}\partial_\mu\partial_\nu h^{\alpha\beta} + \frac{1}{8}h\partial_\mu\partial_\nu h \\ & + \frac{1}{8}\eta_{\mu\nu}\left(h^{\alpha\beta}\square h_{\alpha\beta} - \frac{1}{2}h\square h\right) \end{aligned}$$

important

$$\begin{aligned} & -\frac{1}{4}\left(h_{\mu\rho}\square h^\rho_\nu + h_{\nu\rho}\square h^\rho_\mu - h_{\mu\nu}\square h\right) \\ & + \frac{1}{8}\partial_\mu\partial_\nu\left\{h_{\alpha\beta}h^{\alpha\beta} - \frac{1}{2}hh\right\} - \frac{1}{16}\eta_{\mu\nu}\square\left\{h_{\alpha\beta}h^{\alpha\beta} - \frac{1}{2}hh\right\} \\ & - \frac{1}{4}\partial_\alpha\left[\partial_\nu\left\{h_{\mu\beta}h^{\alpha\beta}\right\} + \partial_\mu\left\{h_{\nu\beta}h^{\alpha\beta}\right\}\right] \\ & + \frac{1}{2}\partial_\alpha\left[h^{\alpha\beta}\left(\partial_\nu h^\lambda_{\mu\beta} + \partial_\mu h^\lambda_{\nu\beta}\right)\right] \end{aligned}$$

Variations
on shell

Second
Canonical quantization of weak field gravity

Solutions: Transverse-traceless

$$\square h_{\mu\nu} = 0 \quad ; \quad \partial^\mu h_{\mu\nu} = 0 \quad , \quad h^{\lambda}_{\lambda} = 0$$

Recall photon polarization vector

$$E_\mu(\lambda) = \frac{1}{\sqrt{2}} (0, 1, \pm i, 0) \quad \text{motion in } z \text{ direction}$$

$$E_\mu^*(\lambda) E^\mu(\lambda) = -1 \quad \text{but} \quad E_\mu(\lambda) E^\mu(\lambda) = 0$$

$$h_{\mu\nu} E^\mu = 0$$

Then we can use

$$\underline{h_{\mu\nu} = A E_{\mu\nu} e^{-ik \cdot x}}$$

$$E_{\mu\nu}^{(++)} = E_\mu^{(+)} E_\nu^{(+)} \quad] \star \star$$

Two polarizations

Second quantization

$$p_\mu = (p, \vec{p})$$

$$h_{\mu\nu}(x) = \sum_{\lambda=\pm\pm} \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} [a(p, \lambda) E_{\mu\nu}(p, \lambda) e^{-ip\cdot x} + h.c.]$$

$$T_{\mu\nu} = -\frac{1}{2} h_{\alpha\beta} \partial_\mu \partial_\nu h^{\alpha\beta} + \frac{1}{8} h \partial_\mu \partial_\nu h + \dots$$

$$H = \int d^3 x T_{00}$$

$$= \sum_{\lambda} \int \frac{d^3 p}{(2\pi)^3} \omega_p [a^\dagger(p, \lambda) a(p, \lambda) + \frac{1}{2}]$$

$$\begin{matrix} a a' \\ a^\dagger a^\dagger \end{matrix} ?$$

Weak Field QFT

Start with

$$S = \int d^4x \sqrt{-g} \left[-\frac{2}{k^2} R + \mathcal{L}_m \right] \quad + \text{expand} \quad g_{\mu\nu} = \eta_{\mu\nu} + \lambda h_{\mu\nu}$$

We find

$$-\frac{2}{k^2} \sqrt{-g} R = -\frac{2}{k} [\partial_\lambda \partial^\lambda h^{\mu\nu} - \square h] + \frac{1}{2} [\partial_\lambda h_{\mu\nu} \partial^\lambda \bar{h}^{\mu\nu} - 2 \partial^\lambda \bar{h}_{\mu\lambda} \partial_\nu \bar{h}^{\nu\lambda}]$$

↑
 total deriv.
 ↴
 KE pieces

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} h^\lambda_\lambda$$

$$+ O(h^3)$$

Fix gauge: $\cancel{\partial_\mu h^{\mu\nu}}$ gauge parameter

$$\mathcal{L}_{gf} = \xi \partial_\mu \cancel{h^{\mu\nu}} \partial^\lambda \cancel{h_{\lambda\nu}}$$

This cancels second term

$$\mathcal{L}_g = \frac{1}{2} \partial_\mu h_{\mu\nu} \partial^\lambda h^{\mu\nu} - \frac{1}{4} \partial_\lambda h \partial^\lambda h - \frac{\kappa}{2} h^{\mu\nu} T_{\mu\nu}$$

Integrate by parts

$$\mathcal{L} = \frac{1}{2} h_{\mu\nu} \left[\square \left(I^{\mu\nu\alpha\beta} - \frac{1}{2} \gamma^{\mu\nu} \gamma^{\alpha\beta} \right) \right] h_{\alpha\beta} - \frac{\kappa}{2} h^{\mu\nu} T_{\mu\nu}$$

$\nwarrow \quad \nwarrow$

$$I^{\mu\nu\alpha\beta} = \frac{1}{2} (\gamma_{\mu\alpha} \gamma_{\nu\beta} + \gamma_{\nu\alpha} \gamma_{\mu\beta})$$

(ghosts later)

$$\begin{aligned} \partial_\mu \cancel{h^{\mu\nu}} &= 0 \\ &= \partial_\mu h^{\mu\nu} - \frac{1}{2} \partial_\nu h^\lambda_\lambda \end{aligned}$$

$\xi = 1 \Rightarrow \text{harmonic}$

\nwarrow matters

Propagator:

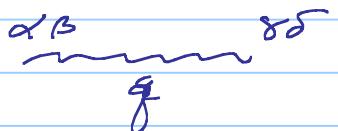
Invert tensor part $\left[I^{\alpha\beta} - \frac{1}{2} \gamma^{\alpha\nu} \gamma^{\beta\rho} \right] \left[a \gamma_{\alpha\rho} + b \gamma_{\beta\rho} \right] = I^{\alpha\beta}$

$$\Rightarrow a=1, b=-\frac{1}{2}$$

Then $i D^{\alpha\beta\gamma\delta}(x) = \int \frac{d^4 q}{(2\pi)^4} \frac{i}{q^2 + i\varepsilon} e^{-iq \cdot x} P^{\alpha\beta\gamma\delta}$

$$P^{\alpha\beta\gamma\delta} = \frac{1}{2} \left[\gamma^{\alpha\delta} \gamma^{\beta\gamma} + \gamma^{\alpha\gamma} \gamma^{\beta\delta} - \gamma^{\alpha\beta} \gamma^{\gamma\delta} \right]$$

Feynman rules

Propagator  = $\frac{i}{g^2} \frac{p^\alpha p^\beta}{p^2}$

Vertex

$$T = \frac{k}{2} h_{\mu\nu} T^{\mu\nu}$$

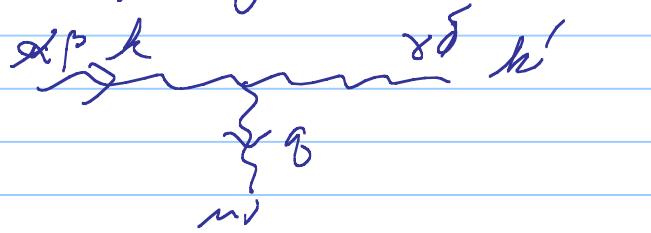
$$T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g_{\mu\nu} [\partial_\lambda \phi \partial^\lambda \phi - m^2]$$

$$\langle p' | T_{\mu\nu} | p \rangle = \frac{-i(p - p')^\mu i x}{(2\pi)^3 (2\omega_1 \omega_2)^{1/2}} \sum$$

Then

$$\frac{p'}{p} = i \frac{k}{2} \left[(p_\mu p'_\nu + p'_\mu p_\nu) - g_{\mu\nu} (p \cdot p' - m^2) \right]$$

Triple graviton vertex



$$\begin{aligned}
 \tau_{\alpha\beta,\gamma\delta}^{\mu\nu} = & \frac{i\kappa}{2} \left(P_{\alpha\beta,\gamma\delta} \left[k^\mu k^\nu + (k-q)^\mu (k-q)^\nu + q^\mu q^\nu - \frac{3}{2} \eta^{\mu\nu} q^2 \right] \right. \\
 & + 2q_\lambda q_\sigma \left[I^{\lambda\sigma}_{\alpha\beta} I^{\mu\nu}_{\gamma\delta} + I^{\lambda\sigma}_{\gamma\delta} I^{\mu\nu}_{\alpha\beta} - I^{\lambda\mu}_{\alpha\beta} I^{\sigma\nu}_{\gamma\delta} - I^{\sigma\nu}_{\alpha\beta} I^{\lambda\mu}_{\gamma\delta} \right] \\
 & + \left[q_\lambda q^\mu \left(\eta_{\alpha\beta} I^{\mu\nu}_{\gamma\delta} + \eta_{\gamma\delta} I^{\lambda\nu}_{\alpha\beta} \right) + q_\lambda q^\nu \left(\eta_{\alpha\beta} I^{\lambda\mu}_{\gamma\delta} + \eta_{\gamma\delta} I^{\lambda\mu}_{\alpha\beta} \right) \right. \\
 & \left. - q^2 \left(\eta_{\alpha\beta} I^{\mu\nu}_{\gamma\delta} + \eta_{\gamma\delta} I^{\mu\nu}_{\alpha\beta} \right) - \eta^{\mu\nu} q^\lambda q^\sigma \left(\eta_{\alpha\beta} I_{\gamma\delta,\lambda\sigma} + \eta_{\gamma\delta} I_{\alpha\beta,\lambda\sigma} \right) \right] \\
 & + \left[2q^\lambda \left(I^{\sigma\nu}_{\alpha\beta} I_{\gamma\delta,\lambda\sigma} (k-q)^\mu + I^{\sigma\mu}_{\alpha\beta} I_{\gamma\delta,\lambda\sigma} (k-q)^\nu \right. \right. \\
 & \left. - I^{\sigma\nu}_{\gamma\delta} I_{\alpha\beta,\lambda\sigma} k^\mu - I^{\sigma\mu}_{\alpha\beta,\lambda\sigma} k^\nu \right) \\
 & + q^2 \left(I^{\sigma\mu}_{\alpha\beta} I_{\gamma\delta,\sigma}^\nu + I_{\alpha\beta,\sigma}^\nu I^{\sigma\nu}_{\alpha\delta} \right) + \eta^{\mu\nu} q^\lambda q_{\text{sigma}} \left(I_{\alpha\beta,\lambda\rho} I^{\rho\sigma}_{\gamma\delta} + I_{\gamma\delta,\lambda\rho} I^{\rho\sigma}_{\alpha\beta} \right) \\
 & + \left\{ \left(k^2 + (k-q)^2 \right) \left(I^{\sigma\mu}_{\alpha\beta} I_{\gamma\delta,\sigma}^\nu + I^{\sigma\nu}_{\alpha\beta} I_{\gamma\delta,\sigma}^\mu - \frac{1}{2} \eta^{\mu\nu} P_{\alpha\beta,\gamma\delta} \right) \right. \\
 & \left. - \left(k^2 \eta_{\gamma\delta} I^{\mu\nu}_{\alpha\beta} + (k-q)^2 \eta_{\alpha\beta} I^{\mu\nu}_{\gamma\delta} \right) \right\} \quad (42)
 \end{aligned}$$

Gravitational Scattering

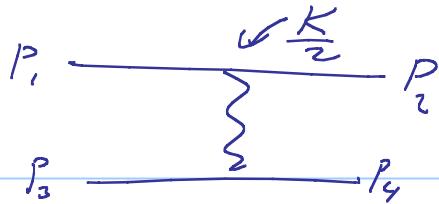
$$-i\mathcal{M} = i\frac{\kappa}{2} \left[(P_1^{\alpha} P_2^{\beta} + P_1^{\beta} P_2^{\alpha}) + \dots \right] - \frac{i\kappa}{8} P_{\text{max}} \frac{i\kappa}{2} \left[P_3^{\alpha} P_4^{\beta} + \dots \right]$$

Non rel. $P_m \rightarrow (m=0)$

$$\mathcal{M} = -\frac{\kappa^2}{4} \frac{m_1^2 m_2^2}{q^2} = -16\pi G \frac{m_1^2 m_2^2}{q^2}$$

$$\text{FT: } \frac{1}{q^2} \rightarrow \frac{1}{4\pi m_1 m_2} ; \text{ norm } \frac{1}{2E_1 E_2} = \frac{1}{4m_1 m_2} \Rightarrow V(r) = -\frac{G m_1 m_2}{r}$$

Completes GR as tree level QFT



Let's try a loop diagram (set $m=0$ for easy power counting)

$$\begin{aligned}
 \text{Diagram} &= \frac{g^4 l}{2\pi^2} \frac{iK}{2} [l_\alpha(l+g)_\beta + l_\beta(l+g)_\alpha] \cdot \frac{i}{l^2(l+g)^2} \frac{iK}{2} [(l+g)_\delta g_\beta + \dots] \\
 &= + \frac{K^3}{4} \frac{\#}{16\pi^2} (g^\alpha g_\mu g^\nu g_\sigma + \eta_{\mu\nu} g^2 g_\sigma) \left[\frac{1}{\epsilon} + \ln g^2 \right]
 \end{aligned}$$

$$\text{QED} \quad \Pi \sim \frac{e^2}{16\pi^2} (g_u g_v - g_{u\nu} g^\nu) \left(\frac{1}{\epsilon} + \ln g \right) \Rightarrow \Delta I = \sum_{F} \frac{\#}{\epsilon} F^\mu F^{\mu\nu} \text{ WF}$$

$$\text{FR} \quad \Pi \sim \frac{K^2}{16\pi^2} (g_u g_v g^\mu g_\sigma) \left(\frac{1}{\epsilon} + \ln g \right) \Rightarrow \Delta I = \underbrace{\partial \partial h \partial \partial h}_{4 \text{ deriv}} \neq R \quad \underbrace{_ _ _ _}_{2 \text{ deriv}}$$

Graviton loop has same power counting ✓



Rosen-Nordström metric

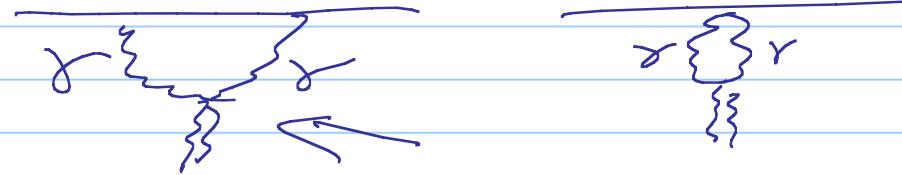
The RN metric describes the gravitational field around a charged, massive point particle.

- What is the T_{μν} of the electromagnetic field around the charged particle?
- Use the Einstein eq in the weak field limit and the harmonic gauge to find the leading term for the metric due to this EM field
- Compare to the leading form of the RN metric in harmonic gauge:

$$g_{00} = 1 - \frac{2Gm}{r} + \frac{G\alpha}{r^2} + \dots$$

$$g_{0i} = 0$$

$$g_{ij} = -\delta_{ij} - \delta_{ij} \frac{2Gm}{r} + G\alpha \frac{r_i r_j}{r^2} + \dots$$



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