PHY-897: Special Topics: Solid State Physics, UMass Amherst, Problem Set #1

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Due: Feb 14 in class.

I. JORDAN-WIGNER TRANSFORMATION

1. Consider a single spin-$\frac{1}{2}$ with Hilbert space $\mathcal{H} = \mathbb{C}^2$ spanned by $|\uparrow\rangle, |\downarrow\rangle$. Show that there is a mapping between this system and a single fermionic level with Hilbert space spanned by $|0\rangle$, $|1\rangle = f^\dagger |0\rangle$ with $\{f^\dagger, f\} = 1$. Write down the correspondence between the spin operators $S^z, S^\pm$ and the fermionic annihilation/creation operator f and $f^\dagger$. (Use a convention where $|\uparrow\rangle$ corresponds to the occupied level $|1\rangle$.)
2. Let us now consider a one dimensional chain of spins $\frac{1}{2}$, $S_i^{x,y,z}$, that we would like to map onto a chain of (spinless) fermions $f_i, f_i^\dagger$. Explain why the correspondence above doesn't trivially generalize to this case.
3. Show that the Jordan-Wigner transformation provides a complete mapping from spins to fermions:
$$\begin{aligned} S_i^z &= n_i - \frac{1}{2}, \\ S_i^+ &= (-1)^{\sum_{j<i} n_j} f_i^\dagger, \\ S_i^- &= (-1)^{\sum_{j<i} n_j} f_i. \end{aligned}$$
Here, $n_i = f_i^\dagger f_i$ is the fermion occupation number on site i, and $(-1)^{\hat{\mathcal{O}}}$ for some operator $\hat{\mathcal{O}}$ is to be understood as $\mathrm{e}^{i\pi\hat{\mathcal{O}}}$. Show that the spin operators satisfy the proper algebra, and in particular, that spin operators acting on different sites commute, as they should.

II. KITAEV CHAIN

The goal of this problem is to study the so-called Kitaev chain, which provides a (mean-field) description of a one-dimensional spinless superconducting wire:
$$H = -t \sum_i \left(c_{i+1}^\dagger c_i + \text{h.c.} \right) - \mu \sum_i c_i^\dagger c_i + \Delta \sum_i \left(c_{i+1}^\dagger c_i^\dagger + \text{h.c.} \right),$$
where the c_i's are fermions, and the last term is a superconducting pairing term that breaks particle number conservation (modulo 2).

1. Consider first $\Delta = 0$: diagonalize H in that limit by going to Fourier space (with periodic boundary conditions).
2. Introduce Majorana fermions using $c_j^\dagger = \frac{1}{2}(a_j + i b_j)$. Compute the algebra formed by a and b, and write H in terms of these operators, restricting yourself to the case $\Delta = t$.
3. Analyze the limiting cases $\Delta = t = 0$, and $\Delta = t$ with $\mu = 0$ in terms of these Majorana fermions. Explain how to construct the eigenstates of H in these simple limits. Consider a Kitaev chain with open boundary conditions, and show that for $\Delta = t$, $\mu = 0$, there is a Majorana zero mode (Majorana operator that commutes with H, and hence has zero energy) at each end of the system.
4. Go back to the general case $\Delta \neq t$. Write the Hamiltonian in k space, and put it in the form
$$H = \frac{1}{2} \sum_k \Psi_k^\dagger H_{\text{BdG}}(k) \Psi_k,$$
where $\Psi_k = \left(c_{-k} \; c_k^\dagger \right)^T$, and $H_{\text{BdG}}(k)$ is a 2×2 matrix called Bogoliubov-de Gennes Hamiltonian.
5. Diagonalize $H_{\text{BdG}}(k)$, and rewrite H into a non-interacting diagonal form $H = \sum_k \epsilon_k (d_k^\dagger d_k - \frac{1}{2})$ where d_k is a new fermionic operator. Plot the single particle energies (dispersion relation) ϵ_k, and show that H becomes gapless for $2\,|t| = |\mu|$