# PHY-602: Statistical Physics, UMass Amherst, Problem Set #1

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Due: Friday, Sept 15. (Late homework receives 50% credit.)

## I. RANDOM WALK IN ONE DIMENSION

Let us consider a random walk in one dimension. At time $t = 0$, the walker is at position $x = 0$. At each time step $\Delta t$, the walker moves either forwards or backwards by an amount $a$ with equal probability. Let $x(t = N\Delta t)$ (with $N \in \mathbb{N}$) be the position of the walker at time $t = N\Delta t$ — corresponding to $N$ time steps.

1. Write $x(t = N\Delta t)$ as a sum of $N$ independent random variables. Using the central limit theorem, compute the mean and the variance of $x(t)$ at long times.
2. Let us recover this result explicitly. What is the probability $P(n, N)$ to find the walker at position $x(t = N\Delta t) = na$ (with $n \in \mathbb{Z}$) after $N$ steps?
3. Using Stirling's formula for $N \gg 1$, expand $\ln P$ assuming that $n/N$ is small (why is this justified?). Show that the resulting distribution is Gaussian. What is the variance of $n$ (or $x = na$)?
4. Yet another way to recover this result is to write down a "rate equation" for $P_N(n)$ (this is more easily generalizable to higher dimensions). Express $P(t = (N+1)\Delta t, x = n)$ after $N+1$ time steps in terms of the probability distribution $P(t = N\Delta t, x)$ after $N$ time steps. Assuming that $\Delta t$ and $a$ are small with $a^2/\Delta t$ fixed, show that this rate equation becomes a diffusion equation
$$\frac{\partial P}{\partial t} = D\frac{\partial^2 P}{\partial x^2},$$
where you will give the expression of $D$ in terms of $a$ and $\Delta t$. The solution of that equation for an initial distribution $P(t = 0, x) = \delta(x)$ satisfies $\langle x(t)^2\rangle = 2Dt$: compare with the variance obtained above.

## II. MAXWELL-BOLTZMANN DISTRIBUTION

In class, we showed that the probability of finding a particle in a classical gas with velocity between $\vec{v} = (v_x, v_y, v_z)$ and $(v_x + dv_x, v_y + dv_y, v_z + dv_z)$ is given by:

$$dP(\vec{v}) = \mathcal{N}\mathrm{e}^{-\frac{mv^2}{2k_BT}}d^3\vec{v}.$$

1. Compute the normalization factor $\mathcal{N}$.
2. What is the mean value of $v = |\vec{v}|$?
3. What is the most probable speed of a particle?
4. Compute the average kinetic energy of a particle.
5. Compute the probability distribution of the kinetic energy of the particles.

## III. CUMULANTS AND CHARACTERISTIC FUNCTION

The characteristic function of a probability distribution $p(x)$ is defined as $\hat{p}(k) = \langle \mathrm{e}^{-ikx}\rangle = \int dx p(x)\mathrm{e}^{-ikx}$. The cumulants $\langle x^n\rangle_c$ of the distribution are defined through $\ln \hat{p}(k) = \sum_{n=1}^{\infty}\frac{(-ik)^n}{n!}\langle x^n\rangle_c$. Using the characteristic function, compute the cumulants and the first few moments of a normal (Gaussian) distribution with mean $\mu$ and variance $\sigma^2$.