PHY-897: Special Topics: Solid State Physics, UMass Amherst, Problem Set #2

Romain Vasseur

Due: Friday March 15.

I. QUANTUM-TO-CLASSICAL CORRESPONDENCE

Let's add a term $-J'\sum_i \sigma_i^x \sigma_{i+1}^x$ to the Hamiltonian $H_{\mathrm{TFIM}} = -J\sum_i \sigma_i^z\sigma_{i+1}^z + g\sigma_i^x$ of the transverse field Ising model.

1. Is the Hamiltonian still exactly solvable using Majorana fermions?
2. Do you expect this term to change the nature of the phases and of the phase transition we studied in class in the case $J' = 0$? Why? Give an example of a quantity that will change as a function of J', and a quantity that won't.
3. Write down a statistical mechanics model in two dimensions for this model. Are the Boltzmann weights still positive?

II. XXZ SPIN CHAIN

Let us consider the so-called XXZ spin$-\frac{1}{2}$ spin chain

$$H = J\sum_i (S_i^x S_{i+1}^x + S_i^y S_{i+1}^y + \Delta S_i^z S_{i+1}^z).$$

1. What are the symmetries of this Hamiltonian (as a function of Δ)?
2. Use a Jordan-Wigner transformation to diagonalize H in the case $\Delta = 0$. What happens to the fermionic problem when $\Delta \neq 0$?

III. FERMIONIC COHERENT STATES AND OVERCOMPLETENESS

Consider a single fermionic level. Show the following identity for the coherent states:

$$\int d\overline{\psi} d\psi \mathrm{e}^{-\overline{\psi}\psi} \left|\psi\right\rangle \left\langle\overline{\psi}\right| = \mathbb{I}.$$

IV. DIRAC FERMIONS

Let's consider the following tight-binding fermionic Hamiltonian

$$H = -t\sum_i \left(c_{i+1}^\dagger c_i + \text{h.c.}\right) - \mu c_i^\dagger c_i.$$

1. For which value of the chemical potential μ is the system an insulator (gapped) vs a metal (gapless)?
2. In the gapless regime, write the Hamiltonian in k space by linearizing the dispersion relation near the two Fermi points $\pm k_F$ (that you will define).
3. Argue that at low energies, the system can be described by the following field theory

   $$H = -iv_F \int dx \left(\psi_R^\dagger \partial_x \psi_R - \psi_L^\dagger \partial \psi_L\right),$$

   where ψ_R and ψ_L are fermionic fields, and v_F is a parameter you will define.
4. Write down the action of this field theory, and the equations of motion of the fields $\psi_{R,L}$.