

# EPFL Lectures

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Note Title

8/10/2016

## Heat Kernel Methods

- alternative to Feynman diagrams calculation in some settings

$$K(x, y, \tilde{z}, \mathcal{D}) = \langle x | e^{-\tilde{z}\mathcal{D}} | y \rangle$$

coeff of heat  
of diffusion

satisfies  $(\partial_{\tilde{z}} + \mathcal{D}) K(x, y, \tilde{z}, \mathcal{D}) = 0$

| if  $\mathcal{D} = -\alpha \nabla^2$

with  $K(x, y, 0, \mathcal{D}) = \delta^d(x-y)$

|  $\Rightarrow$  heat eq G.F.

### History:

- 1800's Math Literature

- Fock, Schwinger - QFT  $\Rightarrow$  Schwinger proper time

- Gravity - DeWitt (Seeley, Gilkey (1975) ...)

↓

Some uses:

1) If  $D = \square + m^2 - i\epsilon$

$$K(x, y, t, \bar{t}, \square + m^2) = \frac{1}{(4\pi)^3} e^{-i\int_{\bar{t}}^t ds \frac{(x-y)^2}{4s} + i\int_s^t ds m^2}$$

(see  
M. Schwartz  
QFT Book)

2) Propagators

$$\frac{i}{A+i\varepsilon} = \int_0^\infty d\tau e^{i\tau(A+i\varepsilon)}$$
$$\Rightarrow iD_F(x-y) = \langle x | \frac{i}{D+m^2+i\varepsilon} | y \rangle = -i \int_0^\infty \frac{d\tau}{16\pi^2 \tau^2} e^{i\int_0^\tau ds \frac{(x-y)^2}{4s} + i\int_0^\tau ds (m^2+i\varepsilon)}$$
$$= -\frac{1}{4\pi} \frac{1}{(x-y)^2 - i\varepsilon} \quad \text{if } m=0$$

### 3) Path integral determinant

$$\text{Use } \ln \frac{b}{a} = \int_0^\infty \frac{d\tilde{\tau}}{\tilde{\tau}} \left( e^{-\tilde{\tau}a} - e^{-\tilde{\tau}b} \right)$$

Then  $\text{Tr } \ln D = -\text{Tr}' \int_{\tilde{\tau}}^{\text{remaining}}$

$$= -\text{Tr}' \int_{\tilde{\tau}} d\tilde{\tau} \times \int d^4x \langle x | e^{-\tilde{\tau}D} | x \rangle + C$$

$\nwarrow K(x, \tilde{\tau})$

### 4) Short distance expansion

$$K(x, \tilde{\tau}) = \frac{i}{(4\pi)^2 k} \frac{e^{-\tilde{\tau}m^2}}{\tilde{\tau} dx} \left[ a_0 + a_1 \tilde{\tau} + a_2 \tilde{\tau}^2 \right]$$

$f(\phi_{BE})$

then

$$\langle N | \text{Tr}' \ln \mathcal{D}|N\rangle = -\frac{i}{(4\pi)} \text{Tr} \sum_{m=0}^{\infty} m^{d-2m} \Gamma(\eta - \frac{d}{2}) \text{Tr}' a_m(x)$$

$\uparrow \eta = 2$        $\uparrow a_2(x) \leftarrow \cancel{\infty}$

## Derivation

from DSM  
(on website)

$$D = \partial_m d^m + T(x)$$

$$d_m = \partial_m \neq P_m(x)$$

Insert complete set of momentum states  $\int \frac{e^{-ip \cdot x}}{(2\pi)^d} dp$

$$\langle x | e^{-iD} | x \rangle = \int d_p \langle x | p \rangle e^{-iD} \langle p | x \rangle$$

$$\langle p | x \rangle = \frac{1}{(2\pi)^{d/2}} e^{ip \cdot x},$$

$$\langle x | x' \rangle = \int \frac{d^d p}{(2\pi)^d} e^{ip \cdot (x' - x)} = \delta^{(d)}(x - x'),$$

$$\langle p' | p \rangle = \int \frac{d^d x}{(2\pi)^d} e^{i(p' - p) \cdot x} = \delta^{(d)}(p' - p)$$

$$d_\mu e^{ip \cdot x} = e^{ip \cdot x} (ip_\mu + d_\mu) ,$$

Use       $d_\mu d^\mu e^{ip \cdot x} = e^{ip \cdot x} (i\bar{p}_\mu + d_\mu)(ip^\mu + d^\mu) ,$

we can then write

$$\begin{aligned} H(x, \tau) &= \int \frac{d^d p}{(2\pi)^d} e^{-\tau[(ip_\mu + d_\mu)^2 + m^2 + \sigma]} \\ &= \int \frac{d^d p}{(2\pi)^d} e^{\tau[p^2 - m^2]} e^{-\tau[d \cdot d + \sigma + 2ip \cdot d]} . \end{aligned} \quad (1.14)$$

The first exponential factor is simply the free field result, while all the interesting physics is in the second exponential. The latter can be Taylor expanded in powers of  $\tau$ , keeping those terms which contribute up to order  $\tau^2$  after integration over momentum. Note that each power of  $p^2$  contributes a factor of  $1/\tau$ . Thus we obtain the expansion

$$\begin{aligned} H(x, \tau) &= \int \frac{d^d p}{(2\pi)^d} e^{\tau(p^2 - m^2)} \left[ 1 - \tau[d \cdot d + \sigma] \right. \\ &\quad + \frac{\tau^2}{2} [(d \cdot d + \sigma)(d \cdot d + \sigma) - 4p \cdot d p \cdot d] \\ &\quad + \frac{4\tau^3}{3!} [p \cdot d p \cdot d (d \cdot d + \sigma) + p \cdot d (d \cdot d + \sigma) p \cdot d \\ &\quad + (d \cdot d + \sigma)p \cdot d p \cdot d] \\ &\quad \left. + \frac{16\tau^4}{4!} p \cdot d p \cdot d p \cdot d p \cdot d + \dots \right] , \end{aligned} \quad (1.15)$$

To evaluate, go Euclidean

With the replacement  $p_\mu p^\mu \rightarrow -|p_E^\mu p_E^\mu| = -p_E^2$ , we obtain

$$\begin{aligned} \int \frac{d^d p_E}{(2\pi)^d} e^{-(p_E^2 + m^2)\tau} &= \int \frac{d\Omega_d}{(2\pi)^d} \int dp_E p_E^{d-1} e^{-(p_E^2 + m^2)\tau} \\ &= \frac{2\pi^{d/2}}{\Gamma(d/2)} \frac{1}{(2\pi)^d} \frac{e^{-m^2\tau}\Gamma(d/2)}{2\tau^{d/2}} \\ &= \frac{1}{(4\pi)^{d/2}} \frac{e^{-m^2\tau}}{\tau^{d/2}}, \end{aligned}$$

$$\begin{aligned} \int \frac{d^d p_E}{(2\pi)^d} e^{-(p_E^2 + m^2)\tau} p_E^\mu p_E^\nu &= \frac{\delta^{\mu\nu}}{d} \frac{1}{(4\pi)^{d/2}} \frac{e^{-m^2\tau}}{\tau^{d/2+1}} \frac{\Gamma(d/2+1)}{\Gamma(d/2)} \quad (1.16) \\ &= \frac{\delta^{\mu\nu}}{2} \frac{e^{-m^2\tau}}{(4\pi)^{d/2}\tau^{d/2+1}}, \end{aligned}$$

$$\begin{aligned} \int \frac{d^d p_E}{(2\pi)^d} e^{-(p_E^2 + m^2)\tau} p_E^\mu p_E^\nu p_E^\lambda p_E^\sigma &= \frac{e^{-m^2\tau}}{(4\pi)^{d/2}\tau^{d/2+2}} \\ &\times \frac{(\delta^{\mu\nu}\delta^{\lambda\sigma} + \delta^{\mu\lambda}\delta^{\nu\sigma} + \delta^{\mu\sigma}\delta^{\lambda\nu})}{4} \end{aligned}$$

Result:

Employing these relations to evaluate Eq. (1.14) gives (to second order in  $\tau$ ),

$$H(x, \tau) = \frac{ie^{-m^2\tau}}{(4\pi)^{d/2}\tau^{d/2}} \times \left[ 1 - \tau\sigma + \tau^2 \left( \frac{1}{2}\sigma^2 + \frac{1}{12}[d_\mu, d_\nu][d^\mu, d^\nu] + \frac{1}{6}[d_\mu, [d^\mu, \sigma]] \right) \right], \quad (1.17)$$

or in the notation of Eq. (1.3),

$$\begin{aligned} a_0(x) &= 1 , & a_1(x) &= -\sigma , \\ a_2(x) &= \frac{1}{2}\sigma^2 + \frac{1}{12}[d_\mu, d_\nu][d^\mu, d^\nu] + \frac{1}{6}[d_\mu, [d^\mu, \sigma]] . \end{aligned} \quad (1.18)$$

Example: QED <sup>Scalar</sup>  $d_\mu = \partial_\mu + e \mathbf{A}_\mu$

$$[d_\mu, d_\nu] = ie F_{\mu\nu}$$

$$\alpha_2 = \frac{1}{12} F^2$$

Get renormalization constant

$$\Delta Z = S d_N^4 F_\mu F^\mu \frac{e^2}{16\pi} \frac{1}{12} P(2 - d_{12})$$

For gravity - Seeley DeWitt coeff. - calculated (Barcel  
+ Davies  
+ ...)

Scalar field in loop  $\leftarrow g^{\mu\nu}$   $\beta = 0$  minimally coupled

$$\mathcal{L} = \partial_\mu \phi \partial^\mu \phi + m^2 \phi^2 + \frac{3}{2} R \phi^2$$

$$a_1 = \left( \frac{1}{6} - \beta \right) R$$

$$a_2 = \frac{1}{180} \left( R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - R_{\mu}^{\nu} R^{\mu\nu} + \frac{5}{2} (6\beta - 1)^2 R^2 - 6 R_{\mu}^{\nu} R^{\mu\nu} \right)$$

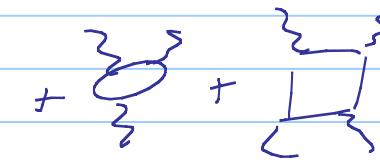
Recall previous "calculation" in our

$$M = \frac{\pi^2}{16\pi^2} \frac{1}{\epsilon} K^2 \left( g_{\mu} g_{\nu} g_{\alpha} g_{\beta} \dots \right)$$

$$\Rightarrow \Delta \mathcal{L} = \frac{\pi^2}{16\pi^2} \frac{1}{\epsilon} \partial_{\mu} \partial_{\nu} K \partial_{\alpha} \partial_{\beta} K$$

Heat kernel gives the full result

Comments:

- 1) Really easy
- 2) All diagrams  $\sim \text{O}(\alpha)$  +  +  + ... div. parts
- 3) Makes general covariance obvious
- 4) Will be used to renormalize loops in GR ✓
- 5) does not capture  $\ln g^2$  effects ]

## Gauss Bonnet

$$\text{in } 4d \quad G = R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} - 4 R_{\mu\nu} R^{\mu\nu} + R^2 = D_\mu J^\mu$$

$$J^\mu =$$

↙

$$\text{and } \frac{1}{8\pi^2} \cdot \int d^4x \sqrt{-g} G = \chi_K \text{ Euler characteristic}$$

Implications 1)  $G$  is surface integral

2) does not influence E of M or matrix elements

3)  $\int d^4x \sqrt{-g} R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta}$  can be rewritten in terms of  $\overbrace{R_{\mu\nu}}^2, \overbrace{R^2}$

Scalar field

$$S_{\text{div}} = \int d^4x F_T - \frac{1}{\varepsilon} \frac{1}{180} \frac{1}{16\pi^2} \left[ 3(R_{\mu\nu} R^{\mu\nu} - \frac{1}{3} R^2) + \frac{5}{2} (G_3 - 1) \bar{R}^2 \right]$$

Perturbative matching

$$\frac{1}{\varepsilon} \left[ g_\mu g_\nu g_\lambda g_\sigma + \dots \right] \longleftrightarrow \left[ a R_{\mu\nu} R^{\mu\nu} + b R^2 \right]$$

# One loop gravity

Pert: +

$\text{t Hooft-Veltman}$

## Heat kernel

$$a_2^{\text{gravity}} = \frac{215}{180} R^2 - \frac{36}{90} R_{\mu\nu} R^{\mu\nu} + \frac{53}{45} R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}$$

$$= \frac{1}{120} R^2 + \frac{7}{20} R_{\mu\nu} R^{\mu\nu}$$

~~and~~

## Divergence

$$S_{\text{div}} = \frac{1}{16\pi^2} \left[ \frac{1}{\epsilon} + \dots \right] \left[ \begin{array}{c} \curvearrowleft \\ \curvearrowright \end{array} \right] -$$

g. g. f. g

Pure gravity is one loop finite

LHV

$$\text{Pure gravity } R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0 \Rightarrow R_{\mu\nu} = 0$$

OK to use eq of motion in  $S_{\text{dfr}}$

— external states satisfy lowest order E of M

$$S_{\text{dfr}} = 0$$

Physically, No UV divergences in absence of matter



$\Rightarrow$  UV finite

But, matter does exist

$\Rightarrow$  world is not 1 loop finite

Two loop divergencies do exist in pure gravity

Geroff Sagnotti

$$\Delta I_{\text{div}} = \frac{209}{2880} \frac{K^2}{(16\pi^2)^2} \left[ \frac{1}{\epsilon} + \dots \right] R^{\mu\nu\rho\sigma} R_{\rho\sigma}^{\alpha\beta} R_{\alpha\beta}^{\delta\gamma}$$

Observe . started with  $R \sim \lambda^2$

one loop  $R^2 \sim \lambda^4$

two loops  $R^3 \sim \lambda^6$

## Summary

- Feynman DeWolfe - gauge fixing
- 1970's pert. theory
- 't Hooft Veltman definitive treatment back ground field, heat kernel

## Bad reputation for quantum gravity

- 1) old style quantization
  - 2) fuzzy thinking on space-time
  - 3) nonrenormalizability
  - 4) Infinites
  - 5) Predictability
  - 6) final theory = ?
  - 7) Non-perturbative, information loss
- } cured by path integral  
}  $\Rightarrow$  See EFT  
} open