

EPFL Lectures

Note Title

2016

- 1) Anomalies in Gravity
- 2) Non local Effective Actions - quantum effects non local
- 3) Beyond Scattering Amplitudes - full curvature effects

Anomalies

- 1) Scale - trace anomaly]
- 2) Conformal
- 3) Axial (not here)

4 Ways to treat

- 1) Feynman diagrams UV
- 2) Path integral UV ←
- 3) Non local effective actions IR ←
- 4) Dispersion relations IR

Scale Symmetry and Trace Anomaly

$$x = \lambda x'$$

$$A_\mu(x) = \lambda^{-1} A_\mu(\lambda x)$$

$$\psi(x) = \lambda^{\frac{3}{2}} \psi(\lambda x)$$

Then

$$\begin{aligned} S &= \int d^4x \left[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi} (iD - m) \psi \right] \\ &= \int d^4x' \lambda^4 \left[-\frac{1}{4\lambda^2} F_{\mu\nu} F^{\mu\nu} + \bar{\psi}' \left(\frac{iD}{\lambda^4} - \frac{m}{\lambda^3} \right) \psi' \right] \end{aligned}$$

invariant if $m=0$

Conserved current: *virial*

$$\tilde{J}_{\text{scal}}^{\mu} = \Theta^{\mu\nu} X_{\nu} - j^{\mu}$$

Noether

E + M

$$\tilde{J}_{\text{scal}}^{\mu} = \underline{\Theta^{\mu\nu}} X_{\nu} - \underline{F^{\mu\nu}} A_{\nu}$$

But ambiguity in $\Theta^{\mu\nu}$ - can use Belinfante $T^{\mu\nu}$ - symmetric

$$\tilde{J}_{\text{scal}}^{\mu} = \underline{T^{\mu\nu}} X_{\nu}$$

$$T_{\mu\nu} = -F_{\mu\lambda} F^{\lambda}_{\nu} + \frac{i}{4} g_{\mu\nu} \epsilon F^{\lambda\sigma} F_{\lambda\sigma}$$

$$\partial_{\mu} \tilde{J}_{\text{scal}}^{\mu} = \overline{T}_{\mu\mu} = 0$$

Path Integral measure

Fujikawa : Anomaly = \mathcal{L} invariant but measure not

Jacobian

$$\Psi(x) = e^{-\alpha b_k} \Psi'(x)$$

$$\bar{\Psi}(x) = \bar{e}^{-\alpha c_k} \bar{\Psi}'(x)$$

$$\text{Then } \int d\Psi [d\bar{\Psi}] = \int d\Psi' [d\bar{\Psi}'] \det[e^{-\alpha}]$$

Regularization

$$\det[e^{-\alpha}] = \det \exp \left[-2 \pi \frac{(\bar{x}/M)^2}{M} \right] \xrightarrow{M \rightarrow \infty}$$

$$= e^{\text{Tr} \ln e^{-\frac{(\bar{x})^2}{M}}} = e^{-S d\bar{x} \ln \langle x | x | e^{(\bar{x}/M)^2} | x \rangle}$$

Use Heat kernel

$$H(x, \tau) = \langle x | e^{-\tau \hat{D}} | x \rangle$$

$$H(x, \tau) = \frac{i \ell}{(4\pi)^2 \tau} \frac{1}{\tau^{1/4}} \left[1 - \tau \Gamma + \tau^{\frac{1}{2}} \left\{ \frac{1}{2} [d_n, d_s] [d^n, d^s] + \frac{1}{2} \sigma^2 \right\} + \dots \right]$$

after some work $[d_n, d_s] [d^n, d^s] \propto \underbrace{\dots}_{\sigma^2} \propto \underbrace{\dots}_{\sigma^2}$

$$\text{Tr} \langle x | e^{-\frac{(D_F)^2}{\tau}} | x \rangle = \underbrace{\frac{i M^4}{4\pi^2}}_{+} + \underbrace{\frac{i \ell^3}{16\pi^2} F_{\mu\nu} F^{\mu\nu}}$$

Trace anomaly via PT

(PBM)

$$\text{For fermion } \Theta^{\mu\nu} = i \frac{1}{2} \bar{\psi} \gamma^\mu \overleftrightarrow{D}^\nu \psi , \quad \Theta^{\mu\mu} = \bar{\psi} i \not{D} \psi$$

$$\text{and under } \psi \rightarrow e^{-\alpha/\hbar} \psi'$$

Then PI test

$$\begin{aligned} Z[h+\alpha] &= \int d\psi d\bar{\psi} e^{i S[\psi, \bar{\psi}] [L_D(\psi) + (h+\alpha)\Theta^{\mu\mu}]} \\ &= \int d\psi d\bar{\psi} e^{i S[\psi, \bar{\psi}] [\tilde{L}_D(\psi') + h\Theta^{\mu\mu}]} \\ &= \int d\psi' d\bar{\psi}' e^{i S[\psi', \bar{\psi}'] [\tilde{L}_D(\psi') + h\Theta^{\mu\mu}]} \end{aligned}$$

Match 1st + 3rd lines

$$\begin{aligned} i \int d^4x \alpha \Theta^{\mu\mu} &= \ln j = i \int d^4x \left[\alpha \frac{e^2}{12\pi^2} F_{\mu\nu} F^{\mu\nu} \right] \\ \Rightarrow \boxed{\Theta^{\mu\mu} = \frac{e^2}{12\pi^2} F_{\mu\nu} F^{\mu\nu}} \end{aligned}$$

Non local effective action + anomaly

- Is anomaly a robust prediction of low energy EFT ?

Derivation was UV effect

- EFT logic - Unknown physic in UV !

-

Anomalies are also IR effects

Recall

Background Field Method

Abbott paper on
web page

Pedagogic Example: QED with massless scalar
- also prep for nonlocal eff actions later

Start from:

$$\mathcal{L} = (D_\mu \phi)^* D^\mu \phi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

$$D_\mu = \partial_\mu + i e A_\mu$$

Integrate by parts

$$\mathcal{L} = - \phi^* D_\mu D^\mu \phi = - \phi^* \mathcal{O} \phi$$

$$D_\mu D^\mu = \square + 2ieA_\mu \partial^\mu + ie(\partial_\mu A^\mu) - e^2 A_\mu A^\mu$$

$$= \square + \mathcal{R}(x)$$

Recall

Do Path Integral

- keeping $A_\mu(x)$ as background field

$$\int d\phi d\phi^* e^{-i \int d^4x \phi^* \mathcal{O} \phi} = \frac{N}{\det \mathcal{O}} = N e^{-\text{Tr} \ln \mathcal{O}}$$
$$= N e^{-\int d^4x \langle x | \text{Tr} \ln \mathcal{O} | x \rangle}$$

Perturbative evaluation

$$\text{Tr} \ln \mathcal{O} = \text{Tr} \ln (\mathbb{I} + \mathcal{N})$$

$$= \text{Tr} \ln \left[\mathbb{I} \left(1 + \frac{1}{\mathbb{I}} \mathcal{N} \right) \right] = \text{Tr} \ln \mathbb{I} + \text{Tr} \ln \left(1 + \frac{1}{\mathbb{I}} \mathcal{N} \right)$$

$$= \text{Tr} \ln \mathbb{I} + \text{Tr} \left(\frac{1}{\mathbb{I}} \mathcal{N} - \frac{1}{2} \frac{1}{\mathbb{I}} \mathcal{N} \frac{1}{\mathbb{I}} \mathcal{N} + \dots \right)$$

Recall

These are calculable:

$$\text{Use } \langle x | \frac{1}{\square} | y \rangle = i \Delta_F(x-y)$$

Then 1st order:

$$\text{Tr} \left(\frac{1}{\square} N \right) = \int d^4x : \Delta_F(x-x) N(x)$$

$$\begin{aligned} & \text{and } \Delta_F(0) = \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2} \rightarrow 0 \text{ en dim reg} \\ & \text{so sum} \end{aligned}$$

Second order

$$\frac{1}{2} \text{Tr} \left(\frac{1}{\square} N \frac{1}{\square} N \right) = \frac{1}{2} \int d^4x d^4y : \Delta_F(x-y) N(y) : \Delta_F(y-x) N(x) \quad \text{no sum}$$

Recall

Calculation

- some int. by parts - isolate $A_\mu(x) M^{\mu\nu}(x-y) A_\nu(y)$

- some trick $\Delta_F(x) \partial_\mu \partial_\nu \Delta_F(x) = (d \partial_\mu \partial_\nu - g_{\mu\nu} \square) \frac{\Delta_F^2(x)}{4(d-1)}$

Result

$$-\partial_\mu = \frac{e^2}{2} \int d^dy A_\mu(x) M_{\mu\nu}(x-y) A_\nu(y)$$

$$\not{M}_{\mu\nu} = (g_{\mu\nu} \square - \partial_\mu \partial_\nu) \frac{\Delta_F^2(x-y)}{4(d-1)}$$

Then more int. by parts

$$\Delta Z = -ie^2 \int d^dx d^dy F_{\mu\nu}(x) \frac{\Delta_F^2(x-y)}{4(d-1)} F^{\mu\nu}(y)$$

Recall

To evaluate this

$$D^2(g) = \int d^4x e^{+ig \cdot A} \Delta_F^2(x)$$



$$= -\frac{i}{16\pi^2} \left[\frac{2}{4-\delta} - \delta + \ln 4\pi - \ln g^2/\mu^2 \right]$$



$$\Delta_F^2(x-y) = \int \frac{d^4q}{(2\pi)^4} e^{-iq \cdot (x-y)} \left(-\frac{i}{16\pi^2} \right) \left[\frac{1}{\epsilon} \dots - \ln g^2/\mu^2 \right]$$

$$= -i \frac{1}{16\pi^2} \left[\frac{1}{\epsilon} + \dots \right] \delta^4(x-y) + \frac{i}{16\pi^2} L(x-y)$$

FT of $\ln g^2/\mu^2$

↑ divergences are local

Recall

One loop effective action

$$1 + \frac{1}{12} \frac{1}{16\pi^2} \left[\frac{1}{\epsilon} + \dots \right]$$

$$S = S_0 + \Delta S = \int d^4x - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} Z_3^{-1} + b \underbrace{\int d^4x d^4y}_{\beta \text{ function}} F_{\mu\nu}(x) L(x-y) F^{\mu\nu}(y)$$

Compact notation

$$\int d^4q e^{i g (x-y)} \ln \frac{1}{\mu^2} = \langle x | \ln \frac{1}{m^2} | y \rangle$$

$$S = \int d^4x - \frac{1}{4} F_{\mu\nu} \left[1 + b e^{2 \ln \frac{\square}{\mu^2}} \right] F^{\mu\nu}$$

$\frac{1}{12\pi^2}$ for fermions

or with $e A_\mu \rightarrow A_\mu$

$$S = \int d^4x - \frac{1}{4} F_{\mu\nu} \left[\frac{1}{e^2(\mu)} + b \ln \frac{\square}{\mu^2} \right] F^{\mu\nu}$$

Note running coupling $\xrightarrow{-\mu^2}$ independent β fits coeff.

Connection to trace anomaly

$\ln \square$ is not scale invariant

$$\ln \square \rightarrow \ln \square - \ln \lambda^2$$

$$S \rightarrow S d^4x \lambda^4 - \frac{1}{4} F_{\mu\nu} \left[\frac{1}{e^2 m^2} + b \ln \square - b \ln \lambda^2 \right] F^{\mu\nu}$$

So that $\partial_\mu \overline{T}_{\text{scale}}^\mu = \overline{T}_\mu^\mu = \frac{b}{2} F_{\mu\nu} \bar{F}^{\mu\nu} = \frac{be^2}{2} F_{\mu\nu} F^{\mu\nu}$ \nwarrow usual norm

Anomaly does not follow from any local L \leftarrow

- but is described by nonlocal L

; IR/UV correspondence

Conformal symmetry and gravity

Symmetry trans

$$g'_{\mu\nu}(x) = e^{2\phi(x)} g_{\mu\nu}(x)$$

$$\phi'(x) = -e^{-\rho(x)} \phi(x)$$

↓
Sym

If this is a symmetry

$$\delta S = \int d^4x \left[\frac{\delta S_m}{\delta \phi} \delta \phi + \frac{\delta S_m}{\delta g_{\mu\nu}} \delta g_{\mu\nu} \right]$$

Eq M

$$\frac{\delta S_m}{\delta g_{\mu\nu}} \stackrel{?}{=} \delta g_{\mu\nu} = 2\alpha g^{\mu\nu}$$

$$g^{\mu\nu} T_{\mu\nu} = T_m = 0.$$

Einstein gravity is itself not conformally invariant

$$R' = e^{-2\phi} [R + 6 \Box \phi]$$

But gauge fields are
massless fermions are
massless scalars can't

Scalar fields

Consider

$$S = \int d^4x \sqrt{-g} \frac{1}{2} \left[g^{mn} \partial_m \phi \partial_n \phi - \frac{3}{2} R \phi^2 \right]$$

$$\text{E-L eq } (\square + \frac{3}{2} R) \phi = 0 \quad (\xi = \frac{1}{6})$$

But \square is not conformally invariant

$$\square = \frac{1}{\sqrt{-g}} \partial^m \left(g^{mn} \partial_n \right) \rightarrow$$

Then

$$\begin{aligned}
 (\square + \frac{3}{2} R) \phi &\rightarrow e^{-2\sigma} \left[\square + 2(\Box\sigma) \partial^m \right] e^{-\sigma} \phi + \frac{3}{2} e^{-2\sigma} [R + 6(\Box\sigma)] \phi \\
 &= e^{-2\sigma} e^{-\sigma} \left[\square - \Box\sigma - 2\partial_m \sigma \partial^m + 2\partial_m \sigma \partial^m + \frac{3}{2} R + 6\Box\sigma \right] \phi \\
 &= e^{-3\sigma} (6\Box\sigma - 11(\Box\sigma)) \phi \implies \xi = \frac{1}{6} \text{ is conformally invariant}
 \end{aligned}$$

recall

Exercise: Calculate
 $T_{\mu\nu}$ from $\frac{\delta S}{\delta g^{\mu\nu}}$
+ show that $\frac{3}{2} R \phi^2$
contributes at zero
order as $\partial_m \phi \partial^m \phi$

Scalar action

$$S = \int d^4x \sqrt{-g} \left[g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{1}{6} R \phi^2 \right]$$

Conformal anomaly (scalar loops)

- formalism already calculated

$$J = \det \begin{bmatrix} -\nabla \\ \ell \end{bmatrix} = e^{\text{Tr} \ln(-\nabla - \ell^{-\frac{D^2}{m^2}})} = e^{-S_d^4 x \alpha_2(x)}$$

Same calculation as before

$$T^{\mu}_{\mu} = \frac{1}{16\pi^2} \text{Tr} \alpha_2 = \frac{1}{16\pi^2} \cdot \frac{1}{18} \left[R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - R_{\mu\nu} R^{\mu\nu} + \square R \right]$$

↑
in general

scalars

Exercise (cont)
Check $T^{\mu}_{\mu} = 0$ for $\xi = \frac{1}{6}$

The Weyl Tensor

$$C_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} - \frac{1}{(d-2)} \left(g_{\mu\alpha} R_{\nu\rho} - g_{\nu\alpha} R_{\mu\rho} - g_{\mu\rho} R_{\nu\alpha} + g_{\nu\rho} R_{\mu\alpha} \right) \\ + \frac{1}{(d-1)(d-2)} (g_{\mu\alpha} g_{\nu\rho} - g_{\mu\rho} g_{\nu\alpha}) R$$

Special since

Squared:

$$C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 2R_{\mu\nu} R^{\mu\nu} + \frac{1}{3} R^2 \quad (4d)$$

$$\text{In action } E = R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 4R_{\mu\nu} R^{\mu\nu} + R^2 = 0$$

$$\Rightarrow C^2 - E = 2(R_{\mu\nu} R^{\mu\nu} - \frac{1}{3} R^2)$$

$$C^2 - \frac{1}{3} E = \frac{2}{3} (R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - R_{\mu\nu} R^{\mu\nu})$$

Anomaly & C^2

$$T_m^{\mu} = \frac{1}{16\pi^2} \frac{1}{270} C^2$$

Conformally invariant action

$$S_{\text{inv}} = \int d^4x F_8 C_{\mu\nu\alpha\beta} C^{\mu\nu\alpha\beta}$$

$\underbrace{\quad}_{\partial(\partial)^4}$

Also: FLRW spacetime have $C_{\mu\nu\alpha\beta} = 0$

Nonlocal effective actions

Barvinchuk Vilkovisky & ...

$$\text{We saw renormalization } \Delta \mathcal{L} = \frac{1}{16\pi^2} \frac{1}{\epsilon} [\alpha R^2 + \beta R_{\mu\nu} R^{\mu\nu}]$$

Recall that

$$\Rightarrow S = \int d^4x \mathcal{L}_S \left[C_1(\mu) R^2 + \alpha R \ln \frac{1}{\mu^2} R + C_2(\mu) R_{\mu\nu} R^{\mu\nu} + \beta R_{\mu\nu} \ln \frac{1}{\mu^2} R^{\mu\nu} \right]$$

[But extra term

$$R_{\mu\nu\rho\sigma} \ln \frac{1}{\mu^2} R^{\mu\nu\rho\sigma} - 4R_{\mu\nu} \ln \frac{1}{\mu^2} R^{\mu\nu} + R \ln \frac{1}{\mu^2} R \neq 0 \text{ in action}$$

$$\text{although } R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 4R_{\mu\nu} R^{\mu\nu} + R^2 = \text{total diver.}]$$

"on web page
Paper with Basan")

General results

$$S_{QL} = \int d^4x \sqrt{g} [\bar{\alpha} R \log\left(\frac{\square}{\mu_1^2}\right) R + \bar{\beta} C_{\mu\nu\alpha\beta} \log\left(\frac{\square}{\mu_2^2}\right) C^{\mu\nu\alpha\beta} + \bar{\gamma} (R_{\mu\nu\alpha\beta} \log(\square) R^{\mu\nu\alpha\beta} - 4R_{\mu\nu} \log(\square) R^{\mu\nu} + R \log(\square) R)] . \quad (1)$$

	α	β	γ	$\bar{\alpha}$	$\bar{\beta}$	$\bar{\gamma}$
Scalar	$5(6\xi - 1)^2$	-2	2	$5(6\xi - 1)^2$	3	-1
Fermion	-5	8	7	0	18	-11
Vector	-50	176	-26	0	36	-62
Graviton	430	-1444	424	90	126	298

TABLE I: Coefficients of different fields. All numbers should be divided by $11520\pi^2$.

General BV program. "Expansion in the curvatures" - .

Non local

$$\underbrace{R \ln D R}_{\text{Nonlocal}} , \underbrace{R^2 \frac{1}{R} R}_{\text{Nonlocal}} \leftarrow$$

Nonlocal expansion in the curvatur

Coefficients are calculable

(divergences are local)

Complicated at third order

$$\begin{aligned}
 \text{Tr}K(s) = & \frac{1}{(4\pi s)^\omega} \int dx g^{1/2} \text{tr} \left\{ \hat{1} + s \hat{P} \right. \\
 & + s^2 \sum_{i=1}^5 f_i(-s \square_2) \mathfrak{R}_1 \mathfrak{R}_2(i) \\
 & + s^3 \sum_{i=1}^{11} F_i(-s \square_1, -s \square_2, -s \square_3) \mathfrak{R}_1 \mathfrak{R}_2 \mathfrak{R}_3(i) \\
 & + s^4 \sum_{i=12}^{25} F_i(-s \square_1, -s \square_2, -s \square_3) \mathfrak{R}_1 \mathfrak{R}_2 \mathfrak{R}_3(i) \\
 & + s^5 \sum_{i=26}^{28} F_i(-s \square_1, -s \square_2, -s \square_3) \mathfrak{R}_1 \mathfrak{R}_2 \mathfrak{R}_3(i) \\
 & + s^6 F_{29}(-s \square_1, -s \square_2, -s \square_3) \mathfrak{R}_1 \mathfrak{R}_2 \mathfrak{R}_3(29) \\
 & \quad \left. + O[\mathfrak{R}^4] \right\}.
 \end{aligned}$$

$$\begin{aligned}
 F_2(\xi_1, \xi_2, \xi_3) = & F(\xi_1, \xi_2, \xi_3) \left[\frac{4\xi_1 \xi_2 \xi_3}{3\Delta^3} (-3\xi_1^2 \xi_2 - 3\xi_1^2 \xi_3 + 2\xi_1 \xi_2 \xi_3 + 3\xi_3^3) \right. \\
 & + \frac{4}{\Delta^2} (-\xi_1^2 \xi_2 - \xi_1^2 \xi_3 + 2\xi_1 \xi_2 \xi_3 + \xi_3^3) \Big] \\
 & + f(\xi_1) \frac{8\xi_1 \xi_2 \xi_3}{\Delta^3} (\xi_1^2 - \xi_2^2 + 2\xi_2 \xi_3 - \xi_3^2) \\
 & + \left(\frac{f(\xi_1) - 1}{\xi_1} \right) \frac{4\xi_1}{\Delta^2} (3\xi_1^2 - 2\xi_1 \xi_2 - \xi_2^2 - 2\xi_1 \xi_3 + 2\xi_2 \xi_3 - \xi_3^2) \\
 & \left. - 2 \frac{1}{\xi_1 - \xi_2} \left(\frac{f(\xi_1) - 1}{\xi_1} - \frac{f(\xi_2) - 1}{\xi_2} \right) \right),
 \end{aligned}$$

$$\begin{aligned}
 F_3(\xi_1, \xi_2, \xi_3) = & F(\xi_1, \xi_2, \xi_3) \left[\frac{2\xi_1 \xi_2}{\Delta^2} (\xi_1 - \xi_2 - \xi_3) (-\xi_1 + \xi_2 - \xi_3) \right. \\
 & - \frac{2}{\Delta} (\xi_1 + \xi_2 - \xi_3) \Big] + f(\xi_1) \frac{4\xi_1 \xi_2}{\Delta^2} (-\xi_1 + \xi_2 - \xi_3) \\
 & + f(\xi_2) \frac{4\xi_1 \xi_2}{\Delta^2} (\xi_1 - \xi_2 - \xi_3) \\
 & + f(\xi_3) \frac{1}{\Delta^2} (\xi_1^3 - \xi_1^2 \xi_2 - \xi_1 \xi_2^2 + \xi_2^3 - 3\xi_1^2 \xi_3 \\
 & \left. + 6\xi_1 \xi_2 \xi_3 - 3\xi_2^2 \xi_3 + 3\xi_1 \xi_3^2 + 3\xi_2 \xi_3^2 - \xi_3^3) \right),
 \end{aligned}$$

Example: Gravity and E + M

- Add gravity to previous QED example:

- scalar with conformal coupling

$$g_{\mu\nu} = \gamma_{\mu\nu} + h_{\mu\nu}$$

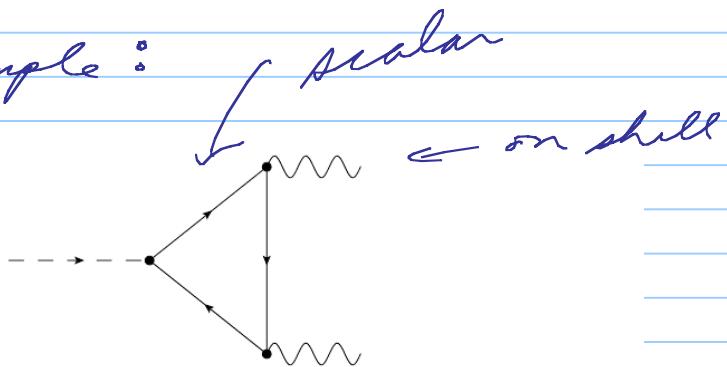


FIG. 1: Triangle diagram.

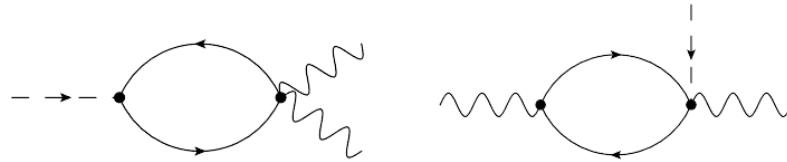


FIG. 2: Bubble diagrams.

Linear terms in $h_{\mu\nu}$

- form effective actions (non local)

↙ ↘ *classical*

$$\Gamma^{(1)}[A, h] = -\frac{1}{2} \int d^4x h^{\mu\nu} \left[b_s \log \left(\frac{\square}{\mu^2} \right) T_{\mu\nu}^{cl} - \frac{b_s}{2} \frac{1}{\square} \tilde{T}_{\mu\nu}^s \right]$$

$$\tilde{T}_{\mu\nu}^s = 2\partial_\mu F_{\alpha\beta}\partial_\nu F^{\alpha\beta} - \eta_{\mu\nu}\partial_\lambda F_{\alpha\beta}\partial^\lambda F^{\alpha\beta}$$

$\frac{1}{8} \square$ ()

Conformal anomaly lives in the non-local terms
Infinitesimal conformal transformations

$$g_{\mu\nu} \rightarrow (1 + 2\sigma)g_{\mu\nu} \quad \text{or} \quad h_{\mu\nu} \rightarrow h_{\mu\nu} + 2\sigma\eta_{\mu\nu}$$

This leads to a change in the action:

$$\Gamma^{(1)}[A, h] \rightarrow \Gamma^{(1)}[A, h] - b_i \int d^4x \sigma \frac{1}{\square} (\partial_\lambda F_{\mu\nu} \partial^\lambda F^{\mu\nu})$$

If we use (for onshell photons)

$$\partial_\lambda F_{\alpha\beta} \partial^\lambda F^{\alpha\beta} = \frac{1}{2} \square (F_{\mu\nu} F^{\mu\nu})$$

Then we get the conformal anomaly for QED

$$T_\mu^\mu = \frac{b_i}{2} \eta^{\mu\alpha} \eta^{\nu\beta} F_{\mu\nu} F_{\alpha\beta} = \beta \bar{F}^2$$

Now - expansion in the curvature

- match to a generally covariant action

$$\Gamma_{\text{anom.}}[g, A] = \int d^4x \sqrt{g} \left[n_R F_{\rho\sigma} F^{\rho\sigma} \frac{1}{\square} R + n_C F^{\rho\sigma} F^\gamma_\lambda \frac{1}{\square} C_{\rho\sigma\gamma}^\lambda \right]. \quad \xleftarrow{\hspace{1cm}} \text{BV}$$

matches for

$$n_R^{(s,f)} = -\frac{\beta^{(s,f)}}{12e}, \quad n_C^s = -\frac{e^2}{96\pi^2}, \quad n_C^f = \frac{e^2}{48\pi^2}$$

This is the BV expansion in the curvature at third order

Local curvature expansion vs Nonlocal expansions in curvature

→ local \Rightarrow energy expansion , unknown c_i
non local ~ all the same order
- coeff known] pure quantum effects

How useful?

Summary Anomalies and Nonlocal actions

Beyond scattering amplitudes \Rightarrow non local actions

Expansion in curvature

related to anomalies

Still only lightly explored