EPFL Lectures

Note Title 8/10/2016

Next:

- Background field method
- Gauge fixing + ghosts in GR
- Heat kernel methods

Background Field Method

Abbott paper on web page.

Pedagogic Example: QED with massless scalar
- also prep for nonlocal eff action later

Start from: $\mathcal{L} = (D_\mu \phi)^* D^\mu \phi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$

$A_\mu = $ B.F.

$D_\mu = \partial_\mu + ie A_\mu$

Integrate by parts
$\mathcal{L} = - \phi^* D_\mu D^\mu \phi = - \phi^* \mathcal{O} \phi$

$D_\mu D^\mu = \Box + 2ie A_\mu \partial_\mu + ie(\partial_\mu A^\mu) - e^2 A_\mu A_\mu$

$\equiv \Box + V(A)$

Do Path Integral

– keeping $A_\mu(x)$ as background field

$$\int d\phi\, d\phi^* \, e^{-i\int d^4x\, \phi^* \mathcal{O} \phi} = \frac{N}{\det \mathcal{O}} = N e^{-\mathrm{Tr} \ln \mathcal{O}}$$

A

$$= N e^{-\int d^4x \langle x | \mathrm{Tr} \ln \mathcal{O} | x \rangle}$$

Perturbative evaluation

e, e^2

$$\mathrm{Tr} \ln \mathcal{O} = \mathrm{Tr} \ln (\Box + v)$$

$$= \mathrm{Tr} \ln \left[\Box \left(1 + \frac{1}{\Box} v \right) \right] = \mathrm{Tr} \ln \Box + \mathrm{Tr} \ln \left(1 + \frac{1}{\Box} v \right)$$

$$= \underbrace{\mathrm{Tr} \ln \Box}_{\text{drop}} + \mathrm{Tr} \left(\frac{1}{\Box} v - \frac{1}{2} \frac{1}{\Box} v \frac{1}{\Box} v + \cdots \right)$$

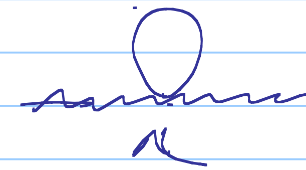
These are calculable:

Use $\langle x | \frac{1}{\Box} | y \rangle = i\Delta_F(x-y)$

Then 1st order:

$$\int d^4x \, \langle x | \mathrm{Tr}\left(\frac{1}{\Box} V\right) | x \rangle = \int d^4x \; i\Delta_F(x-x)\, V(x)$$

A^2

 $= 0$

$$\Delta_F(0) = \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2} \to 0 \text{ in dim reg.}$$

Second order

$$\frac{1}{2} \mathrm{Tr}\left(\frac{1}{\Box} V \frac{1}{\Box} V\right) = \frac{1}{2} \int d^4x \, d^4y \; i\Delta_F(x-y)\, V(y)\; i\Delta_F(y-x)\, V(x)$$

Calculation

- some int. by parts - isolate $A_\mu(x)\, M^{\mu\nu}(x-y)\, A_\nu(y)$

- some trick $\Delta_F(x)\, \partial_\mu \partial_\nu\, \Delta_F(x) = \left(d\, \partial_\mu \partial_\nu - g_{\mu\nu} \Box\right) \dfrac{\Delta_F^2(x)}{4(d-1)}$

Result

$$\text{~}\mathcal{O}\text{~} = \frac{e^2}{2} \int d^4x\, d^4y\, A_\mu(x)\, M_{\mu\nu}(x-y)\, A_\nu(y)$$

$$M_{\mu\nu} = \left(g_{\mu\nu}\Box - \partial_\mu \partial_\nu\right) \frac{\Delta_F^2(x-y)}{4(d-1)}$$

Then more int. by parts

$$\Delta\mathcal{L} = -i \int d^4x\, d^4y\, F_{\mu\nu}(x)\, \frac{\Delta_F^2(x-y)}{4(d-1)}\, F^{\mu\nu}(y)$$

To evaluate this

$$D^2(q) \equiv \int d^4x \, e^{iq\cdot x} \, \Delta_F^2(x)$$

$$= \frac{-i}{16\pi^2}\left[\frac{2}{4-d} - \gamma + \ln 4\pi - \ln q^2/\mu^2\right]$$

$$\Delta_F^2(x-y) = F.T.\left[\frac{1}{\varepsilon} - \ln q^2/\mu^2\right]$$

$$= \underbrace{\frac{-i}{16\pi^2}\left[\frac{1}{\varepsilon} - \dots\right]\delta^4(x-y)}_{\text{Divergences are } \underline{\text{local}}} + \underbrace{\frac{1}{16\pi^2} L(x-y)}_{\text{Finite, not local}}$$

F.T. of $\ln q^2/\mu^2$

One loop effective action

$$S = S_0 + \Delta S = \int d^4x \; -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} Z_3^{-1}$$

$$-\frac{1}{4} F^2$$

$$+ b e^2 \int d^4x \, d^4y \; F_{\mu\nu}(x) \, L(x-y) \, F^{\mu\nu}(y)$$

β function

Generalize

$$\mathcal{L} = \phi^* \left[d_\mu d^\mu + \sigma(x) \right] \phi \qquad \phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \vdots \end{pmatrix}$$

$$d_\mu = \partial_\mu + \Gamma_\mu(x) \quad \text{— matrix}$$

$$\Delta S = \int d^4x\, d^4y\, \mathrm{Tr}\left[\Gamma_{\mu\nu}(x) \frac{\Delta_F^2(x-y)}{4(d-1)} \Gamma^{\mu\nu} + \frac{1}{2} \sigma(x) \Delta_F^2(x-y) \sigma(y) \right]$$

with $\Gamma_{\mu\nu} = \partial_\mu \Gamma_\nu - \partial_\nu \Gamma_\mu + [\Gamma_\mu, \Gamma_\nu]$

Divergences are local:

$$\Delta S_{div} = \int d^4x \frac{1}{16\pi^2}\left[\frac{1}{\varepsilon} + \ldots\right] \mathrm{Tr}\left[\frac{1}{12} \Gamma_{\mu\nu} \Gamma^{\mu\nu} + \frac{1}{2}\sigma^2(x)\right] \checkmark$$

Advantages B.F.

- can deal with actions
- can retain symmetries easier
- renorm of nonlinear theories is easy
- scattering amplitude - many at once

Note: can be a single field $\phi = \bar{\phi} + \delta\phi$

(see DSM App B on course web page)

G+L

Background Field GR

't Hooft Veltman

Expand $g_{\mu\nu} = \bar{g}_{\mu\nu} + \kappa h_{\mu\nu}$ (exact)

background field

quantum piece

$$g^{\mu\nu} = \bar{g}^{\mu\nu} - \kappa h^{\mu\nu} + \kappa^2 h^{\mu\lambda} h_\lambda{}^\nu + \cdots$$

Raise and lower indices with $\bar{g}$

Now need to expand:

$$\Gamma_{\mu\nu}{}^\lambda = \bar{\Gamma}_{\mu\nu}{}^\lambda + \underbrace{\Gamma_{\mu\nu}{}^\lambda}_{\mathcal{O}(h)} + \underbrace{\underline{\underline{\Gamma_{\mu\nu}{}^\lambda}}}_{\mathcal{O}(h^2)}$$

$$R = \bar{R} + \underline{R} + \underline{\underline{R}}$$

$$\Gamma_{\mu\nu}{}^{\lambda} = \frac{1}{2} g^{\lambda\sigma}\left[\partial_{\mu} g_{\sigma\nu} + \cdots\right.$$

Expansions are straightforward

Ex $$\underline{\Gamma}_{\mu\nu}{}^{\lambda} = \frac{1}{2}\bar{g}^{\lambda\sigma}\left[\partial_{\mu} h_{\nu\sigma} + \partial_{\nu} h_{\mu\sigma} - \partial_{\sigma} h_{\mu\nu}\right] - \frac{1}{2} h^{\lambda\sigma}\underbrace{\left[\partial_{\mu}\bar{g}_{\sigma\sigma} + \partial_{\nu}\bar{g}_{\mu\sigma} - \partial_{\sigma}\bar{g}_{\mu\nu}\right]}_{\propto \bar{\Gamma}_{\mu\nu}{}^{\lambda}}$$

$$= \frac{1}{2}\underline{\bar{g}}^{\lambda\sigma}\left[\underline{\bar{D}_{\mu} h_{\sigma\nu} + \bar{D}_{\nu} h_{\mu\sigma} - \bar{D}_{\sigma} h_{\mu\nu}}\right]$$

$\leftarrow$ covariant w.r.t $\bar{g}$

Important point:

- all terms are covariant w.r.t $\bar{g}$ **
- displays gauge inv. of formalism at each step

Gauge invariance in BF

$$x^\mu \to x'^\mu = x^\mu + \xi^\mu(x)$$
$$dx'^\mu = (\delta^\mu_\nu + \partial_\nu \xi^\mu)\, dx^\nu$$

Now $g_{\mu\nu}(x) \to J^{-1\alpha}_{\mu}\, g_{\alpha\beta}(x')\, J^{-1\beta}_{\nu} = g_{\mu\nu}(x') - g_{\alpha\nu}\partial_\mu \xi^\alpha - g_{\mu\alpha}\partial_\nu \xi^\alpha$

$g_{\mu\nu}(x') = g_{\mu\nu}(x) + \xi^\alpha \partial_\alpha g_{\mu\nu}(x)$

So we have

$$g'_{\mu\nu} = g_{\mu\nu} - g_{\alpha\nu}\partial_\mu \xi^\alpha - g_{\alpha\mu}\partial_\nu \xi^\alpha + \xi^\alpha \partial_\alpha g_{\mu\nu}$$

Now plug in $g_{\mu\nu} = \bar{g}_{\mu\nu} + \kappa h_{\mu\nu}$ + expand

$$h_{\mu\nu} \to h_{\mu\nu} + \bar{D}_\mu \xi_\nu + \bar{D}_\nu \xi_\mu$$

covariant deriv.

Result

$$\mathcal{L} = -\frac{2}{\kappa^2}\sqrt{-g}R = \sqrt{-g}\Big[-\frac{2}{\kappa^2}\bar{R} - \frac{1}{\kappa}\left[h^\alpha{}_\alpha \bar{R} - 2\bar{R}^\alpha{}_\nu h^\nu{}_\alpha\right]$$

$$+\frac{1}{2}\bar{D}_\alpha h_{\mu\nu}\bar{D}^\alpha h^{\mu\nu} - \frac{1}{2}\bar{D}_\alpha h^\lambda{}_\lambda \bar{D}^\alpha h^\sigma{}_\sigma$$

$$+\bar{D}_\nu h^\lambda{}_\lambda D^\beta h^\nu_\beta - \bar{D}_\nu h_{\alpha\beta}\bar{D}^\alpha h^{\nu\beta}$$

$$-\bar{R}\left[\frac{1}{4}(h^\lambda{}_\lambda)^2 - \frac{1}{2}h^\alpha{}_\beta h^\beta{}_\alpha\right] + h^\lambda{}_\lambda h^\alpha{}_\nu \bar{R}_\alpha{}^\nu + 2h^\nu{}_\beta h^\beta{}_\alpha \bar{R}^\alpha{}_\nu\Big]$$

Comments

1) First order term vanishes by E.O.M. $\frac{2}{K} h^{\alpha\beta}\left[R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R\right] = 0$

2) $h_{\mu\nu}$ is now just an ordinary field

Gauge fixing

Weak field: $\partial^\mu h_{\mu\nu} - \frac{1}{2}\partial_\nu h^\lambda{}_\lambda = 0$

add to Lagrangian $\mathcal{L} = \ldots + \frac{1}{2}\left[\quad\quad\quad\right]^2$

Generalization of this would be

$$C_\nu = \left[\bar{D}^\mu h_{\mu\nu} - \frac{1}{2}\bar{D}_\nu h^\lambda{}_\lambda\right] \longrightarrow \frac{1}{2} C_\nu C^\nu$$

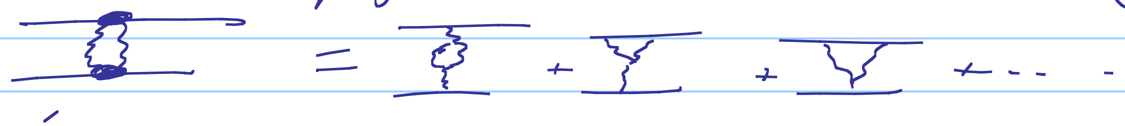
using background field $\bar{g}_{\mu\nu}$ everywhere

$\Longrightarrow$ covariant wrt $\bar{g}$

Faddeev Popov overview (for some)

Feynman first did this for QG
- 2 DOF physical
- but more in loops

** can reconstruct loops from trees (Feynman tree theorem)



onshell
Transverse
phys. DOF

But for YM + QG get wrong result!

- introduce "artificial, dopey particle" to get right answer

Feynman (1963) exerpts

This made me investigate the entire subject in great detail to find out what the trouble is. I discovered in the process two things. First, I discovered a number of theorems, which as far as I know are new, which relate closed loop diagrams and diagrams without closed loop diagrams (I shall call the latter diagrams "trees"). The unitarity relation which I have just been describing, is one connection between a closed loop diagram and a tree; but I found a whole lot of other ones, and this gives me more tests on my machinery. So let me just tell you a little bit about this theorem, which gives other rules. It is rather interesting. As a matter of fact, I proved that if you have a diagram with rings in it there are enough theorems altogether, so that you can express any diagram with circuits completely in terms of diagrams with trees and with all momenta for tree diagrams in physically attainable regions and on the mass shell. The demonstration is remarkably easy. There are several ways of demonstra-

There is another theory, more well-known to meson physicists, called the Yang-Mills theory, and I take the one with zero mass; it is a special theory that has never been investigated in great detail. It is very analogous to gravitation; instead of the coordinate transformation group being the source of everything, it's the isotopic spin rotation group that's the source of everything. It is a non-linear theory, that's like the gravitation theory, and so forth. At the suggestion of Gell-Mann I looked at the theory of Yang-Mills with zero mass, which has a kind of gauge group and everything the same; and found exactly the same difficulty. And therefore in meson theory it was not strictly unknown difficulty, because it should have been noticed by meson physicists who had been fooling around the Yang-Mills theory. They had not noticed it because they're practical, and the Yang-Mills theory with zero mass obviously does not exist, because a zero mass field would be obvious; it would come out of nuclei right away. So they didn't take the case of zero mass and investigate it carefully. But this disease which I discovered here is a disease which exist in other theories. So at least there is one good thing: gravity isn't alone in this difficulty. This observation that

Well, what then, now you have the difficulty; how do you cure it? Well I tried the following idea: I assumed the tree theorem to be true, and used it in reverse. If every closed ring diagram can be expressed as trees, and if trees produce no trouble and can be computed, then all you have to do is to say that the closed loop diagram is the sum of the corresponding tree diagrams, that it should be. Finally in each tree diagram for which a graviton line has been opened, take only real transverse graviton to represent that term. This then serves as the definition of how to calculate closed-loop diagrams; the old rules, involving a propagator $1/k^2+i\varepsilon$ *etc.* being superseded. The advantage of this is, first, that it will be gauge invariant, second, it will be unitary, because unitarity is a relation between a closed diagram and an open one, and is one of the class of relations I was talking about, so there's no difficulty when I made it gauge invariant. But then secondly, you must subtract from the answer, the result that you get by imagining that in the ring which involves only a graviton going around, instead you calculate with a different particle going around, an artificial, dopey particle is coupled to it. It's a vector particle, artificially coupled to the external field, so designed as to correct the error in this one. The forms are evidently invariant,

fermion

DeWitt: Because of the interest of the tricky extra particle that you mentioned at the end, and its possible connection, perhaps, with some work of Dr Białynicki-Birula, have you got far enough on that so that you could repeat it with just a little more detail? The structure of it and what sort of an equation it satisfies, and what is its propagator? These are technical points, but they have an interest.

Feynman: Give me ten minutes. And let me show how the analysis of these tree diagrams, loop diagrams and all this other stuff is done mathematical way. Now I will show you that I too can write equations that nobody can understand. Before I do that I should

DeWitt (GR) and Fadeev-Popov (YM) formalized this

Logic:

Insert unity in a clever way

- a gauge condition and all gaauge copies

all give same result

like $\int dx^i \, \delta(f_j(x)) \det\left(\frac{\partial f_j}{\partial x_i}\right) = 1$

- integrate over gauge copies

- integrate constraint

- replace $\det\left|\frac{\partial f_j}{\partial x_i}\right|$ by [illegible] over ghost fields

<u>More detail</u> Feynman - Dewitt - Fadeev - Popov ghosts

QED $A^\theta_\mu = A_\mu + \partial_\mu \theta$

$$\text{add } 1 = \int [d\theta(x)]\, \delta(F(A^\theta_\mu) - F)\, \Delta(A) \qquad \det\left(\frac{\partial f}{\partial \theta}\right)\Big|$$

in general $\Delta(A)$ is independent of θ, can factor out $\underline{[d\theta]}$

can choose $F = F(x)$ + exponentiate

$$\int dF(x)\, \delta(f(A) - F(x))\, e^{-\frac{i}{2\xi}\int d^4x\, F^2(x)} = e^{-i\int d^4x \frac{\xi}{2} f^2(A)}$$

gauge constraint in action

Evaluate FP determinant with ghost field

$$\det\left|\frac{\partial f}{\partial \theta}\right| = \int dc\, d\bar{c}\, e^{i\int d^4x\, \bar{c}\, \frac{\partial f}{\partial \theta} c(x)}$$

QED $\frac{\partial f}{\partial \theta} \sim$ ind of A

Gravity

$$C_\mu = \left[\bar{D}^\mu h_{\mu\nu} - \frac{1}{2}\bar{D}_\nu h^\lambda{}_\lambda\right]$$

add $\quad \Delta(h)\,\delta(C_\nu - F_\nu)$

exponentiate

$$\int [dF_\nu]\, e^{i\int d^4x\, F_\nu F^\nu}\,\delta(C-F) = e^{i\int d^4x\, \mathcal{L}_{gf}}$$

with

$$\mathcal{L}_{gf} = \frac{1}{2} C_\nu C^\nu$$

such that

$$\mathcal{L} = -\frac{2}{\kappa^2}\underline{\underline{R}} + \frac{1}{2}\left[\bar{D}^\mu h_{\mu\nu} - \frac{1}{2}\bar{D}_\nu h^\lambda{}_\lambda\right]^2$$

Ghosts

$$C_\nu = \bar{D}^\mu h_{\mu\nu} - \frac{1}{2}\bar{D}_\nu h^\lambda{}_\lambda$$

Gauge trans: $h_{\mu\nu} \longrightarrow h_{\mu\nu} + \bar{D}_\nu \xi_\mu + \bar{D}_\mu \xi_\nu$

$$\delta C_\mu = \bar{D}^\mu(\bar{D}_\nu \xi_\mu + \bar{D}_\mu \xi_\nu) - \frac{2}{2}\bar{D}_\nu \bar{D}_\mu \xi^\mu$$

$$= \bar{D}^\mu \bar{D}_\mu \xi_\nu - [\bar{D}_\nu, \bar{D}_\mu]\xi^\mu$$

$$= (\bar{g}_{\mu\nu}\bar{D}^2 + \bar{R}_{\nu\mu})\xi^\nu$$

$$\frac{\partial f}{\partial \sigma}$$

F.P. Determinant

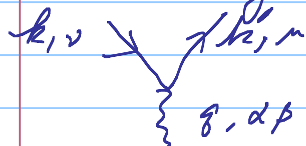
$$\Delta(h) = \det \frac{\partial G_\nu}{\partial \xi_\alpha} = \det\left(\bar g_{\nu\alpha} \bar D^2 + \bar R_{\nu\alpha}\right)$$

Ghost fields η^μ ← fermionic vector

Then

$$\mathcal{L}_{ghost} = \bar\eta^\nu \left[\bar g_{\nu\alpha} \bar D^2 + \bar R_{\nu\alpha} \right] \eta^\alpha$$

New Feynman rule: (near flat space



$$= -i\left[k_\alpha k_\beta \eta_{\mu\nu} + q\cdot k\, \eta_{\mu\alpha} \eta_{\nu\beta} + q_\mu k_\beta \eta_{\nu\alpha} - q_\nu k_\beta \eta_{\mu\alpha} + q^2 \eta_{\mu\alpha} \eta_{\nu\beta} \right]$$

↖ exercise

Summary

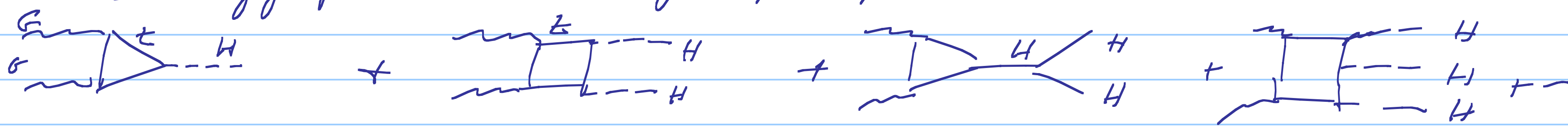
$$Z = \int [dh_{\mu\nu}][d\eta][d\bar{\eta}]\, e^{i\int d^4x \sqrt{g}\, [\underline{\underline{\mathcal{L}}}_G(h) + \mathcal{L}_{gf}(h) + \mathcal{L}(\eta,\bar{\eta},h)]}$$

$$\underline{\underline{\mathcal{L}_G}} + \mathcal{L}_{gf} = \frac{1}{2}\bar{D}_\alpha \underline{h_{\mu\nu}}\, \bar{D}^\alpha h^{\mu\nu}$$

$\mathcal{L}_{gh}$ = above

With matter add $\int [d\phi]$

Problem Higgs production through top loop



Use the background field method to show that the top quark loop (for $M_H \ll M_t$ as an approximation) results in:

$$\mathcal{L}_{eff} = \frac{\alpha_s}{24\pi} \ln\left(\frac{v + H}{v}\right) F^{a\mu\nu} F^{a}_{\mu\nu}$$

and show that $GG \to HH$ vanishes exactly at threshold $s = (2M_H)^2$
(in this approx.)

Hint: $\mathcal{L} = \frac{\Gamma_t}{\sqrt{2}}(v + H)\bar{t}t = m_t(H)\bar{t}t$, thus