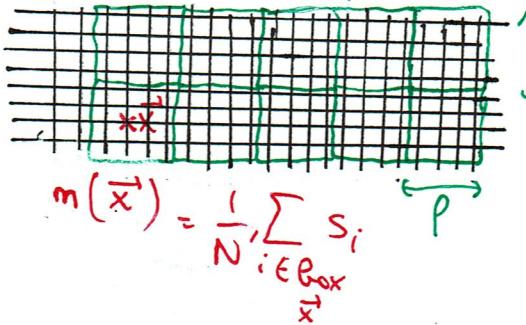


Landau-Ginzburg

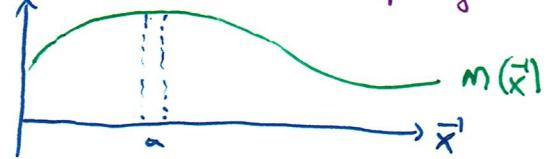
Theory and Fluctuations

Universality: many different systems enjoy the same critical point
 . details of the lattice unimportant \Rightarrow coarse-grained picture

(A) Landau-Ginzburg Free Energy



Boxes of size P^d : N' spins
 with $\xi \gg P \gg a$
 correlation length (ξ) (to be defined more precisely below)
 \uparrow
 Lattice spacing



N' very big: $m \in [-1, 1]$, with $\frac{N}{N'} \gg 1$ (number of boxes) so that \vec{x} is a continuous variable. (A bit hand-wavy, but details of the procedure will not matter \Rightarrow universality again!)

$$\text{Now: } Z = \sum_{\{s_i\}} e^{-\beta \mathcal{H}(\{s_i\})} = \sum_{m(\vec{x})} \underbrace{\left[\sum_{\{s_i\}|m(\vec{x})} e^{-\beta \mathcal{H}(\{s_i\})} \right]}_{\substack{\text{sum over all configurations} \\ \{s_i\} \text{ that give } m(\vec{x}) \text{ after} \\ \text{coarse graining}}} = \sum_{m(\vec{x})} e^{-\beta F[m(\vec{x})]}$$

$F[m(\vec{x})]$: Landau-Ginzburg free energy: functional of $m(\vec{x})$

$$\sum_{m(\vec{x})} = \int \prod_{\vec{x}} dm_{\vec{x}} = \int Dm(\vec{x}) : \text{Functional (or path) integral}$$

sum over all field configurations $m(\vec{x})$

This yields:

$$Z = \int Dm(\vec{x}) e^{-\beta F[m(\vec{x})]} \quad \boxed{\text{looks like a normal partition function! (with microstates = field configuration } m(\vec{x}))}$$

Probability of a configuration $m(\vec{x})$: $p[m(\vec{x})] = \frac{e^{-\beta F[m(\vec{x})]}}{Z}$

We'll take $m(\vec{x}) \in \mathbb{R}$ (can imagine terms in $F[m]$ that will favor $m \in [-1, 1]$)

Notations: . in some books $F[m]$ is denoted by $\mathcal{H}[m]$ (even if microscopic)
 - usually, $F[m]$ contains some entropy terms

We will also write $\beta F[m] = S[m] = \text{Action}$ by analogy with Quantum Field Theory.
 (later in the course, we will also rename $m(\vec{x}) \rightarrow \phi(\vec{x})$) ORDER PARAMETER

How do we compute $F[m(\vec{x})]$?

\Rightarrow guess the answer ("What else could it be?") using constraints:

. Locality: nearest neighbor interactions, F should also be local:

$$F[m(\vec{x})] = \int d^d x \delta[m(\vec{x})] \quad \text{depends on } m(\vec{x}), \nabla m(\vec{x}) \text{ and higher derivatives}$$

Symmetry: Our lattice model has a symmetry $s_i \rightarrow -s_i$ if $B=0$
 (or $s_i \rightarrow s_i$ in general) \Rightarrow should also have $m \rightarrow -m$ symmetry in F .

. Also expect rotation (isotropy) and translation invariance

Analyticity: Consider m small (transition from $m \neq 0$ to $m=0$)

Ferromagnet Paramagnet for Ising

We will also assume $m(\vec{x})$ is slowly varying ($(\nabla m)^4$ smaller than $(\nabla m)^2$ etc...)

\Rightarrow Enough to guess $F[m(\vec{x})]$!

F can only depend on even power of m :

$$\beta[m(\vec{x})] = \text{Cst} - Bm + \alpha_2 m^2 + \alpha_4 m^4 + \frac{1}{2} |\nabla m|^2 + \dots$$

drop:
unimportant

applied field: $\frac{\partial \beta}{\partial B} = -m$

$m^6, m^8, \dots, (\nabla m)^4, \dots$

choose the normalization of m so that this coefficient is $\frac{1}{2}$ (Jacobian in Dm)
unimportant

$$F[m(\vec{x})] = \boxed{S[m(\vec{x})]} = \int d\vec{x} \left[\frac{1}{2} (\nabla m)^2 + \alpha_2 m^2 + \alpha_4 m^4 - Bm + \dots \right]$$

it is sometimes convenient to set $\beta=1$ (absorb it in m) \Rightarrow m dimensionfull: $[S] = \infty$ (dimensionless)
 $[m] = L^{\frac{2-d}{2}}$

If we truncate at $\mathcal{O}(m^4)$: α_4 must be > 0 (otherwise F can become $-\infty$)
 would favor m very large and $\int Dm e^{-S}$ would diverge!

Although we focused on the Ising model (Ferromagnetism), this approach is very general: liquid crystals, superfluids, superconductivity ...

- 2 ingredients:
- order parameter
- symmetry

B - Mean-field approximation and critical exponents

We still have to compute the (path) integral: $Z = \int Dm e^{-S[m]}$

if we knew how to do that, theoretical physics would be much easier!

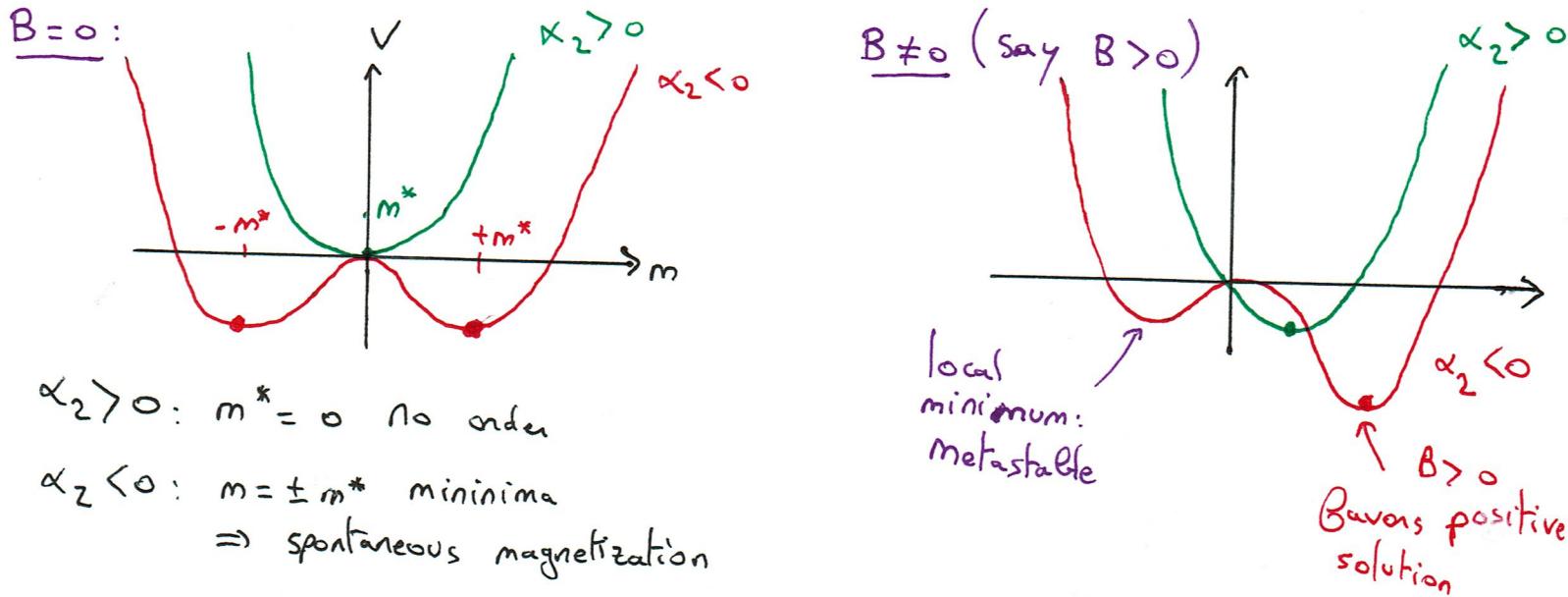
Mean-field approximation: neglect fluctuations, assume that integral is dominated by saddle point: $Z \approx e^{-S[m^*]}$ up to irrelevant prefactors, where m^* minimizes the free energy (or the action).

Then $\langle \mathcal{O}(m) \rangle = \frac{1}{Z} \int Dm \mathcal{O}(m) e^{-S[m]} \underset{\text{mean field}}{\approx} \mathcal{O}(m^*)$ with \mathcal{O} = observable that depends on m

Now, $S[m]$ is minimum for $m^* = \text{cst}$ (uniform magnetization): $|\nabla m^*|^2 = 0$

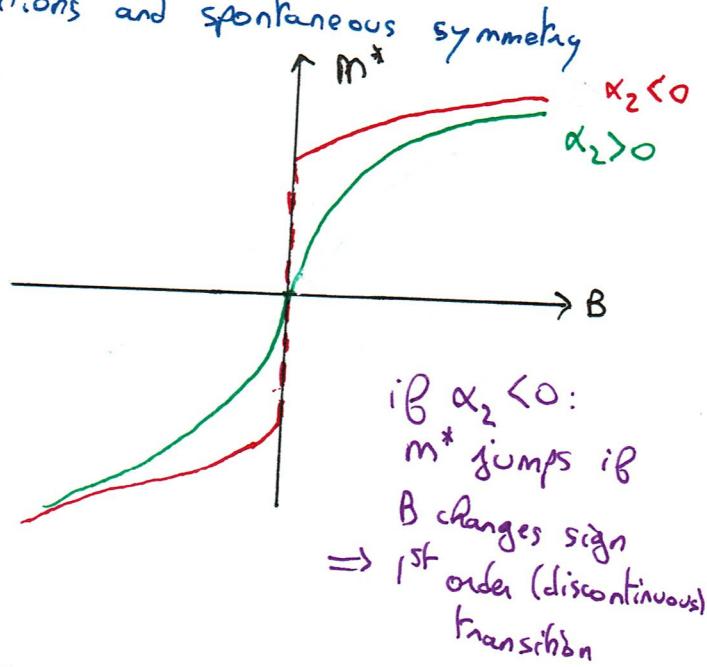
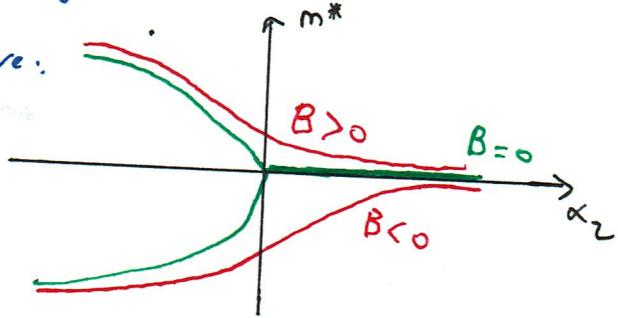
Then m^* minimizes

$$V(m) = \alpha_2 m^2 + \alpha_4 m^4 - Bm$$

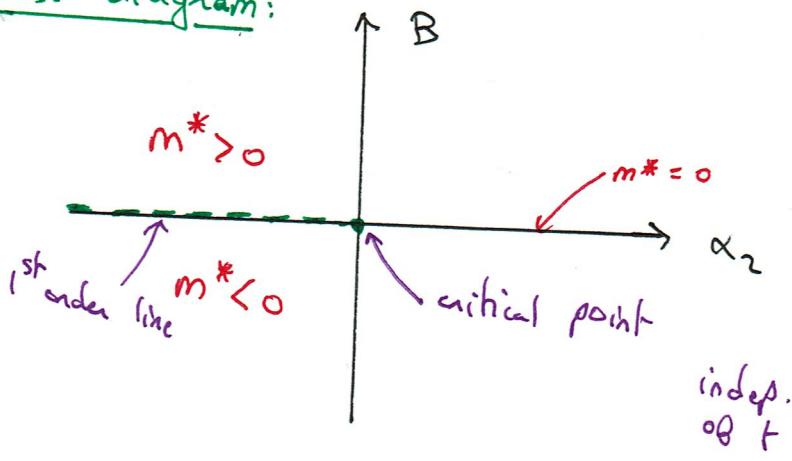


\Rightarrow very simple picture of phase transitions and spontaneous symmetry breaking for $B=0$.

We have:



Phase diagram:



Coefficients: α_2, α_4 etc depend on T
 $(\text{or on } K = \beta J \Rightarrow \text{set } J=1, K_B=1)$

α_2 changes sign at $T=T_c$:

$$\alpha_2 = a_0 \frac{T-T_c}{T_c} + \mathcal{O}(T^2)$$

$$\alpha_4 = U + \mathcal{O}(T)$$

Remark: if we had a term $\alpha_3 m^3$ in $B[m]$, the transition as a function of m would be a say $\alpha_3 < 0$

α_2 is generically first order (jump in the order parameter) (check this!) [5]

Critical exponents: $\frac{\partial B}{\partial m} \Big|_{m^*} = 2a_0 + m^* + 4\upsilon m^{*3} - B$

$$\text{if } B=0: m^*=0 \text{ or } m^* = \pm \sqrt{\frac{a_0|t|}{2\upsilon}} \sim \pm \sqrt{T_c - T} \text{ if } T < T_c (t < 0)$$

$$T = T_c: m^* \sim B^{1/3} \Rightarrow \delta = 3$$

$$\text{Susceptibility: } \chi = \frac{\partial m}{\partial B} \Big|_{B=0} \Rightarrow (2a_0 t + 12\upsilon m^{*2}) \chi = 1$$

$$\begin{aligned} \rightarrow \text{if } t > 0: m^* &= 0 \text{ and } \chi = \frac{1}{2a_0 t} \\ \rightarrow \text{if } t < 0: m^* &= -\frac{a_0 t}{2\upsilon} \text{ and } \chi = -\frac{1}{4a_0 t} \end{aligned} \left. \right\} \chi \sim \frac{1}{|T - T_c|} \quad \gamma = 1 \quad (\gamma_+ = \gamma_-)$$

Specific Heat: $B=0$ and use $C_v = \frac{\partial E}{\partial T} = -T \frac{\partial^2 F}{\partial T^2}$ ($F = E - TS$)

$$F = a_0 t m^{*2} + \upsilon m^{*4} \left\{ \begin{array}{l} = 0 \text{ if } t > 0 \\ = -\frac{(a_0 t)^2}{2\upsilon} + \frac{(a_0 t)^2}{4\upsilon} = -\frac{(a_0 t)^2}{4\upsilon} \text{ if } t < 0 \end{array} \right.$$

$$\Rightarrow \frac{C_v}{V} \approx -\frac{T_c}{V} \frac{\partial^2 F}{\partial T^2} \sim \begin{cases} \frac{a_0^2}{2\upsilon} & \text{if } t < 0 \\ 0 & \text{if } t > 0 \end{cases}$$

$$\frac{C_v}{V} \text{ discontinuous: } \begin{cases} \text{more generally:} \\ F \sim A|T - T_c|^{2-\alpha} \\ C_v \sim \frac{1}{|T - T_c|^\alpha} \end{cases}$$

We recover mean-field exponents: independent from dimension d underlying lattice etc. We will see below that the correct exponents do depend on dimensionality.

C Correlation Functions

The Ginzburg-Landau formalism also allows us to extract correlation functions.

Recall that we expect:

(away from $T=T_c$)

with $\xi \sim |T-T_c|^{-\nu}$

$$G(\vec{r} - \vec{r}') = \langle m(\vec{r}) m(\vec{r}') \rangle - \langle m(\vec{r}) \rangle \langle m(\vec{r}') \rangle$$

$$\sim e^{-|\vec{r} - \vec{r}'|/\xi}$$

ξ correlation length
(characteristic length of correlations)

- In stat. mech, recall that averages can be computed by taking derivatives of $\log Z$ (or F) with respect to some conjugate field: $\langle E \rangle = -\frac{\partial \log Z}{\partial \beta}$

$$m = -\frac{\partial F}{\partial B} \text{ etc...}$$

- Introduce inhomogeneous field: $Z[R] = \int Dm e^{-S[m] - \int d^n r R(\vec{r}) m(\vec{r})}$

now: $\langle m(\vec{r}) \rangle = \left. \frac{\delta \log Z}{\delta R(\vec{r})} \right|_{R=0}$ means that $\delta \log Z = \int d^n r \langle m(\vec{r}) \rangle \delta R(\vec{r}) + \dots$

for small $R = \delta R(\vec{r})$

in general: $\frac{\delta F}{\delta R(\vec{r})} = \lim_{\varepsilon \rightarrow 0} \frac{F[R(\vec{r}) + \varepsilon \delta(\vec{r})] - F[R(\vec{r})]}{\varepsilon}$ $\left[\begin{array}{l} \frac{\delta}{\delta R(\vec{r})} \int d^n r R(\vec{r}) = 1 \\ \frac{\delta R(\vec{r}')}{\delta R(\vec{r})} = \delta(\vec{r}' - \vec{r}) \end{array} \right]$

Now it's easy to check that:

$$G(\vec{r} - \vec{r}') = \left. \frac{\delta \langle m(\vec{r}) \rangle}{\delta R(\vec{r}')} \right|_{R=0} = \left. \frac{\delta^2 \log Z}{\delta R(\vec{r}') \delta R(\vec{r})} \right|_{R=0}$$

also called $\chi(\vec{r} - \vec{r}')$
generalized susceptibility

This is an example of Fluctuation-dissipation relation

Should remind you of: $\langle E^2 \rangle - \langle E \rangle^2 = k_B T^2 \frac{\partial \langle E \rangle}{\partial T}$ (canonical)

$$\langle N^2 \rangle - \langle N \rangle^2 = k_B T \frac{\partial \langle N \rangle}{\partial T} \text{ etc...}$$

(grand canonical)

[7]

Saddle point (Mean Field approximation): (with uniform field $B=0$)

$$\delta \left(S[m] - \int d\vec{r} R(\vec{r}) m(\vec{r}) \right) = 0 \Rightarrow -\nabla^2 m^*(\vec{r}) + 2t m^*(\vec{r}) + 4U m^3(\vec{r}) = R(\vec{r})T$$

\downarrow

From integration by parts

$$\langle m(\vec{r}) \rangle \approx m^*(\vec{r}) : G(\vec{r} - \vec{r}') = \frac{\delta m^*(\vec{r}')}{\delta R(\vec{r})} \Big|_{R=0} \Rightarrow (-\nabla^2 + \xi^{-2}) G(\vec{r}) = T \delta(\vec{r})$$

set $\vec{r}' = 0$

With $\xi^{-2} = 2t + 12Um^2$: $1/(\text{length})^2$ by dimensional analysis! (†)

We see that $G(\vec{r})$ is the Green's function of the operator $(-\nabla^2 + \xi^{-2})$:

$$\boxed{\frac{G(\vec{r} - \vec{r}')}{T} = \int \frac{d^d k}{(2\pi)^d} \frac{e^{-i\vec{k} \cdot (\vec{r} - \vec{r}')}}{|\vec{k}| < \Lambda \quad K^2 + \xi^{-2}}}$$

$K =$ wave vector or momentum
 [not important here, but will be crucial later]

Even though we're using a continuum description, we have to remember that we started from a lattice model: $|\vec{k}| < \Lambda \sim 1/a$
 UV cutoff

solve (†) by Fourier Transform

Let $\eta = |\vec{r} - \vec{r}'|$. If $\xi^{-2} = 0$ (or if $\eta \ll \xi \Rightarrow K \gg \xi^{-1}$)

$$\text{We have } G(\vec{r}) = T \int \frac{d^d k}{(2\pi)^d} \frac{e^{-i\vec{k} \cdot \vec{r}}}{K^2} \sim \eta^{-(d-2)}$$

$$\Rightarrow G(\vec{r}) \sim \frac{1}{\eta^{d-2}} \Rightarrow \eta = 0$$

(This also follows from dimensional analysis since $[m] = L^{-(d-2)/2} \sqrt{T}$)

For $\eta \gg \xi$, one can show that

$$G(\vec{r}) \approx e^{-\eta/\xi}$$

as expected
 for $\eta \gg \xi$ up to subleading (power law) terms

This is known as the Ornstein-Zernike correlation.

We thus identify: $\xi = \frac{1}{\sqrt{2t + 12Um^{*2}}}$ as the correlation length

with $m^* = 0$ if $t > 0$ ($T > T_c$) and $m^{*2} = -\frac{a_0 t}{2U}$

$$\xi \sim \frac{1}{\sqrt{|T - T_c|}}$$

$$\Rightarrow \gamma = 1/2$$

D) Fluctuations and upper-critical dimension

Mean-Field theory predicts the critical exponents:

$$\alpha = 0, \beta = 1/2, \delta = 3, \gamma = 1$$

$$\eta = 0, \nu = 1/2$$

Unfortunately, this disagrees with experiments and numerical simulations in dimensions $d=2$ and $d=3$, but is quantitatively correct for $d > 4$!

\downarrow upper critical dimension

D.I) Ginzburg Criterion: recall that we supposed that $Z = \int dm e^{-\beta F[m]}$

is dominated by a saddle point $m^* = 0$ for $t > 0$, $m = \pm m^* \neq 0$ for $t < 0$.

This is OK if the fluctuations $\delta m = m(\vec{r}) - \langle m \rangle = m(\vec{r}) - \langle m \rangle$ are small.

In other words, we require: $\langle \delta m^2 \rangle \ll \langle m \rangle^2$

This can be estimated as follows: $\langle \delta m(\vec{r}) \delta m(0) \rangle = G(\vec{r})$ decays over a distance ξ

\Rightarrow integrate fluctuations over ball of radius ξ :

Ginzburg ratio:

(for $t < 0$)

$$R = \frac{\int_0^\xi d\vec{r} \langle \delta m(\vec{r}) \delta m(0) \rangle}{\int_0^\xi d\vec{r} \langle m(\vec{r}) \rangle^2}$$

$$= (m^*)^2$$

$$\sim \frac{1}{\xi^d m^{*2}} \int_0^\xi d\eta \eta^{d-1} / \eta^{d-2}$$

$$\sim \xi^{2-d} / (m^*)^2$$

9

Mean-Field theory is self-consistent if $R \ll 1$. It predicts $m^* \sim |T - T_c|^\alpha$

$$\Rightarrow R \sim |T - T_c|^{\frac{d-4}{2}}$$

$d < 4$: R diverges as $T \rightarrow T_c$
 Mean-Field predicts its own failure!
 $d > 4$: $R \rightarrow 0$ as $T \rightarrow T_c$: expect MF to be correct (it is!)

$d_c = 4$ is called the upper critical dimension of the Ising model universality class. (Remark: $d=4$ is a marginal case and has to be treated separately)

(D.II) Fluctuations around the saddle point

We can make this argument more precise by studying the fluctuations around the saddle point. Let $\delta m = m - \underbrace{\langle m \rangle}_{m^*}$ and expand $S = \beta \int d\vec{r} \left[\frac{(\nabla m)^2}{2} + V(m) \right]$

$$\begin{aligned}
 S[m] &= \beta \int d\vec{r} \left[(m^* + \delta m) \left(-\frac{\nabla^2}{2} \right) (m^* + \delta m) + V(m^*) + \delta m \underbrace{V'(m^*)}_{\substack{\text{integration by part} \\ \text{By definition of } m^*}} + \frac{(\delta m)^2}{2} V''(m^*) \right] \\
 &= S[m^*] + \beta \int d\vec{r} \left[\underbrace{\frac{(\nabla \delta m)^2}{2}}_{\substack{\text{integration by part} \\ \text{By definition of } m^*}} + \frac{\delta m(\vec{r})^2}{2} V''(m^*) \right] + \dots \\
 &= \frac{1}{2} \int d\vec{r} d\vec{r}' \delta m(\vec{r}) G^{-1}(\vec{r} - \vec{r}') \delta m(\vec{r})
 \end{aligned}$$

with $G^{-1}(\vec{r} - \vec{r}') = \frac{\delta S}{\delta m(\vec{r}') \delta m(\vec{r})} \Big|_{m=m^*} = \beta \delta(\vec{r} - \vec{r}') \left[-\nabla^2 + V''(m^*) \right]$

Although the functional derivatives make this calculation look complicated, this is basically just a Taylor expansion!

$$\Rightarrow Z \underset{\substack{\uparrow \\ \text{Gaussian approximation}}}{\approx} e^{-S[m^*]} \int D[\delta m] e^{-\frac{1}{2} \int d\vec{r} d\vec{r}' \delta m(\vec{r}') G^{-1}(\vec{r} - \vec{r}') \delta m(\vec{r})}$$

\uparrow saddle point \uparrow Gaussian fluctuations

Within this approximation, the path integral is Gaussian so we can compute it! (see appendix). This yields:

$$\langle \delta m(\vec{r}) \delta m(\vec{r}') \rangle = G(\vec{r} - \vec{r}')$$

↪ consistent with the calculation in section C

$$\text{and } Z \simeq e^{-S[m^*]} \left(\text{Det}[G^{-1}] \right)^{-1/2}$$

↑ ignore 2π factors here

Here, we should think of G^{-1} as an operator ("generalized matrix" $G_{\vec{r}, \vec{r}'}^{-1}$) and $\text{Det } G^{-1} = \prod_K \lambda_K$ with λ_K the eigenvalues of G^{-1} . G^{-1} is diagonal in Fourier space with $\lambda_{\vec{k}} = \beta(k^2 + \xi^{-2})$ so that:

$$F = -T \log Z = F[m^*] + \frac{T}{2} \sum_{\vec{k}} \log \left(\frac{k^2 + \xi^{-2}}{T} \right)$$

mean-field result

contribution due to Gaussian fluctuations around the saddle point.

$$\xi^{-2} = 2T + 92 \nu [m^*]^2$$

$$= A_{\pm} H \quad \begin{pmatrix} A_+ = 2 \\ A_- = 4 \end{pmatrix}$$

Let's compute the corrections to the specific heat:

$$C_V = -\frac{T}{V} \frac{\partial^2 F}{\partial T^2} = C_{\text{Mean Field}} + \frac{1}{2} \int_{|\vec{K}| < \Lambda} \frac{d^d K}{(2\pi)^d} \left[1 - \frac{A_{\pm} T}{T_c} \frac{2}{k^2 + \xi^{-2}} + \left(\frac{A_{\pm}}{T_c} \right)^2 \frac{T^2}{(k^2 + \xi^{-2})^2} \right]$$

we have an underlying lattice: $K \in$ Brillouin zone

3 contributions

- $\int_{-\Lambda}^{\Lambda} \frac{d^d K}{(2\pi)^d} \frac{1}{2}$: "equipartition", just a finite number for us
 - $\int_{-\Lambda}^{\Lambda} \frac{d^d K}{k^2 + \xi^{-2}} \sim \int_0^{\Lambda} dk \frac{k^{d-1}}{k^2 + \xi^{-2}} = \begin{cases} \sim \Lambda^{d-2} & \text{if } d \geq 2 \quad (\log \Lambda \text{ if } d=2) \\ \sim \int_0^{\infty} \frac{dk}{k^2 + \xi^{-2}} \underset{k \rightarrow \xi^{-1}}{\sim} \xi & \text{if } d=1 \end{cases}$
 - $\int_0^{\Lambda} dk \frac{k^{d-1}}{(k^2 + \xi^{-2})^2} = \begin{cases} \int_0^{\Lambda} dk k^{d-5} \sim \Lambda^{d-4} & \text{if } d \geq 4 \\ \int_0^{\infty} \frac{dk}{(k^2 + \xi^{-2})^2} \sim \xi^{4-d} & \text{if } d < 4 \end{cases}$
- if $d > 4$, finite correction to C_V (UV divergent but finite Λ for us)
- if $d < 4$: $C_V \sim C_V^{\text{MF}} + C \xi^{4-d}$

Using the mean field prediction $\xi \sim |T - T_c|^{-\frac{1}{d_2}}$, we find:

$$C_V = C_V^{\text{Mean-Field}} + \text{Cst} |T - T_c|^{-(2-d_2)} \quad \text{if } d < 4$$

diverges as $T \rightarrow T_c$: more singular than MF prediction! Prediction $\kappa = 2 - d_2$ not reliable, but this tells us that MF fails!

Ultimately, this divergence came from the IR (large distance, small k) behavior of the integrals. This is a general feature: $\int dk K^{d-s}$ IR divergent if $d < d_c = 4$.
Also in perturbation theory below $d_c = 4 \Rightarrow$ need the renormalization group.
For $d > d_c = 4$, there are UV [divergences _{singularities}] in the integrals, but this is not a problem in statistical physics because we have a cutoff $\Lambda \sim \frac{1}{a}$ and mean-field results apply. (\neq in high energy/particle physics! Theory with $d > d_c$ is called non-renormalizable = "problematic")

(E) Lower critical dimension

For $d \leq d_c = 4$, we've seen that fluctuations are important and modify the critical behavior predicted by mean-field theory. It turns out that if $d \leq d_p$ (Lower critical dimension), fluctuations can destroy order: no phase transition at all!

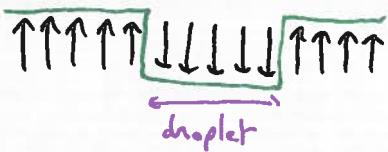
(E.g.: in $d=1$, the Ising model doesn't have a phase transition)
 \hookrightarrow exact solution by transfer matrix calculation

(E.I) Discrete symmetries and Peierls' argument

Lower critical dimension for systems with discrete symmetries (e.g. Ising $G = \mathbb{Z}_2$)
 $s_i \rightarrow -s_i$

$$d_p = 1$$

d=1: Ising model Box concreteness, $T=0$: all spins aligned



Introduce "droplet" with 2 domain walls. Energy cost:

$$\Delta E = 2 \times 2J \text{ independent of the size of the droplet}$$

Degeneracy $\sim N^2$ (each Domain wall can occupy $\mathcal{O}(N)$ positions)

\Rightarrow Free energy:

$$F \sim 4J - 2K_B T \log N$$

$\rightarrow F < 0$ for any $T > 0$
as $N \rightarrow \infty$.

The system will prefer to create droplets of arbitrary sizes.

"Domain walls" proliferate

More precisely: $2n$ domain walls $\Rightarrow \Delta E = 4J_n$

$$S = K_B \log \binom{N}{2n} = K_B [N \log N - 2n \log 2n - (N - 2n) \log (N - 2n)]$$

$$\frac{\partial F}{\partial n} = 0 \Rightarrow \frac{4J}{K_B T} = 2 \log \left[\frac{N-2n}{2n} \right] \Rightarrow \frac{n}{N} \sim \frac{1}{2} e^{-2J/K_B T}$$

density of DWs
 $\neq 0$ for any $T > 0$
No Order!

d=2: ΔE scales with perimeter of droplet, entropy doesn't always win.
Exact solution in $d=2$ (Onsager) \Rightarrow phase transition

E.II Continuous symmetries: Goldstone modes

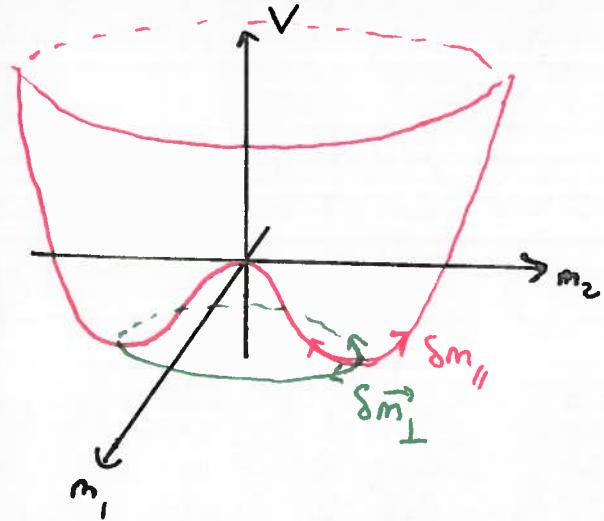
Consider now continuous spins: $H = -J \sum_{(i,j)} \vec{s}_i \cdot \vec{s}_j$ \vec{s}_i : n -component vector
 $O(n)$ symmetry, $\vec{s}_i^2 = 1$
 $\vec{s}_i \rightarrow R \vec{s}_i$, $R \in O(n)$

Ginzburg Landau: order parameter $\vec{m} = \langle \vec{s}_i \rangle$

$$F = \int d^d \eta \left[\frac{1}{2} \sum_{\alpha=1}^n (\vec{\nabla} m_{\alpha})^2 + T \underbrace{\sum_{\alpha} m_{\alpha}^2}_{\vec{m}^2} + U \underbrace{\sum_{\alpha, \beta} m_{\alpha}^2 m_{\beta}^2}_{(\vec{m}^2)^2} + \dots \right]$$

Saddle point: $\langle \vec{m}_* \rangle^2 = -T/2U$ for $T < 0$: infinitely degenerate

Let $\vec{m}_* = \begin{pmatrix} m_* \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ and consider fluctuations $\vec{m} = \begin{pmatrix} m_* + \delta m_{||} \\ \delta m_{\perp} \\ \vdots \\ \delta m_{\perp}^{n-1} \end{pmatrix}$ $n-1$ transverse components



- There are two types of fluctuations: δm_{\parallel} and δm_{\perp} (transverse).

- From our previous analysis, we expect:

$$(-\nabla^2 + V''_{\parallel}(m^*)) \langle \delta m_{\parallel}(\vec{r}) \delta m_{\parallel}(\vec{r}') \rangle = T \delta(\vec{r})$$

$$(-\nabla^2 + V''_{\perp}(m^*)) \langle \delta m_{\perp}(\vec{r}) \delta m_{\perp}(\vec{r}') \rangle = T \delta(\vec{r})$$

with $V''_{\parallel}(m^*)$ finite and $\neq 0$ (curvature in longitudinal direction), but

$$V''_{\perp}(m^*) = 0 : \boxed{\xi_{\perp} = \infty}$$

"massless modes"
 $m = \xi = 0$

Goldstone modes ($n-1$ modes)
(also called spin waves) other examples:
phonons!

- We will show later on that ξ_{\perp} remains infinite beyond the Gaussian approximation.

- Explicitly: $(\nabla \vec{m})^2 = (\nabla \delta m_{\perp})^2 + (\nabla \delta m_{\parallel})^2$ and $\vec{m}^2 = m^{*2} + \delta m_{\parallel}^2 + \delta m_{\perp}^2 + 2 m^* \delta m_{\parallel} \delta m_{\perp}$
 $(\vec{m}^2)^2 = m^{*4} + 6 \delta m_{\parallel}^2 m^{*2} + 2 m^{*2} \delta m_{\perp}^2 + 4 \delta m_{\parallel} m^{*3} + \dots$

$$\Rightarrow V(m) = -\frac{F^2}{4U} + \frac{\delta m_{\parallel}^2}{2} \underbrace{(2t + 12U m^{*2})}_{\xi_{\parallel}^{-2} = 4/H} + \frac{\delta m_{\perp}^2}{2} \underbrace{(2t + 4m^{*2}U)}_{\xi_{\perp}^{-2} = 0} + \dots$$

Mermin-Wagner theorem: In the ordered phase, there are critical ("massless") fluctuations:

$$\langle \delta m_{\perp}^{\alpha}(\vec{r}) \delta m_{\perp}^{\beta}(\vec{r}') \rangle = \int \frac{d^d K}{(2\pi)^d} e^{-i \vec{K} \cdot \vec{r}} \delta_{\alpha\beta}$$

$\xi_{\perp}^{-2} = m^2$ in high energy;

$\alpha = 1, \dots, n-1$
 $\beta = 2$ Lower critical dimensions

Take $\vec{r} = 0$ but keep UV cutoff Λ :

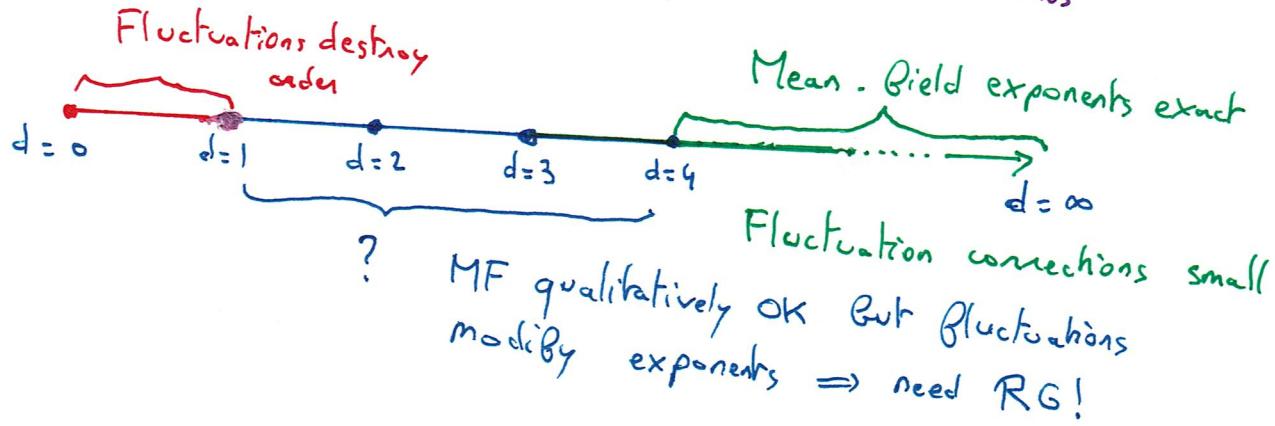
Integral IR divergent (at small K) in $d \leq 2$: Fluctuations diverge: no order!

Goldstone modes destroy the ordered phase if $d \leq d_p = 2$: $\langle \vec{s} \rangle = 0$

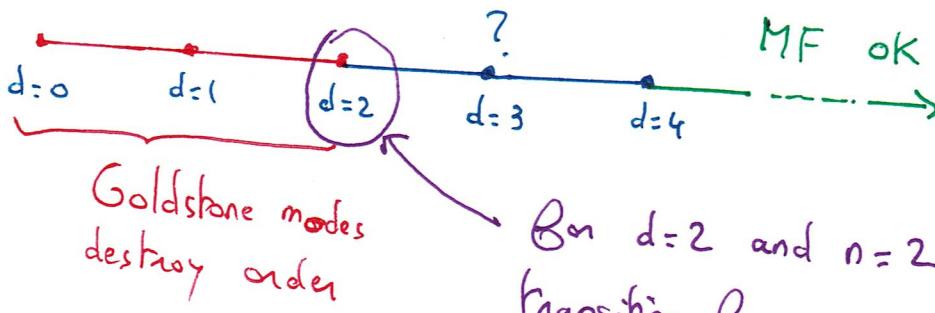
Summary:

Fluctuations can be crucial depending on the dimension d and on the order of the parameter space n (assuming $O(n)$ symmetry)

$n=1$ Discrete symmetry (Ising model), no Goldstone modes



$n \geq 2$ $n-1$ Goldstone modes



For $d=2$ and $n=2$, there is a special transition here: no ordered phase (topological phase transition)

Appendix: Gaussian Integrals

Our starting point is the integral: $\int_{-\infty}^{+\infty} d\phi e^{-\frac{K}{2}\phi^2 - iR\phi} = \sqrt{\frac{2\pi}{K}} e^{-R^2/2K}$

Let: $Z[R] = \langle e^{-iR\phi} \rangle = e^{-R^2/2K}$ with $\langle \dots \rangle = \frac{\int d\phi e^{-\frac{K}{2}\phi^2} (\dots)}{\int d\phi e^{-\frac{K}{2}\phi^2}}$

Then $\langle \phi^n \rangle = i^n \left. \frac{d^n}{dR^n} Z[R] \right|_{R=0}$ ($\log Z[R]$: generating functions of cumulants)

N variables: $\int \prod_{i=1}^N d\phi_i e^{-\sum_{i,j} \phi_i \frac{K_{ij}}{2} \phi_j - i \sum_i R_i \phi_i} = \frac{(2\pi)^{N/2}}{(\text{Det } K)^{1/2}} e^{-\sum_{i,j} R_i K_{ij}^{-1} R_j}$

This can be shown by diagonalizing K (Jacobian = 1) See HW

$$\Rightarrow Z[R] = \langle e^{-i \sum_{i=1}^N R_i \phi_i} \rangle = e^{-\sum_{i,j} R_i \frac{K_{ij}^{-1} R_j}{2}}$$

← For the measure $\propto e^{-\sum_{i,j} \phi_i K_{ij} \phi_j / 2}$

$$\langle \phi_i \phi_j \phi_k \dots \rangle = \frac{1}{(-i)} \frac{d}{dR_i} \frac{1}{(-i)} \frac{d}{dR_j} \frac{1}{(-i)} \frac{d}{dR_k} \dots Z[R]$$

$$\Rightarrow \langle \phi_i \rangle = 0$$

$$\boxed{\langle \phi_i \phi_j \rangle = K_{ij}^{-1}}$$

Higher order correlation functions: $\langle \phi_i \phi_j \phi_k \phi_l \rangle = \underbrace{\langle \phi_i \phi_j \rangle}_{\text{green bracket}} \underbrace{\langle \phi_k \phi_l \rangle}_{\text{green bracket}} + \underbrace{\langle \phi_i \phi_k \rangle}_{\text{red bracket}} \underbrace{\langle \phi_j \phi_l \rangle}_{\text{red bracket}} + \underbrace{\langle \phi_i \phi_l \rangle}_{\text{red bracket}} \underbrace{\langle \phi_j \phi_k \rangle}_{\text{red bracket}}$

This is known as Wick's theorem

Functional Gaussian integral: $\int D\phi e^{-\frac{1}{2} \int d\vec{n} d\vec{n}' \phi(\vec{n}) K(\vec{n}, \vec{n}') \phi(\vec{n}')} \propto (\text{Det } K)^{-1/2}$

$\int \prod_{\vec{x}} d\phi_{\vec{x}}$ Kernel

And by analogy with above: $\langle e^{-i \int d\vec{n} R(\vec{n}) \phi(\vec{n})} \rangle = e^{-\frac{1}{2} \int d\vec{n} d\vec{n}' R(\vec{n}) K^{-1}(\vec{n}, \vec{n}') R(\vec{n}')} \equiv Z[R]$

Here, the inverse Kernel K^{-1} is defined as:

$$\int d\vec{n} \cdot K(\vec{n}, \vec{n}') K^{-1}(\vec{n}', \vec{n}'') = \delta(\vec{n} - \vec{n}'')$$

We have: $\langle \phi(\vec{n}) \phi(\vec{n}') \rangle = - \frac{\delta}{\delta R(\vec{n})} \frac{\delta}{\delta R(\vec{n}')} Z[A] \Big|_{A=0}$

$$\Rightarrow \boxed{\langle \phi(\vec{n}) \phi(\vec{n}') \rangle = K^{-1}(\vec{n}, \vec{n}')}}$$

Wick's theorem can be used to compute higher order correlation functions.

In most cases: $K(\vec{n}, \vec{n}') = K(\vec{n} - \vec{n}')$ so K^{-1} can be computed using the Fourier transform: $\hat{K}(\vec{k}) = \int K(\vec{n} - \vec{n}') e^{i\vec{k} \cdot \vec{n}} d\vec{n} \rightarrow \text{diagonalizes } K$

$$= \frac{1}{\hat{K}^{-1}(\vec{k})}$$

$$\Rightarrow K^{-1}(\vec{n} - \vec{n}') = \int \frac{d^d k}{(2\pi)^d} \frac{e^{-i\vec{k} \cdot (\vec{n} - \vec{n}')}}{\hat{K}(\vec{k})}$$

Example: $Z = \int D\phi e^{-\frac{1}{2} \int d^d n [(\nabla\phi)^2 + \xi^{-2} \phi^2]} \Rightarrow K(\vec{n} - \vec{n}') = \delta(\vec{n} - \vec{n}') \left[-\nabla^2 + \xi^{-2} \right]$

K^{-1} satisfies: $(-\nabla^2 + \xi^{-2}) K^{-1}(\vec{n}) = \delta(\vec{n})$

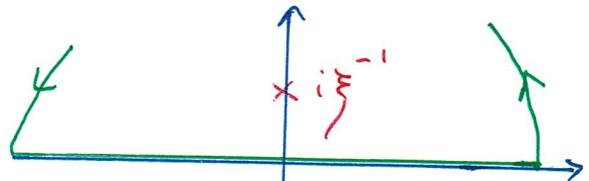
eigenvalues: $\vec{k}^2 + \xi^{-2}$
 $\text{Def } K = \prod_{\vec{k}} (\vec{k}^2 + \xi^{-2})$

$$\Rightarrow \boxed{\langle \phi(\vec{n}) \phi(0) \rangle = \int \frac{d^d k}{(2\pi)^d} \frac{e^{-i\vec{k} \cdot \vec{n}}}{\vec{k}^2 + \xi^{-2}}} \quad \begin{matrix} \text{Two-point} \\ \text{Function} \\ (\text{"propagator"}) \end{matrix}$$

We also have $Z = \left(\prod_{\vec{k}} (\vec{k}^2 + \xi^{-2}) \right)^{-1/2} \Rightarrow -\log Z = \frac{V}{2} \int \frac{d^d k}{(2\pi)^d} \log [\vec{k}^2 + \xi^{-2}]$
up to (infinite!) constant

Appendix : Green's Function in 3D

$$\begin{aligned}
 G(\vec{n}) &= \frac{1}{V} \int \frac{d^3 k}{(2\pi)^3} \frac{e^{-i\vec{k}\cdot\vec{n}}}{k^2 + \xi^{-2}} = \frac{1}{4\pi^2} \int_0^\pi d\theta \sin\theta \int_0^\infty dk \frac{k^2}{k^2 + \xi^{-2}} e^{-ikn \cos\theta} \\
 &= \frac{1}{4\pi^2} \int_0^\infty dk \frac{k}{k^2 + \xi^{-2}} \left(\frac{e^{ikn} - e^{-ikn}}{i n} \right) \\
 &= -\frac{i}{4\pi^2} \int_{-\infty}^{+\infty} \frac{dk}{n} \frac{k e^{ikn}}{k^2 + \xi^{-2}} \\
 &\quad \hookrightarrow k^2 + \xi^{-2} = (k + i\xi^{-1})(k - i\xi^{-1}) \\
 &= -\frac{i \times (2\pi i)}{4\pi^2 n} \frac{i\xi^{-1}}{2i\xi^{-1}} e^{-n/\xi} = \frac{1}{4\pi n} e^{-n/\xi}
 \end{aligned}$$



"Yukawa"
Screened
Coulomb
Potential

Remark: $\xi = \infty \Rightarrow \nabla^2 \left(-\frac{1}{4\pi n} \right) = \delta^{(3)}(\vec{n}) \quad \text{✓}$