PHY-602: Statistical Physics, Midterm

Show all your work for maximum credit.

I. THERMODYNAMICS *(25 POINTS)*

1. Consider an ideal gas at temperature T_i in a thermally isolated container of volume V_i. The volume of the container is instantaneously increased so the gas expands into vacuum to a final volume $V_f = 2V_i$, still within adiabatic walls. What is the temperature of the gas after this sudden expansion process? Compute the increase of entropy associated with this irreversible expansion.

2. Show the Maxwell relation $\left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial p}{\partial T}\right)_V$.

3. The equation of state of an ultrarelativistic gas is $p = \frac{E}{3V}$. Show that adiabatic processes for such a gas are characterized by $PV^{4/3} = \text{const.}$

II. NUCLEAR SPINS IN A SOLID *(25 POINTS)*

A solid contains N non-interacting spin-1 nuclei which can each be labelled by a quantum number $m = -1, 0, 1$. Because of the interactions within the solid, a nucleus in the state $m = +1$ or $m = -1$ has the same energy $\epsilon > 0$, while its energy in the state $m = 0$ is zero.

1. What is the free energy of a system of N nuclei at temperature T?

2. Compute the heat capacity of this system and discuss its high- and low-temperature limits.

3. What is the entropy of this system, again as a function of T?

III. DNA ZIPPER *(25 POINTS)*

A (very) simplified theory for the unwinding of two-stranded DNA molecules models the system as a zipper that has N links, each of which has two states: closed (energy 0) and open (energy $\epsilon > 0$). The DNA molecule ("zipper") can only unzip from its left end, and link s can only open if all the links to the left of it are already open.

1. Assuming that the zipper is in contact with a thermal reservoir at temperature T, show that the partition function of the zipper reads

$$Z = \frac{1 - \mathrm{e}^{-\beta\epsilon(N+1)}}{1 - \mathrm{e}^{-\beta\epsilon}}. \tag{1}$$

2. Find the average number of open links. Sketch your result as a function of temperature and discuss the high and low temperature limits.

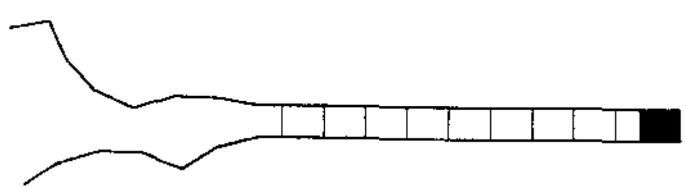

FIG. 1: Open and closed links in a single-ended zipper. Figure from C. Kittel, American Journal of Physics 37, 917 (1969).

IV. IDEAL GAS WITH ADSORBING SURFACE *(25 POINTS)*

Consider a classical ideal gas of N (indistinguishable) particles of mass m at a temperature T and pressure p. The particles can either move freely in a volume V, or be adsorbed on a surface with area A with an attractive potential ϕ. The particles on the surface form a two-dimensional ideal gas, where the energy of an adsorbed particle is $\epsilon_{\mathbf{p}} = \frac{\mathbf{p}^2}{2m} - \phi$ where $\mathbf{p}$ is the two-dimensional momentum and ϕ is the binding potential. Let N_s be the number of particles adsorbed on the surface, and N_g the number of remaining particles moving freely in the volume V (so that $N = N_g + N_s$).

1. Compute the free energy of the free (non-adsorbed) gas with N_g particles.
2. What is the chemical potential μ_g of the free gas particles, as a function of N_g, V and T?
3. Compute the free energy of the two-dimensional gas of N_s adsorbed particles.
4. Show that the chemical potential μ_s of the adsorbed particles reads

$$\mu_s = -\phi - k_B T \ln \frac{A}{N_s \lambda_T^2},$$

   with λ_T the thermal de Broglie wavelength.
5. The free gas particles and the absorbed gas particles are in equilibrium. What is the density N_s/A of the adsorbed particles? Express your results in terms of the pressure p, the binding potential ϕ and temperature T.