PHY-602: Statistical Physics, UMass Amherst, Problem Set #8

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Due: Wednesday, Nov 8. (Late homework receives 50% credit.)

I. FREE EXPANSION OF AN ULTRA-RELATIVISTIC FERMI GAS

Consider an ideal, ultra-relativistic gas of N indistinguishable spin-$\frac{1}{2}$ fermions with energy $\epsilon(\vec{p}) = pc$, initially at equilibrium in a volume V_i at zero temperature, $T_i = 0$.

1. Calculate the initial total energy E_i of this zero-temperature, three-dimensional ultra-relativistic Fermi gas.
2. The initial confining walls are then instantaneously removed, and the gas then expands into a vacuum to a much larger final volume V_f (enclosed by thermally insulating walls) upon which it internally equilibrates – due to some weak interactions. V_f is so large that quantum statistics can be ignored, and the final state of the gas can be treated as classical and non-interacting (though still ultra-relativistic). What is the final temperature T_f of the gas?
3. Compute the change in entropy ΔS of the gas due to this expansion.

II. WHITE DWARFS

When a star has exhausted its nuclear fuel, the thermal radiative pressure is no longer sufficient to stand against the gravitational force trying to pull it together. If its mass is not much larger than the solar mass $M_\odot$, it will then collapse to a white dwarf star with a radius much smaller than the solar radius and with an internal pressure now due to degenerate electrons. It is therefore possible for a burnt-out star to maintain itself against complete collapse under gravity via the degeneracy pressure of its constituent electrons. White dwarfs are extremely dense and their Fermi temperature $T_F \sim 10^9 K$ is much larger than their temperature $\sim 10^7 K$ so that we can safely neglect all finite-temperature corrections for the degenerate electron gas.

1. Consider a white dwarf of radius R and mass M with approximately constant density, and two nucleons per electron. Recall the expression of the Fermi energy and the energy due to the $T = 0$ degenerate electron gas E_{kin} assuming the electrons are non-relativistic.
2. Show that the gravitational energy of the white dwarf is given by
$$E_{\text{grav}} = -\frac{3}{5}\frac{GM^2}{R}.$$
3. Show that the equilibrium radius $R^\star$ of a white dwarf that minimizes the total energy $E_{\text{kin}} + E_{\text{grav}}$ scales as $R^\star \sim M^{-1/3}$.
4. This means that white dwarfs shrink when mass is added! At some point, the non-relativistic approximation will break down. Assuming now ultra-relativistic electrons, compute the new Fermi energy and the total ground state energy E_{kin} of the electron gas.
5. Argue that if
$$M \geq M_c \sim \left(\frac{\hbar c}{G}\right)^{3/2} \frac{1}{m_n^2}, \tag{1}$$
with m_n the mass of a nucleon, the degenerate electron star is unstable against collapse. This is called the Chandrasekhar limit with $M_c \sim 1.4 M_\odot$. A star that violates the limit will collapse into a neutron star or black hole, depending on whether neutron degeneracy pressure can hold up the star.