PHY-817: Advanced Statistical Physics, UMass Amherst, Take-home Exam

Romain Vasseur

Due: Wednesday, May 2.

I. CORRECTION TO SCALING IN THE 3D ISING MODEL

In class, we have argued that irrelevant perturbations lead to corrections to scaling. For the three dimensional Ising model, the singular part of the free energy at zero field ($h = 0$) should scale as

$$f \sim |t|^{2-\alpha} \left(A + B\,|t|^{\theta} + \dots \right),$$

where $B\,|t|^{\theta}$ is the leading correction to scaling as $t \to 0$. Estimate the exponent θ for the Ising universality class in $d = 3$ using the ϵ expansion.

II. SUPERFLUID TRANSITION IN $2 + \epsilon$ DIMENSIONS

Let us consider an XY model with $O(2)$ symmetry in $d = 2 + \epsilon$ expansion.

1. What is the renormalization group equation for the stiffness K of a Gaussian model in $2 + \epsilon$ dimensions?
2. Assuming that the RG equations for the XY model in $2 + \epsilon$ dimensions are simple deformations of those found in class for $d = 2$ (without worrying about what the physical interpretation of y is in this case!), show that there is a non-trivial fixed point describing the finite temperature phase transition in this model.
3. Obtain the eigenvalues at this fixed point to lowest non-trivial order in ϵ, and estimate the exponents ν and α for the superfluid transition in $d = 3$ from these results.

III. LOGARITHMIC CORRECTIONS AT THE 3D ISING TRICRITICAL POINT

Recall that the Ising tricritical point is described by a ϕ^6 theory

$$S = \int d^d r \left[\frac{1}{2}(\nabla\phi)^2 + \frac{t}{a^2}\phi^2 + \frac{u}{a^{4-d}}\phi^4 + \frac{g}{a^{2(3-d)}}\phi^6 \right],$$

with a a UV cutoff, and where $t = 0$ and $u = 0$ at the tricritical point. You have shown in the problem sets that the upper critical dimension of this model is $d_c = 3$, and you've computed the critical exponents of this model below three dimensions to order $\mathcal{O}(\epsilon = 3 - d)$. In this problem, we are interested in the scaling at the tricritical point in three dimensions $d = 3$. We recall the RG flow equations that you have computed in the problem set #4

$$\frac{dt}{d\ell} = 2t + \dots,$$
$$\frac{du}{d\ell} = u - 960gu + \dots,$$
$$\frac{dg}{d\ell} = -2400g^2 + \dots$$

The perturbations t and u are both relevant at the tricritical point, while g is said to be *marginally irrelevant* since it is marginal to linear order, but irrelevant once non-linear corrections are included.

1. Let us set $u = 0$. How does the correlation length diverge as $t \to 0$?
2. Let us now set $t = 0$, but u and g non-zero and small. Solve the RG flow equations to find the renormalized couplings $u(\ell)$ and $g(\ell)$ in terms of u and g.

3. As usual, we imagine running the RG until a scale $\ell_\star$ such that $u(\ell_\star) = \mathcal{O}(1)$. Write down a transcendental equation for $\ell_\star$ and find its asymptotic solution as $u \to 0$.

4. Conclude that the correlation length scales as

$$\xi \underset{u\to 0}{\sim} \frac{(-\log u)^\theta}{u},$$

where you will compute the exponent θ (corresponding to logarithmic corrections to the mean-field prediction because of the marginally irrelevant variable g).

IV. PERTURBED CONFORMAL FIELD THEORIES

In two dimensions, conformal invariance can be used to compute the spectrum of scaling operators (and their associated scaling dimensions) for many physically relevant models. Let us consider the Ising multicritical point

$$S_p = \int d^2x \left(\frac{1}{2}(\nabla\phi)^2 + g\phi^{2(p-1)}\right),$$

where $p = 3$ corresponds to the usual ϕ^4 Ising theory, $p = 4$ is the Ising tricritical point, and larger values of p correspond to higher order multicritical points (in the language of conformal field theory, this family of theories with $p = 3, 4, 5, \dots$ is known as the "minimal models"). Remarkably, the RG fixed point action $S_p^\star$ can be characterized exactly using conformal invariance, and the scaling dimensions of all the operators are known exactly. For example, the dimension of the $:\phi^2:$ scaling operator is given by $\Delta_t = \frac{4}{p(p+1)}$.

In this problem, we are interested in the fate of the multicritical point $S_p^\star$ perturbed by the (relevant) operator $:\phi^{2(p-2)}:$ with scaling dimension $\Delta = 2 - \frac{4}{p+1}$

$$S = S_p^\star + \frac{\lambda}{a^{2-\Delta}} \int d^2x :\phi^{2(p-2)}:,$$

with a a UV cutoff. We will be interested in the limit $p \gg 1$, and we introduce a small parameter $\epsilon = \frac{2}{p+1}$ so that $\epsilon \ll 1$. The OPE of the scaling operators at the $S_p^\star$ fixed point are also known exactly, and to leading order in ϵ, they read

$$:\phi^{2(p-2)}: \times :\phi^{2(p-2)}: = 1 + \frac{4}{\sqrt{3}} :\phi^{2(p-2)}: + \dots,$$
$$:\phi^2: \times :\phi^{2(p-2)}: = \frac{\epsilon^2}{\sqrt{3}} :\phi^2: + \dots$$

Note that the OPE coefficients are highly non-trivial and non-integers!

1. Ignoring the other couplings, write down an RG flow equation for the coupling λ and identify a perturbative fixed point for $\epsilon \ll 1$.

2. What is the scaling dimension of $:\phi^{2(p-2)}:$ at this new fixed point?

3. Compute the new scaling dimension Δ_t' of $:\phi^2:$ at the new fixed point. (*Hint: since $\Delta_t = \mathcal{O}(\epsilon^2)$, you will need to keep track of ϵ^3 terms*).

4. Show that your result for Δ_t' is compatible with the new fixed point being $S_{p-1}^\star$.