

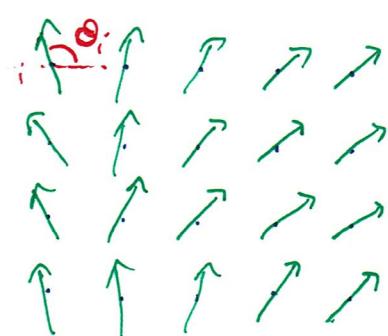
The 2D XY model and  
Topological phase transitions

Mermin-Wagner theorem: Lower critical dimension for  $O(n)$  model  $d_p = 2$

$\Rightarrow$  No phase transition and only disordered phase in 2D?

This chapter: not if  $n=2$ ! critical phase at low T (power-law  $\langle \vec{S}_i \cdot \vec{S}_j \rangle$  at low T) (Nobel Prize 2016)

(A) High and low temperature expansions



$$\vec{S}_i = \begin{pmatrix} \cos \theta_i \\ \sin \theta_i \end{pmatrix} \quad \vec{S}_i \cdot \vec{S}_j = \cos(\theta_i - \theta_j)$$

$$\Rightarrow H_{xy} = -J \sum_{\langle i,j \rangle} \cos(\theta_i - \theta_j)$$

$\theta \in [0, 2\pi)$

and  $Z = \int \prod_{i=1}^N \frac{d\theta_i}{2\pi} e^{-\beta H_{xy}}$  with  $K = \beta J$

High T ( $K \ll 1$ ):  $\langle \vec{S}_i \cdot \vec{S}_j \rangle = \langle \cos(\theta_i - \theta_j) \rangle = \frac{1}{Z} \int D\theta \cos(\theta_i - \theta_j) e^{-\beta H}$

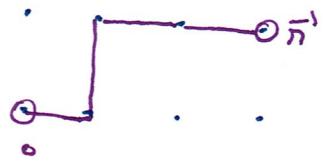
$$= \frac{1}{Z} \int D\theta \cos(\theta_i - \theta_j) \underbrace{\prod_{\langle i,j \rangle} \left[ 1 + K \cos(\theta_i - \theta_j) + O(K^2) \right]}_{\text{expand this product for } K \ll 1}$$

Graphical representation:

$$\begin{array}{c} \bullet \xrightarrow{\delta} \bullet \\ \vdots \qquad \qquad \qquad \bullet \xrightarrow{\delta} \bullet \\ \vdots \qquad \qquad \qquad \vdots \end{array} = K \cos(\theta_i - \theta_j)$$

$$\vdots \qquad \qquad \qquad \bullet \xrightarrow{\delta} \bullet = 1$$

$$\int_0^{2\pi} \frac{d\phi_i}{2\pi} \cos(\phi_i - \phi_j) = 0 \Rightarrow \text{need to connect } o \text{ and } \vec{n} \text{ with a path of } K \cos(\phi_i - \phi_j)$$



$$\text{Use: } \int_0^{2\pi} \frac{d\phi_2}{2\pi} \cos(\phi_1 - \phi_2) \cos(\phi_2 - \phi_3) = \frac{1}{2} \cos(\phi_1 - \phi_3)$$

$\Rightarrow$  each bond contributes a factor of  $K/2$

Final integral gives  $\int_0^{2\pi} \frac{d\phi_0 d\phi_n}{(2\pi)^2} (\cos(\phi_0 - \phi_n))^2 = \frac{1}{2}$

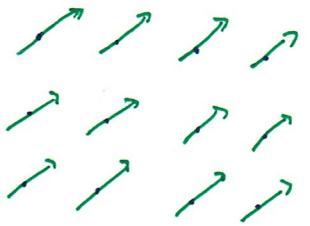
This gives

$$\langle \vec{S}_0 \cdot \vec{S}_n \rangle \approx \left(\frac{K}{2}\right)^n = e^{-n/\xi} \quad \text{with } \xi = \frac{1}{\ln(2/K)}$$

assume shortest path is unique

$\rightarrow$  disordered R/T phase with exponentially decaying correlations

Low T ( $K \gg 1$ ): infinitely degenerate GS: all spins aligned



Look at small fluctuations around GS: spin waves  
( $=$  Goldstone modes)

$$-\beta \mathcal{H}_{xy} \approx K \sum_{\langle i,j \rangle} \left( 1 - \frac{(\phi_i - \phi_j)^2}{2} + \dots \right) \approx -\frac{K}{2} \int d^2x (\nabla \phi)^2 + \dots$$

so that  $\langle \vec{S}_0 \cdot \vec{S}_n \rangle = \text{Re} \langle e^{i(\phi_0 - \phi_n)} \rangle = \text{Re} e^{-\frac{1}{2} \langle (\phi_0 - \phi_n)^2 \rangle}$

$\square$  Gaussian action ( $\phi \in \mathbb{R}$  in this limit)

$$S = \frac{K}{2} \int d^2x (\nabla \phi)^2 = \frac{K}{2} \int d^2x \phi (-\nabla^2) \phi \Rightarrow -K \nabla^2 \langle \phi(n) \phi(0) \rangle = \delta(n)$$

$\square$  here  $K$  can't be absorbed in a redefinition of  $\phi$ ,  $\phi$  is an angle!

$$\Rightarrow \langle \phi(n) \phi(0) \rangle = -\frac{1}{2\pi K} \log n = G(n)$$

$$G(0) = -\frac{1}{2\pi K} \log \frac{a}{\square} \quad \begin{matrix} \square \\ \text{IR cutoff} \end{matrix} \quad \begin{matrix} \square \\ \text{UV cutoff} \end{matrix}$$

$$\Rightarrow \langle \vec{S}_0 \cdot \vec{S}_n \rangle \approx e^{G(n) - G(0)} = \left(\frac{a}{n}\right)^{1/2\pi K}$$

L3

Remark:  $G(0) = \frac{1}{K} \int \frac{dp^2}{(2\pi)^2} \frac{1}{p^2} = \frac{1}{2\pi K} \int_{2\pi/L}^{2\pi/a} \frac{dp}{p} = \frac{1}{2\pi K} \log \frac{L}{a}$  as claimed

$$g = \alpha^2 \tilde{g}$$

$$\frac{dg}{dp} = -2\tilde{g} + \dots$$

What about corrections to the Gaussian action?

$O(2) = U(1)$  symmetry  $\phi \rightarrow \phi + \alpha$ : no  $\phi^2, \phi^4, \dots$  terms!

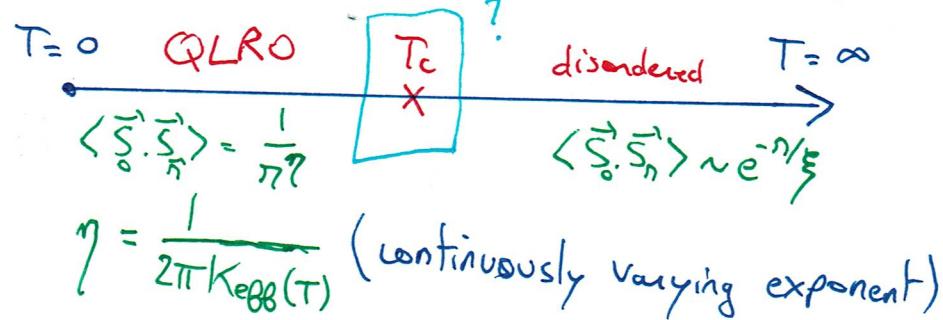
$S = \frac{K}{2} \int dx (\nabla \phi)^2 + g \int dx (\nabla \phi)^4 + \dots$

$K = \text{marginal}$  irrelevant

$\Rightarrow$  stable low T phase with algebraic correlations: Quasi Long Range Order (QLRO)

interactions only renormalize  $K \rightarrow K_{\text{eff}}$  (stiffness)

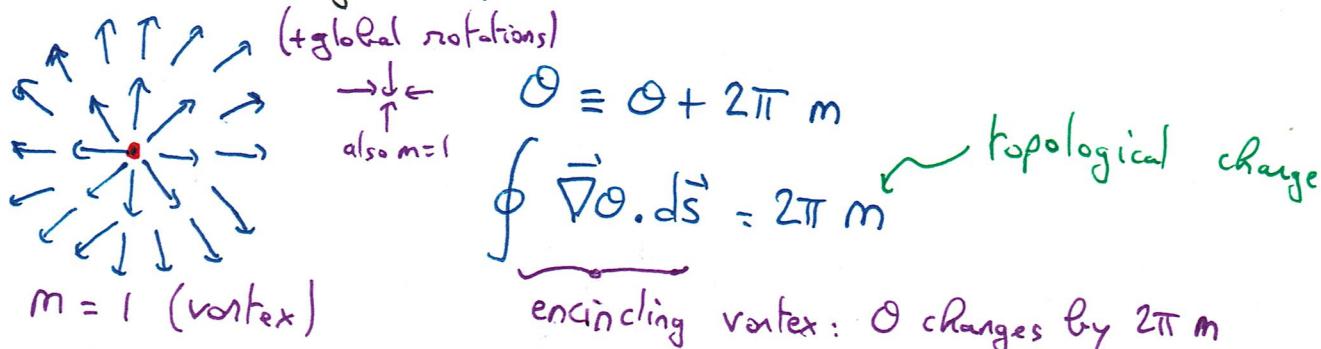
Very specific to  $n=2$ : For  $n \geq 3$ , the  $\neq$  Goldstone mode "branches" interact and lead to a disordered phase (cf. non-linear sigma model)

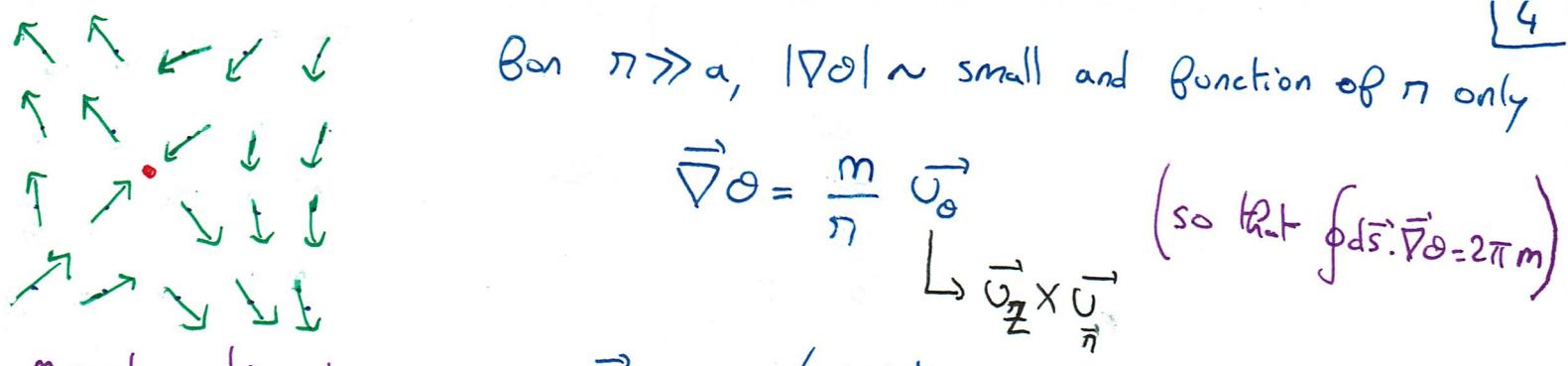


## (B) Topological defects: vortices

- How do we go from QLRO to a disordered phase?
- Perturbative RG: Higher order terms in gradient expansion irrelevant
- BUT: we assumed  $O(n)$  smooth and forgot that  $\phi \in [0, 2\pi]$

$n=2$  and  $d=2$ : Topological defects that cannot be continuously deformed to the uniform configuration!





For  $n \gg a$ ,  $|\nabla\phi| \sim$  small and function of  $n$  only 4

$$\vec{\nabla}\phi = \frac{m}{n} \vec{U}_\phi \quad \left( \text{so that } \oint d\vec{s} \cdot \vec{\nabla}\phi = 2\pi m \right)$$

$\downarrow \vec{U}_z \times \vec{U}_{\frac{n}{m}}$

$m = -1$  anti-vortex

$$\Rightarrow \vec{\nabla}\phi = \frac{m}{n} \begin{pmatrix} -\gamma/n \\ \gamma/n \end{pmatrix} = -m \vec{\nabla}x (\log n \vec{U}_z)$$

- Note: continuum approximation fails near "core" of vortex (lattice important)

- Energy cost (action) of a vortex:  $S_m = S_{\text{core}}^m(a) + \frac{K}{2} \int_{\gamma > a} dx (\nabla\phi)^2$

$\downarrow$  lattice dependent

$$\Rightarrow S_m = S_{\text{core}}^m(a) + \frac{K}{2} \int_a^L 2\pi n dn \frac{m^2}{n^2} = S_{\text{core}}^m(a) + \underbrace{\pi K m^2 \log\left(\frac{L}{a}\right)}_{\text{large energy cost}}$$

$\downarrow$  single! (see below)

as  $T \rightarrow 0$ : Vortices can't spontaneously form.

- Single vortex with charge  $m$ :  $Z_1(m) \approx \left(\frac{L}{a}\right)^2 e^{-S_m}$  action:  $\beta E_m^{2-\pi K m^2}$

$\uparrow$  configurational entropy

- $K$  large ( $T \rightarrow 0$ ): energy dominates,  $Z_1(m) \xrightarrow[L \rightarrow \infty]{} 0$ , single-vortex configurations have negligible weight

- $K < K_c = \frac{2}{\pi m^2}$ : entropy favors vortex formation!

First vortices:  $m = \pm 1$

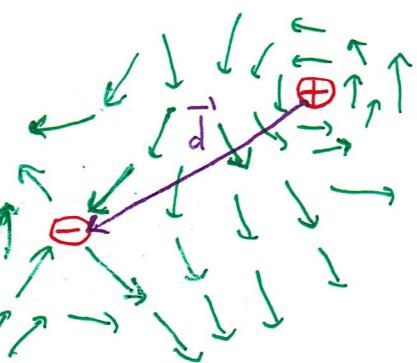
$$K_c = \frac{2}{\pi}$$

Kosterlitz-Thouless transition

$K < K_c$ : vortices proliferate

exact(!), see below

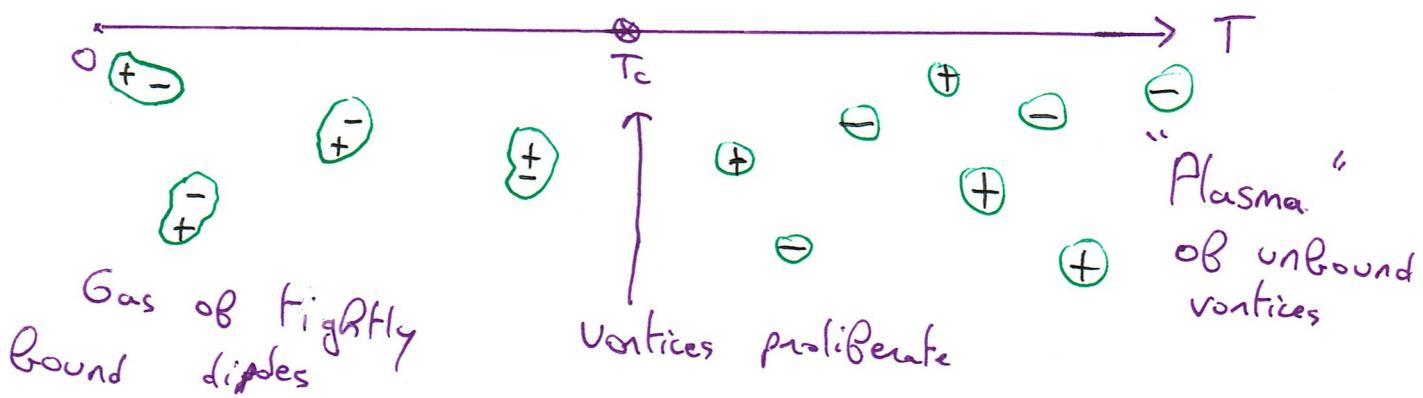
• In fact, tightly bound pairs of vortices (dipoles) appear at any  $T > 0$ : 5



$$\begin{aligned}\vec{\nabla}\phi_{\text{dipole}} &= \vec{\nabla}\phi_+(\vec{r} - \frac{\vec{d}}{2}) + \vec{\nabla}\phi_-(\vec{r} + \frac{\vec{d}}{2}) \\ &= -\vec{\nabla} \times \left[ \vec{v}_z \underbrace{\log\left(\frac{|\vec{r}-\vec{d}_z|}{|\vec{r}+\vec{d}_z|}\right)}_{\approx \log(1 - \frac{\vec{d} \cdot \vec{r}}{\pi^2})} \right] \\ &= \vec{\nabla} \times \left[ \vec{v}_z \frac{\vec{d} \cdot \vec{r}}{\pi^2} \right] \sim \frac{d}{\pi^2} \text{ as } r \rightarrow \infty\end{aligned}$$

$$\Rightarrow \int (\nabla\phi)^2 d^2r \sim \int \frac{d^2r}{\pi^4}$$

as  $r \rightarrow \infty$ : Finite energy cost for bound pairs vortex-antivortex.  
↳ Relative Boltzmann weight



### C Coulomb Gas Formulation and Sine-Gordon Theory

Need to make this picture precise: interactions between vortices?

Write:  $\vec{\nabla}\phi = \underbrace{\vec{\nabla}\phi}_{\text{Vortices}} - \vec{\nabla} \times (\vec{v}_z \psi)$

$\left. \begin{array}{l} \text{Solv. Waves} \\ \text{Vorticity free: } \vec{\nabla} \times (\vec{\nabla}\phi) = 0 \end{array} \right\}$

equation of motion  $\vec{\nabla}^2\phi = 0$

$\left. \begin{array}{l} \text{Stokes' theorem} \\ \oint d\vec{s} \cdot \vec{\nabla}\phi = \int_{\Sigma} d\vec{x} \vec{v}_z \cdot \underbrace{(\vec{\nabla} \times \vec{\nabla}\phi)}_{\vec{v}_z \vec{\nabla}^2\psi} \end{array} \right\}$

$\vec{\nabla} \times (\vec{\nabla}\phi) = (\vec{\nabla}^2\psi) \vec{v}_z$

For a single vortex, we saw  $\Psi = m \log r \Rightarrow$  by superposition

$$\Psi(\vec{x}) = \sum_i m_i \log |\vec{x} - \vec{x}_i|$$

$$\nabla^2 \Psi = 2\pi \sum_i m_i \delta^{(2)}(\vec{x} - \vec{x}_i)$$

potential due to set of vortices  $\{m_i\}$

$m_i$  = topological charge

$$E_i = E_{ij} \partial_j \phi \Rightarrow E = \nabla \Psi$$

$$\nabla \times E = 0 \quad (\nabla^2 \phi = 0)$$

$$\nabla \cdot E = \nabla^2 \Psi = P_{\text{vortices}}$$

$$(\vec{\nabla} \Psi)^2$$

$$\underbrace{\qquad\qquad\qquad}_{(\vec{\nabla} \Psi)^2}$$

$$\underbrace{\qquad\qquad\qquad}_{\text{integration by parts}}$$

$$= -\phi \vec{\nabla} \cdot (\vec{\nabla} \times \vec{v}_z \Psi) = 0$$

$\Rightarrow \phi$  and  $\Psi$  decouple!

and

$$\begin{aligned} \frac{K}{2} \int d\vec{x} (\vec{\nabla} \Psi)^2 &= -\frac{K}{2} \int d\vec{x} \Psi \nabla^2 \Psi = -\frac{K}{2} \int d\vec{x} \left( \sum_i m_i \log |\vec{x} - \vec{x}_i| \right) \times \\ &\quad \left( 2\pi \sum_j m_j \delta(\vec{x} - \vec{x}_j) \right) \\ &= -2\pi^2 \sum_{i,j} m_i m_j K \underbrace{\frac{\log |\vec{x}_i - \vec{x}_j|}{2\pi}}_{C(\vec{x}_i - \vec{x}_j)} \end{aligned}$$

$C(\vec{x}_i - \vec{x}_j)$  = Coulomb potential in 2d

$$\sum_{i,j} = 2 \sum_{i < j} + \sum_{i=j} : i=j \text{ terms diverge, again this is an artifact of the continuum picture that breaks down near cores.} \Rightarrow \text{include "core energy" (unknown)}$$

$$\Rightarrow \beta H = \underbrace{\frac{K}{2} \int d\vec{x} (\nabla \phi)^2}_{\text{Spin waves}} + \sum_i \beta \epsilon_m^o - 4\pi^2 \underbrace{\sum_{i < j} m_i m_j C(\vec{x}_i - \vec{x}_j) K}_{\text{vortices interact via Coulomb interactions!}}$$

$$\text{microstates } \{\phi, \vec{v}\} \rightarrow \begin{cases} \text{spin waves } \phi(\vec{x}) \\ \text{charges } \{m_i\} + \text{location of vortices} \end{cases}$$

$$Z_{xy} \simeq \underbrace{\int D\phi(\vec{x}) e^{-\frac{K}{2} \int d^2x (\nabla\phi)^2}}_{Z_{SW} \text{ spin waves}} \underbrace{\sum_{\{m_i\}} \int \prod_{i=1}^N dx_i e^{-\sum_i \beta_i E_m^0 + 4\pi^2 K \sum_{i,j} C(\vec{x}_i - \vec{x}_j) m_i m_j}}_{Z_Q}$$

$Z_Q$ : Vortices = grand canonical gas of charged particles with 2D Coulomb interactions

- Moreover:
- vortices come in pairs  $+,-$ :  $\sum_i m_i = 0$  (neutral gas)
  - we can restrict to  $m_i = \pm 1$  (higher order vortices irrelevant)  
in the RG sense: see below
- set  $e^{-\beta E_m^0} = \gamma$ : "Fugacity"

$$Z_Q = \sum_N \frac{1}{(\frac{N}{2})!^2} \int \prod_{i=1}^N dx_i \gamma^N e^{4\pi^2 K \sum_{i,j} m_i m_j C(\vec{x}_i - \vec{x}_j)}$$

$\uparrow \quad \uparrow$

$m_i = \pm 1$        $|\vec{x}_i - \vec{x}_j| > \text{a } \sim \text{ vortex core}$       Coulomb Gas  
 $\sum_i m_i = 0$       permutations of  $+$  and  $-$  charges

"Sine-Gordon Theory": Because  $Z_{SW}$  is analytic, we can focus on  $Z_Q$

- It is useful to reformulate  $Z_Q$  in terms of a Field Theory
- Claim:  $S_{SG} = \int d^2x \left[ \frac{K_{\text{dual}}}{2} (\nabla\varphi)^2 - g \cos\varphi \right]$   $\varphi$ : angle  
 $\hookrightarrow$  "dual" angle field

Perturbation theory in  $g$ :  $g \cos\varphi = g \frac{e^{i\varphi} + e^{-i\varphi}}{2}$

$$Z_{SG} = \int D\varphi e^{-S_{SG}} = \int D\varphi e^{-S_{SG}^0} \sum_{N=0}^{\infty} \left( \frac{\int d^2x \cos\varphi}{N!} \right)^N$$

$$\Rightarrow Z_{SG} = Z_{SG}^0 \left( \sum_{N=0}^{\infty} \frac{(g/2)^N}{N!} \int \prod_{i=1}^N dx_i \sum_{\{m_i = \pm 1\}} \langle e^{i \sum_{i=1}^N m_i \varphi(\vec{x}_i)} \rangle_0 \right)$$

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$$\text{Here : } \langle \dots \rangle_0 = \frac{1}{Z_{SG}^0} \int D\varphi e^{-\underbrace{\int d\vec{x} \frac{K_{dual}}{2} (\nabla \varphi)^2}_{S_0}} (\dots)$$

• Because of the symmetry  $\varphi \rightarrow \varphi + \delta$  of  $S_0$ , we have  $\langle e^{i \sum_j m_j \cdot \varphi_j} \rangle_0 = e^{i \sum_j m_j \delta} \times \langle e^{i \sum_j m_j \cdot \varphi_j} \rangle_0$

$\Rightarrow \boxed{\sum_{j=1}^N m_j = 0}$

• Also, since  $S_0$  is Gaussian:  $\langle e^{i \sum_j m_j \cdot \varphi_j} \rangle_0 = e^{-\frac{1}{2} \sum_{\alpha, \beta} m_\alpha m_\beta \langle \varphi(\vec{x}_\alpha) \varphi(\vec{x}_\beta) \rangle_0}$   
 with  $\langle \varphi(\vec{x}_\alpha) \varphi(\vec{x}_\beta) \rangle_0 = -\frac{1}{2\pi K_{dual}} \log |\vec{x}_\alpha - \vec{x}_\beta|$  (cf. High T expansion)

$\Rightarrow Z_{SG}^0 = Z_Q$  with  $Bogoliubov \gamma = \frac{g}{2}$  and  $\frac{1}{K_{dual}} = 4\pi^2 K$

$\binom{N}{N/2}$  identical terms in expansion:  $\frac{1}{N!} \binom{N}{N/2} = \frac{1}{(N/2)!^2}$  ✓  $\quad (\alpha = \beta \text{ terms can be absorbed in a redefinition of } \alpha)$

$\Rightarrow S_{SG} = \int d^2x \left[ \frac{1}{8\pi^2 K} (\nabla \varphi)^2 - \gamma (e^{i\varphi} + e^{-i\varphi}) \right]$  Sine Gordon Theory

$\uparrow \quad \uparrow$   
 $m=+1 \quad m=-1$   
 vortex antivortex

- In this "dual" language, vortex = local field  $V_m = e^{im\varphi}$  !  $m = \text{charge}$
- $Z_{SG}^0 = \int D\varphi e^{-\frac{1}{8\pi K^2} \int d^2x (\nabla \varphi)^2} = \text{Det} \left[ -\frac{1}{4\pi K^2} \nabla^2 \right] = \left( \frac{1}{4\pi K^2} \int d^2x \right)^{-1/2}$  Unimportant, smooth function of  $K$
- $\cos(m\varphi)$  can be interpreted as a "field" for  $\varphi$  that favors  $m\varphi = 2\pi K$  with  $K = 0, 1, \dots, m-1$ .

## D Renormalization Group analysis

. Using the sine-Gordon formulation, we can compute the two-point function of the vertex operator  $V_m = e^{im\varphi}$ : (at the fixed point  $\gamma = 0$ )

$$\langle e^{im\varphi(\vec{x})} e^{-im\varphi(\vec{y})} \rangle_0 = e^{-\frac{m^2}{2} \underbrace{\langle (\varphi(\vec{x}) - \varphi(\vec{y}))^2 \rangle_0}_{2\pi K \log |\vec{x} - \vec{y}|}} = \left( \frac{a}{|\vec{x} - \vec{y}|} \right)^{2\pi K m^2}$$

$\Rightarrow$  Scaling dimension  $\Delta_m = \pi K m^2$

- ✓ relevant if  $\pi K m^2 < d = 2$
- ✗ irrelevant if  $\pi K m^2 > 2$

$\Rightarrow$  Recover result from our hand-wavy argument!

Focus on  $m=1$   
( $m>1$  less relevant)

$$\boxed{\frac{dy}{d\rho} = (2 - \pi K)y}$$

$y$  flows to 0 (irrelevant)  
if  $K > K_c = \frac{2}{\pi}$

. From our OPE perspective,  $Ky$  term indicates a nontrivial OPE coefficient between  $V_m, V_{-m}$  and the energy operator  $(\nabla\varphi)^2$ . Expect  $y^2$  term in  $\frac{dK}{d\rho}$  (recall that without vortices,  $\frac{dK}{d\rho} = 0$ : exactly marginal)

. Here, follow the Coulomb Gas picture

### Effective interaction between test charges:

Compute  $\langle e^{i\varphi(\vec{x})} e^{-i\varphi(\vec{x}')}\rangle$  perturbatively in  $y$ :

Expect:

- if  $K > K_c$ ,  $\langle e^{i\varphi(\vec{x})} e^{-i\varphi(\vec{x}')}\rangle = \left( \frac{a}{|\vec{x} - \vec{x}'|} \right)^{2\pi K_{\text{eff}}}$
- if  $K < K_c$ : perturbation theory breaks down, plasma phase  
 $\Rightarrow \sim e^{-|\vec{x} - \vec{x}'|/\xi}$  screening of Coulomb interactions

[ effective Coulomb interaction, renormalized by  $y$  ]

$$\langle \underline{V(\vec{x}, \vec{x}')} \rangle = \frac{\langle V(\vec{x}, \vec{x}') \rangle_0 + \gamma^2 \int d\vec{y} d\vec{y}' \langle V(\vec{x}, \vec{x}') V(\vec{y}, \vec{y}') \rangle_0 + \dots}{1 + \gamma^2 \int d\vec{y} d\vec{y}' \langle V(\vec{y}, \vec{y}') \rangle_0 + \dots}$$

$\underline{\quad}$

•  $\mathcal{O}(\gamma)$  terms = 0 by neutrality condition

$$\langle V(\vec{x}, \vec{x}') V(\vec{y}, \vec{y}') \rangle_0 = \langle e^{i\varphi(\vec{x}) - i\varphi(\vec{x}')} e^{i\varphi(\vec{y}) - i\varphi(\vec{y}')} \rangle_0 = e^{-\sum_{\alpha<\beta} \underbrace{\langle \varphi_\alpha \varphi_\beta \rangle_0}_{\alpha=1,2,3,4} m_\alpha m_\beta} \frac{4\pi^2 K C_{\alpha\beta}}$$

↳ six terms:  $(y, y'), (x, x'), (y, x'), (y', x)$   
 $(x, y), (x', y')$

$$(m_x, m_y) = (+, -) \text{ or } (-, +)$$

$$(m_x, m_y) = (++) ; (-, -)$$

$$\Rightarrow \langle V(\vec{x}, \vec{x}') \rangle = \underbrace{\langle V(\vec{x}, \vec{x}') \rangle_0}_{e^{-4\pi^2 K C(\vec{x}-\vec{x}')}} \left[ 1 + \gamma^2 \int d\vec{y} d\vec{y}' e^{-4\pi^2 K C(\vec{y}-\vec{y}')} \underbrace{\left( e^{4\pi^2 K D(x, x', y, y')} - 1 \right)}_{\text{internal attraction}} + \dots \right]$$

$$\begin{aligned} \text{with } D(x, x', y, y') &= C(\vec{x} - \vec{y}) + C(\vec{x}' - \vec{y}') - C(\vec{y} - \vec{x}') - C(\vec{y}' - \vec{x}) &= \text{interaction} \\ &= C(\vec{x} - \vec{R} - \vec{n}_2) + C(\vec{x}' - \vec{R} + \vec{n}_2) - C(\vec{x}' - \vec{R} - \vec{n}_2) - C(\vec{x} - \vec{R} + \vec{n}_2) & \text{between} \\ &\simeq -\vec{n}_1 \cdot \vec{\nabla} C(\vec{x} - \vec{R}) + \vec{n}_1 \cdot \vec{\nabla} C(\vec{x}' - \vec{R}) + \mathcal{O}(n^3) & \text{changes} \\ &\vec{y} = \vec{R} + \vec{n}_2 \\ &\vec{y}' = \vec{R} - \vec{n}_2 \end{aligned}$$

$$\text{so that: } e^{4\pi^2 K D} - 1 \simeq -4\pi^2 K \underbrace{\vec{n} \cdot \vec{\nabla} (C(\vec{x} - \vec{R}) - C(\vec{x}' - \vec{R}))}_{\int d\vec{n} \text{ of this} = 0} + 8\pi^4 K^2 (\vec{n} \cdot \vec{\nabla} (\dots)) + \mathcal{O}(n^3)$$

$$\int d\vec{y} d\vec{y}' = \int d\vec{n} d\vec{n}'$$

$$\text{and } \int d\vec{n} (\vec{n} \cdot \vec{G}(\vec{R}))^2 = \int d\vec{n} \frac{n^2 G^2}{2}$$

$$\langle V(\vec{x}, \vec{x}') \rangle = e^{-4\pi^2 K C(\vec{x}-\vec{x}')} \left[ 1 + 2\pi \gamma^2 \int d\vec{n} n e^{-4\pi^2 K C(n)} \frac{8\pi^4 K^2 n^2}{2} F(\vec{x}, \vec{x}') + \dots \right]$$

$$\text{where } F(\vec{x}, \vec{x}') = \int d\vec{R} (C(\vec{x} - \vec{R}) - C(\vec{x}' - \vec{R}))^2$$

↑  
integration by parts

and  $\nabla^2 C = \delta$

$\overline{UV}$  divergence  
absorb in a

Using this and reexponentiating the  $\gamma$ -series:

$$\langle V(\vec{x}, \vec{x}') \rangle = e^{-4\pi^2 K C(\vec{x} - \vec{x}')} e^{16\pi^5 K^2 \gamma^2 C(\vec{x} - \vec{x}') \int_a^\infty d\eta \eta^3 \left(\frac{\eta}{a}\right)^{-2\pi K}} + \mathcal{O}(\gamma^4)$$

$$= e^{-4\pi^2 K_{\text{eff}} C(\vec{x} - \vec{x}')}$$

$\uparrow$  effective interaction/stiffness renormalized by vertices

with   $K_{\text{eff}} = K - 4\pi^3 K^2 \gamma^2 a^{2\pi K} \int_a^\infty d\eta \eta^{3-2\pi K} + \mathcal{O}(\gamma^4)$

In the E8M language:  $\epsilon = \frac{K}{K_{\text{eff}}} = \text{effective dielectric constant}$   
(renormalized by dipoles)

- Perturbative result makes sense if  $\int_a^\infty d\eta \eta^{3-2\pi K} < \infty : K > K_c = \frac{2}{\pi}$
- For  $K < K_c$ : Perturbation theory breaks down (also true for  $\phi^4$  theory  
IR divergence for  $d < d_c = 4$ )

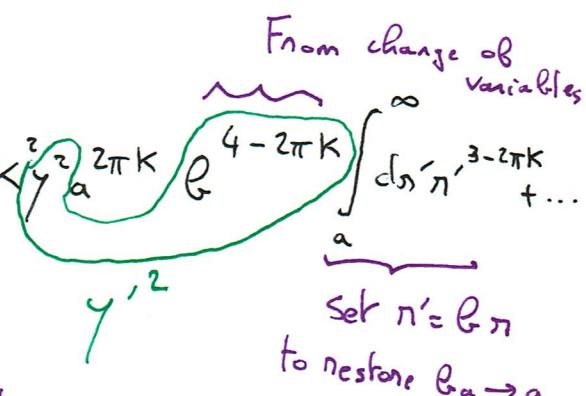
RG picture: rewrite  $\int_a^\infty = \int_a^{B_a} + \int_{B_a}^\infty$

$$K_{\text{eff}} = K \left( 1 - 4\pi^3 K \gamma^2 a^{2\pi K} \underbrace{\left[ \frac{\pi^{4-2\pi K}}{4-2\pi K} \right]_{a}^{B_a}}_{\approx a^4 \delta p} \right) - 4\pi^3 K \gamma^2 a^{2\pi K} \int_a^\infty d\eta \eta^{3-2\pi K} + \dots$$

$$\beta = e^{\frac{\delta I}{I}} = 1 + \delta p$$

$K'$ : renormalization due to dipoles

with  $\eta$  between  $a$  and  $B_a = (1 + \delta p) a$



Change  $a \rightarrow B_a$  can be absorbed in renormalized coupling constants:

$$\gamma' = \gamma (1 + (2 - \pi K) \delta p)$$

$$K' = K (1 - 4\pi^3 K \gamma^2 \delta p) \quad (\text{Finally set } a = 1 \text{ here})$$

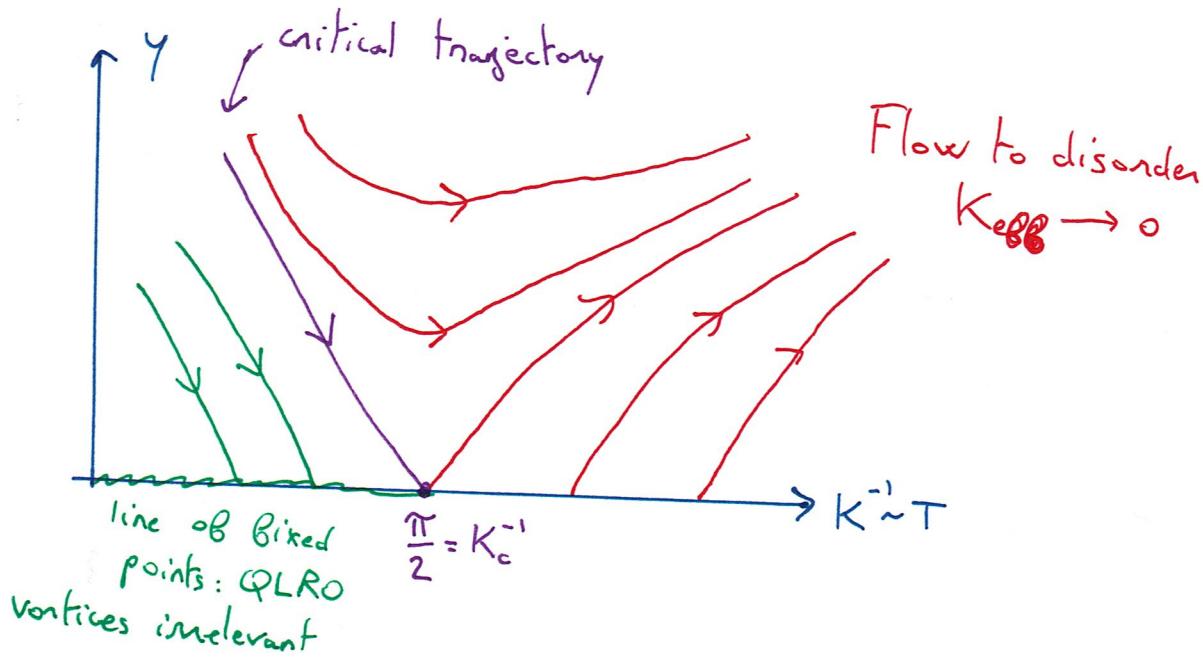
Using  $\frac{dK^{-1}}{dp} = -\frac{1}{K^2} \frac{dK}{dp}$ , this yields

$$\frac{dK^{-1}}{dp} = 4\pi^3 y^2$$

$$\frac{dy}{dp} = (2 - \pi K) y$$

Kosterlitz  
Equations

### E Kosterlitz-Thouless RG Flow



• QLRO phase: vortices irrelevant

$$\langle S_x^\dagger S_0^\dagger \rangle \approx \frac{1}{\pi \eta}$$

But  $y$  renormalizes  $K \rightarrow K_{eff}$  ( $\xi = \infty$  in this phase)

with  $\eta = \frac{1}{2\pi K_{eff}}$

$\downarrow$  non-universal

of spin wave calculation vortices can be ignored

• Transition: Flow along critical trajectory (purple line)

$$K_{eff} = K_c = \frac{2}{\pi} \text{ universal}$$

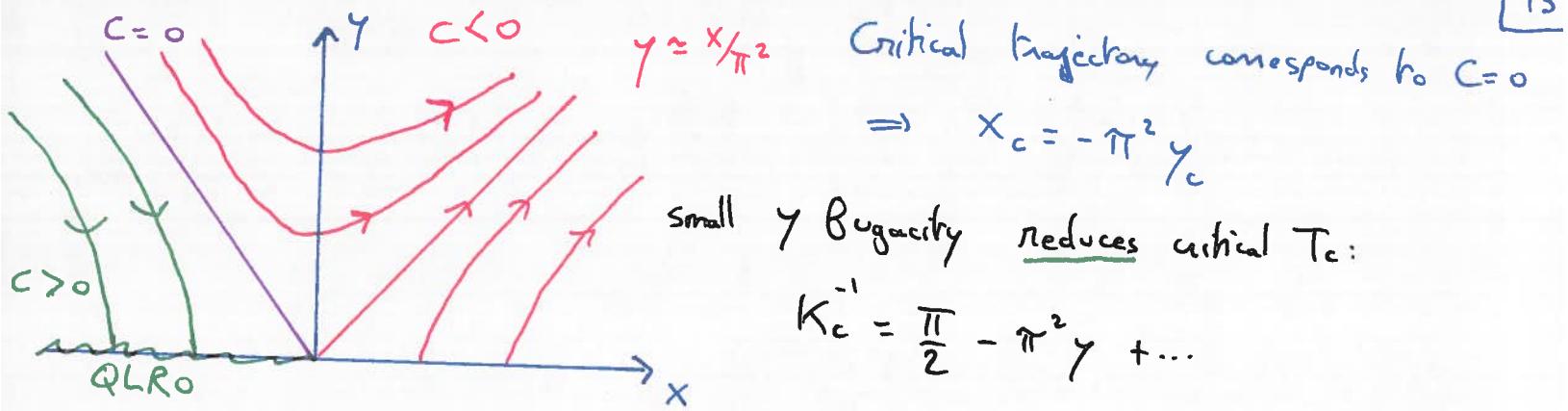
$$\eta = 1/4$$

at the transition

• Near fixed point:  $x = K^{-1} - \pi/2 \Rightarrow \frac{dy}{dp} \approx \frac{4}{\pi} xy + \dots$

$$\text{Now } \frac{d}{dp} (x^2 - \pi^4 y^2) = 0 \Rightarrow x^2 - \pi^4 y^2 = C$$

Hyperbolae with fixed  $C$



Small  $y$  fugacity reduces critical  $T_c$ :

$$K_c^{-1} = \frac{\pi}{2} - \pi^2 y + \dots$$

Set  $c = -c_0(T_c - T)$  (drives transition)  $\rightarrow$

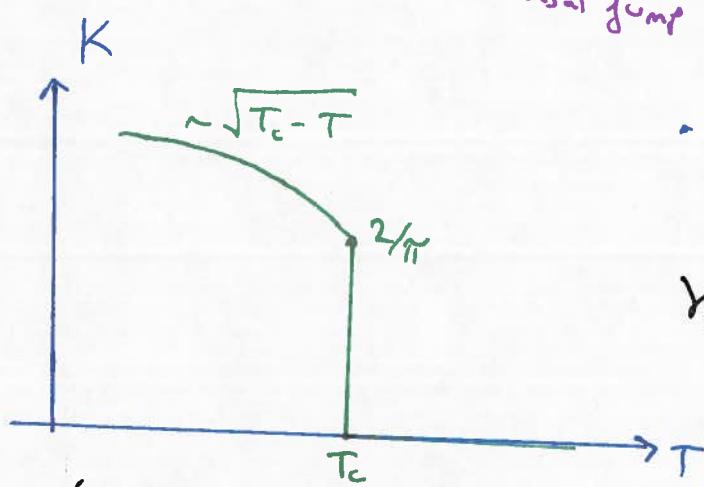
near transition:

$$K_{\text{eff}} = \frac{2}{\pi} + C \text{st} \sqrt{T_c - T}$$

*universal jump*       $\sqrt{\dots}$       *singularity*

$c < 0 (T_c < T)$ :  $K$  flows to 0  
 $c > 0 (T_c > T)$ :  $y$  flows to 0

$$x = -\sqrt{c_0(T_c - T)} = \lim_{P \rightarrow \infty} x(P)$$



(Bishop and Rmpy 1978)

- Famously observed experimentally in superfluid thin films

$$\mathcal{H} = \int d^2x \Psi^\dagger \left( -\frac{\hbar^2}{2m} \nabla^2 \right) \Psi = \mathcal{H}_{xy}(\theta)$$

$\uparrow$  assume 2d system       $\downarrow$   $^3\text{He}$  atom

stiffness  $K = \frac{\hbar^2 \rho_s}{m^2 k_B T}$

$$\rho_s = m \Psi^2: \text{SF density}$$

$\Rightarrow$  phase of superfluids = XY model. SF destroyed by vortices at high T.

Correlation length:  $\xi = \infty$  for  $T \leq T_c$ , finite for  $T > T_c$

$$T > T_c: x^2 + c_0(T - T_c) = y^2 \pi^4 \Rightarrow \frac{dx}{dp} = 4\pi^3 y^2 = \frac{4}{\pi} (x^2 + c_0(T - T_c))$$

$$\Rightarrow \frac{4}{\pi} p = \frac{1}{\sqrt{c_0(T - T_c)}} \left[ \text{Arctan} \left[ \frac{x}{\sqrt{c_0(T - T_c)}} \right] \right]_{x_0}^{x(p)}$$

scale at which  $x(p^*) \sim 1$   
 $p^* \approx c / \sqrt{T - T_c}$

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Recall  $L' = \beta L = (1 + \delta p)L \Rightarrow \xi = a e^{p^*}$  so that  $\xi \sim e^{C/\sqrt{T-T_c}}$

$$\Rightarrow L = L_0 e^{+p}$$

$$\xi(p^*) \sim \mathcal{O}(1) \sim \xi e^{-p^*}$$

$$\xi' = \xi/p$$

Free energy:  $B \sim \xi^{-2} \sim e^{-2C/\sqrt{T-T_c}} + \text{regular}$

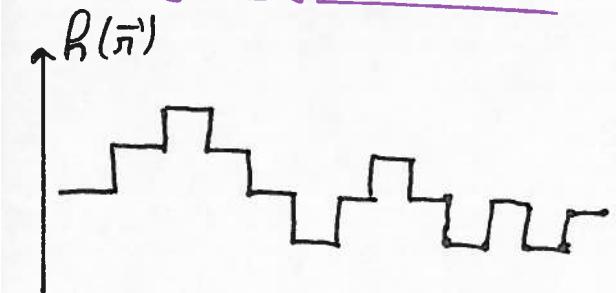
$\uparrow$  all derivatives finite at  $T_c$

Essential singularity, no exponent  $\downarrow$  Here!

## (F) Related Problems

- KT picture of vortex unbinding was a new paradigm for phase transitions without symmetry breaking.
- Related to 2d Coulomb g-s, superfluidity...
- 1d quantum physics: Luttinger liquids (interacting  $e^-$  in 1d)
- Dislocations and melting transitions

## Roughening transitions



2d fluctuating interface with  $R = \text{integer}$   
 $(\alpha=1)$   
 Continuum description:

$$\mathcal{H} = \frac{K}{2} \int d\mathbf{x} (\nabla R)^2 - \lambda \cos(2\pi R) + gR^2$$

$\Rightarrow$  Sine-Gordon theory

- $\rightarrow \lambda$  relevant: smooth interface  $R = \text{integer} = 0$  (uniform)
- $\rightarrow \lambda$  irrelevant: rough interface  $\langle R(n)R(0) \rangle \sim \log n$  ( $K \text{ small}$ )

## Appendix: Alternative derivation of the RG equations

• Fugacity: For a single vertex, we have  $Z_1 = \gamma \left(\frac{L}{a}\right)^{2-\pi K} = \gamma' \left(\frac{L}{Ba}\right)^{2-\pi K}$

$$\Rightarrow \gamma' = e^{2-\pi K} \gamma \Rightarrow \boxed{\frac{d\gamma}{dp} = (2-\pi K)\gamma}$$

• Simplicity: integrate out dipoles between  $a$  and  $Ba$ :  $\int_a^{\infty} \rightarrow \int_a^{Ba}$

$$K' = K - 4\pi^3 K^2 \gamma^2 a^{2\pi K} \int_a^{Ba} dn \pi^{3-2\pi K} + \mathcal{O}(\gamma^4)$$

$$= K \left[ 1 - 4\pi^3 K \gamma^2 \cancel{a^{2\pi K}} \frac{(B \cancel{a}^{4-2\pi K} - 1)}{4-2\pi K} a^{4-2\pi K} \right]$$

$$= K \left( 1 - 4\pi^3 K \gamma^2 a^4 \delta p \right)$$

Because we didn't make  $\gamma$  dimensionless  
in the action.  $\bar{\gamma} = a^2 \gamma$  and  $\bar{\gamma} \rightarrow \gamma$

$$\Rightarrow \boxed{\frac{dK^{-1}}{dp} = 4\pi^3 \gamma^2}$$