PHY-602: Statistical Physics, UMass Amherst, Problem Set #7

Romain Vasseur

Due: Wednesday, Nov 1. (Late homework receives 50% credit.)

I. OCCUPATION OF THE EXCITED STATES IN THE BEC PHASE

Consider N non-interacting, non-relativistic bosons of mass m, in a cubic box of side L. Show that the transition temperature scales as $T_c \sim \frac{N^{2/3}}{mL^2}$ and the 1-particle energy levels scale as $E_n \sim \frac{1}{mL^2}$. Show that in the BEC phase ($T < T_c$), the mean occupancy of the first excited 1-particle state is large, but not as large as the number of particles in the ground state.

II. GENERALIZED BOSE GAS

Consider an ideal d-dimensional Bose gas with dispersion relation, $\epsilon(k) \propto k^s$ where $\epsilon(k)$ is the energy of a single particle state with wavevector k.

1. Determine for what values of d and s, there will be Bose-Einstein condensation at finite temperature $T > 0$. (*Hint: study the behavior of N as $\mu \to 0$*).

2. Show that the pressure p is related to the energy density according to

$$p = \frac{s}{d}\frac{E}{V},$$

   where E is the energy of the system and $V = L^d$ its volume.

3. Show that if Bose condensation does occur, then the heat capacity in the BEC phase scales as $C_v \sim k_B \left(\frac{T}{T_c}\right)^\gamma$ where you will give the exponent γ in terms of s and d.

III. 2D BOSE GAS IN A HARMONIC TRAP

Bose-Einstein condensates are realized experimentally using ultracold atoms that are trapped in optical lattices (an effective potential for the atoms emerges from the interference of laser beams). Consider an ideal two-dimensional Bose gas in a harmonic trap with potential $V(\vec{q}) = \frac{1}{2}m\omega^2 q^2$ with $\vec{q} = (x, y)$.

1. What are the single particle energy levels of a single boson in this two-dimensional harmonic trap? Show that the density of states reads

$$g(E) = \frac{E}{(\hbar\omega)^2}.$$

   (*Hint: consider first $\Sigma(E) = \#$ states with energy $\leq E$*).

2. Briefly argue that there will be Bose-Einstein condensation in this problem.