

PHY-817: Advanced Statistical Physics, UMass Amherst, Problem Set #1

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Due: Wednesday, Feb 7 in class.

I. CORRELATION FUNCTIONS IN THE ONE-DIMENSIONAL ISING MODEL

The Ising model $\mathcal{H}(\{\sigma_i\}) = -J \sum_{\langle i,j \rangle} \sigma_i \sigma_j - h \sum_i \sigma_i$ has a simple exact solution in one dimension. The goal of this problem is to remind you of the transfer matrix formalism that you saw the standard statistical mechanics course, and to use it to compute the correlation length ξ of that model. For simplicity, we will focus on the case $h = 0$.

1. Introduce a 2×2 *transfer matrix* \hat{T} whose matrix elements $\hat{T}_{\sigma_i, \sigma_{i+1}}$ are given by the Boltzmann weight interaction between nearest neighbor spins. Show that the partition function of an Ising model with N spins and periodic boundary conditions $\sigma_{N+1} \equiv \sigma_1$ is given by $Z = \text{Tr } \hat{T}^N$.
2. Deduce the exact expression of the free energy per spin $f = \lim_{N \rightarrow \infty} F/N$ in the thermodynamic limit.
3. Show that the spin-spin correlation function within the transfer matrix formalism is given by

$$\langle \sigma_i \sigma_{i+r} \rangle = \frac{1}{Z} \text{Tr} \left(\sigma^z \hat{T}^r \sigma^z \hat{T}^{N-r} \right),$$

where σ^z is the usual Pauli matrix.

4. Show that $G(r) = \langle \sigma_i \sigma_{i+r} \rangle$ decays exponentially with r , and compute the correlation length ξ as a function of temperature.

II. BLUME-CAPEL MODEL: MEAN-FIELD THEORY

Consider a generalization of the Ising model in which the spin variables can take three values $\sigma_i = 0, \pm 1$. This model can either be thought of as a classical spin-1 model, or as an Ising model with vacancies corresponding to $\sigma_i = 0$. The Hamiltonian reads

$$\mathcal{H}(\{\sigma_i\}) = -J \sum_{\langle i,j \rangle} \sigma_i \sigma_j - h \sum_i \sigma_i + \Delta \sum_i \sigma_i^2.$$

1. Work out the mean-field theory for this model and derive a self-consistency equation for the magnetization $m = \langle \sigma_i \rangle$. (Do not try to approximate the σ_i^2 term.) You do not need to analyze the solutions of this equation.
2. What is the mean field free energy $F[m]$ as a function of m ? Show that the mean-field solution for the magnetization minimizes $F[m]$.

III. MEAN-FIELD THEORY AS A VARIATIONAL APPROACH

Consider a statistical model with Hamiltonian \mathcal{H} , and free energy $F = -T \log Z$. In the following we will assume for concreteness that the microstates of this model are spin configurations $\{\sigma_i\}$. We would like to approximate \mathcal{H} by a simpler (typically non-interacting) Hamiltonian \mathcal{H}_0 , whose free energy $F_0 = -T \log Z_0$ is easier to compute.

1. Show the so-called Bogoliubov inequality

$$F \leq F_0 + \langle \mathcal{H} - \mathcal{H}_0 \rangle_0,$$

where $\langle \dots \rangle_0 = \frac{1}{Z_0} \sum_{\{\sigma\}} (\dots) e^{-\beta \mathcal{H}_0}$ is the thermal average with respect to the Hamiltonian \mathcal{H}_0 . (*hint: you'll need to use the convexity property of the exponential function.*)

2. Let $\mathcal{H} = -J \sum_{\langle i,j \rangle} \sigma_i \sigma_j$ be the Hamiltonian of an Ising model on a lattice with coordination number z , and $\mathcal{H}_0 = -h_{\text{eff}} \sum_i \sigma_i$. Find the value of the magnetic field h_{eff} that minimizes the right hand side of this inequality.

IV. ISING MODEL ON A COMPLETE GRAPH (NOT GRADED)

Consider an Ising model with infinite range interactions: each spin interacts with the same strength with all the other spins in the lattice with N sites. The Hamiltonian reads

$$\mathcal{H} = -\frac{J}{2N} \sum_{i,j=1}^N \sigma_i \sigma_j - h \sum_{i=1}^N \sigma_i,$$

with $\sigma_i = \pm 1$ and where the i, j sums run over all spins. The factor of N in front of the interaction term is chosen to get a sensible thermodynamic limit.

1. Show that \mathcal{H} can be expressed in terms of the total spin $S = \sum_i \sigma_i$ only.
2. Show the identity

$$e^{\frac{\beta J}{2N} S^2} = \sqrt{\frac{N\beta J}{2\pi}} \int_{-\infty}^{\infty} dm e^{-\frac{N\beta J}{2} m^2 + \beta J S m},$$

and use it to perform the sum over the spins $\sum_{\{\sigma_i\}}$ in the partition function. Express the partition function as

$$Z = \sqrt{\frac{N\beta J}{2\pi}} \int_{-\infty}^{\infty} dm e^{-N\beta f(m)} \text{ with } f(m) = \frac{Jm^2}{2} - k_B T \ln(2 \cosh[\beta(Jm + h)]).$$

3. Deduce the exact free energy per spin and the mean magnetization per spin $\langle s \rangle$ in the thermodynamic limit $N \rightarrow \infty$. Compare with the mean-field solution of the Ising model discussed in class.

Note: the trick used in question 2 is a powerful formal technique known as the Hubbard-Stratonovich transformation. We will use it later this semester.