PHY-602: Statistical Physics, UMass Amherst, Problem Set #3

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Due: Wednesday, Oct 4. (Late homework receives 50% credit.)

I. PRINCIPLE OF MAXIMUM ENTROPY AND GENERALIZED GIBBS ENSEMBLE

In this problem, our goal is to use the principle of maximum entropy and the Gibbs entropy $S = -k_B \sum_n p_n \ln p_n$ for a probability distribution p_n to deduce the form of all ensembles. Recall that we showed in class that when restricted to states of fixed energy E, the entropy is maximized by the microcanonical ensemble.

1. Show that at fixed average energy $\langle E \rangle = \sum_n p_n E_n$, the entropy is maximized by the canonical ensemble. Argue that maximizing the entropy subject to this constraint is equivalent to minimizing the free energy.
2. Show that at fixed average energy $\langle E \rangle$ and particle number $\langle N \rangle$, the entropy is maximized by the grand canonical ensemble. What is the interpretation of the Lagrange multipliers in this case?
3. Consider a system with M conserved quantities $Q^{(i)}$ with $i = 1, 2, \ldots, M$. Using the principle of maximum entropy, compute the probability to find this system in a state n where the conserved quantities take the values $Q_n^{(i)}$.

II. ULTRA-RELATIVISTIC GAS

Consider an ultra-relativistic gas of N classical particles obeying the energy-momentum relation $E = pc$, where c is the speed of light and $pc \gg mc^2$ with m the mass of the particles. Compute the internal energy and the equation of state of this gas.

III. SURFACE ADSORPTION

We consider a gas of ^{4}He atoms at a pressure of $p = 1$ atm and temperature of $T = 300K$ in contact with an adsorbing surface. The atoms in the gas can be bound to surface sites with an attractive potential of $\phi = 10$ meV. Each site on the surface can only adsorb a single atom, and the surface sites are far enough apart so that we can neglect the interactions between them. What fraction of the surface sites will be occupied?

IV. QUANTUM HARMONIC OSCILLATORS AND STRINGS

1. Compute the free energy of a quantum harmonic oscillator with frequency ω, and energy levels $E_n = \hbar\omega\left(n + \frac{1}{2}\right)$ with $n = 0, 1, 2, \ldots$ Find the average energy and the heat capacity as a function of temperature T.
2. Compute the partition function and the heat capacity of a classical harmonic oscillator and compare your results with the quantum case.
3. A quantum string can vibrate at frequencies $\omega_p = p\omega$ with $p = 1, 2, \ldots,$ where each vibration mode can be treated as a quantum oscillator with energy levels $E_{n,p} = \hbar\omega_p n$ with $n = 0, 1, 2, \ldots$ (we have chosen the zero of energy at $n = 0$). Express the free energy as a sum over the different modes, and argue that in the high temperature limit, the sum can be replaced by an integral. Show that the free energy at high temperature reads
$$F \approx -\frac{\pi^2 k_B^2 T^2}{6\hbar\omega}.$$
hint: You may need $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ *to evaluate the integral.*
4. Compute the entropy, the average energy and the heat capacity of a quantum string at high temperature.