Majorana Field Theory

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- In this chapter, we're going to take the continuum limit of the Ising model. This will lead to our first emergent QFT: the Majorana Field theory, a.k.a: Ising CFT

  ↑ Conformal Field Theory

(I) Continuum Limit of the Ising Chain

We showed that the transverse Field Ising chain can be rewritten in the fermionic language as: (up to a Jordan-Wigner transformation)

$$H = J\sum_k 2(g - \cos ka)\, n_k - i \sin ka \left(c_{-k}^{\dagger} c_k^{\dagger} + c_{-k} c_k\right) + \text{cst}$$

↑ lattice spacing a (was set to 1 in previous chapters)

$$L = Na \qquad x_j = aj$$

- Long distance (low-energy) limit: $\xi \gg a$ (close to phase transition)

Let $\boxed{\psi(x_j) = \frac{1}{\sqrt{a}} c_j}$ as $a \to 0$

$$\{c_i^{\dagger}, c_j^{\dagger}\} = \delta_{i,j}.$$

$$\Rightarrow \{\psi^{\dagger}(x_i), \psi(x_j)\} = \lim_{a\to 0} \frac{\delta_{i,j}}{a} = \delta(x_i - x_j)$$

$$\left(\sum_i \delta_{i,0}\, f(x_i) = f(0) = \sum_i a\, f(x_i) \frac{\delta_{i,0}}{a} = \int dx\, f(x)\, \delta(x) \right)$$

Now: $$c_k = \frac{1}{\sqrt{N}} \sum_j c_j e^{-ikx_j} = \sqrt{\frac{a}{L}} \sum_j \sqrt{a}\, \psi(x_j) e^{-ikx_j}$$

$$\longrightarrow \int dx\, e^{-ikx}\, \psi(x)/\sqrt{L}$$

Plugging this in H and working with $ka \ll 1$ (to be justified more precisely in the next chapter)

$$J\sum_k \big(g - \underbrace{\cos ka}_{\sim 1}\big) c_k^\dagger c_k \to J(g-g_c) \sum_k c_k^\dagger c_k = \int dx\, \psi^\dagger(x)\psi(x)$$

using $k \leftrightarrow -i\partial_x$, we see that we dropped a term $\sim \partial_x \psi^\dagger \partial_x \psi$

$\to$ gradient expansion

we'll show later that these terms can be ignored (irrelevant in the RG language)

$$iJ\sum_k \sin ka \; c_{-k} c_k = iJa \sum_k k\, c_{-k} c_k \to \frac{c}{2}\int dx\, \psi\, \partial_x \psi$$

$(c = 2Ja)$

Putting everything together:

$$\boxed{H \simeq \frac{c}{2}\int dx \left[\psi^\dagger \partial_x \psi^\dagger - \psi\, \partial_x \psi\right] + \Delta \int dx\, \psi^\dagger \psi} + \underbrace{\ldots}_{\text{Higher order term}}$$

$\Delta = 2J(g-1)$

$\{\chi(x), \chi(y)\} = \delta(x-y)$

Let $\psi^\dagger = \frac{\chi + i\eta}{\sqrt{2}}$, $\psi = \frac{\chi - i\eta}{\sqrt{2}}$ (χ, η : Majorana Field)

$$H = \frac{c}{2}\int dx\, i\left(\chi\, \partial_x \eta + \eta\, \partial_x \chi\right) + i\Delta \int dx\, \eta(x)\chi(x)$$

Finally, let's perform a last rotation: $\chi = \frac{\chi_+ + \chi_-}{\sqrt{2}}$, $\eta = \frac{\chi_+ - \chi_-}{\sqrt{2}}$

kinetic term:

$\{\chi_\pm(x), \chi_\pm(y)\} = \delta(x-y)$

$$\Rightarrow \boxed{H = \frac{c}{2} i \int dx \left[\chi_+ \partial_x \chi_+ - \chi_- \partial_x \chi_-\right] + i\Delta \int dx\, \chi_+ \chi_-}$$

$\chi_\pm$ describe chiral, right / left moving Majorana fields.
At the critical point: $\Delta = 0$ and $\chi_\pm$ decouple.

- **Heinsenberg equation of motion**: $\partial_t \chi_\pm = i[H, \chi_\pm]$

$$\int dx\, \chi_+(x)\,\partial_x \chi_+(x)\,\chi_+(y) = -\int dx\, \chi_+(x)\chi_+(y)\,\partial_x \chi_+(x)$$

$\{\partial_x \chi_+(x), \chi_+(y)\} = \delta(x-y)$

$$+ \int dx\, \chi_+(x)\,\partial_x \delta(x-y)$$

$$= \int dx\, \chi_+(y)\,\chi_+(x)\,\partial_x \chi_+(x) - 2\,\partial_x \chi_+(y)$$

$$\Rightarrow \quad \boxed{\begin{aligned} \partial_t \chi_+ &= +c\,\partial_x \chi_+ + \Delta \chi_- \\ \partial_t \chi_- &= -c\,\partial_x \chi_- - \Delta \chi_+ \end{aligned}}$$

At the critical point ($\Delta = 0$): $\chi_\pm = f(x \pm ct)$

χ_+ : left mover
χ_- : right mover

- **Relativistic invariance**: Let $\gamma^0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ $\quad \gamma^1 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

set $c = 1$

so that $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$ $\qquad \eta^{\mu\nu} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

$\hat{\chi} = \begin{pmatrix} \chi_+ \\ \chi_- \end{pmatrix}$ Majorana spinor $\qquad \bar{\hat{\chi}} = \hat{\chi}^T \gamma^0 = (-\chi_- \;\; \chi_+)$

$$\not{\partial}\hat{\chi} = \begin{pmatrix} 0 & -(\partial_t + \partial_x) \\ +\partial_t - \partial_x & 0 \end{pmatrix}\begin{pmatrix} \chi_+ \\ \chi_- \end{pmatrix} = \begin{pmatrix} -(\partial_t + \partial_x)\chi_- \\ +(\partial_t - \partial_x)\chi_+ \end{pmatrix}$$

$\not{\partial} = \gamma^\mu \partial_\mu$

$$\boxed{(\not{\partial} - \Delta)\,\hat{\chi} = 0}$$

Majorana Equation
(in 1+1d!)

mass $m = \Delta$

II Path integral formulation

In order to compute correlation functions, it will be convenient (as often in quantum mechanics) to use a path (functional) formulation.

• Fermionic Coherent states: $\hat{\psi}|\psi\rangle = \psi|\psi\rangle$

$\hat{\psi}$: operator; ψ: "number"

* Eigenvalue of annihilation operator

However: there's something weird about ψ: $\hat{\psi}^2|\psi\rangle = \psi^2|\psi\rangle = 0$

$\Rightarrow \psi^2 = 0$

ψ can't be an usual $\mathbb{C}$ number! $\Rightarrow$ Grassmann number

$\{\psi_i, \psi_j\} = 0$ and $\{\psi_i, \hat{\psi}_j\} = 0$

* $|\psi\rangle = |0\rangle - \psi\underbrace{|1\rangle}_{\hat{\psi}^\dagger|0\rangle}$ $\left[\hat{\psi}|\psi\rangle = +\psi\hat{\psi}\underbrace{|1\rangle}_{\hat{\psi}^\dagger|0\rangle} = \psi|0\rangle = \psi|\psi\rangle\right]$

$= e^{-\psi\hat{\psi}^\dagger}|0\rangle$

For a discrete number of modes: $\boxed{|\psi\rangle = e^{-\sum_i \psi_i \hat{\psi}_i^\dagger}|0\rangle}$

$|\psi\rangle$: vector

Similarly: $\langle\bar{\psi}|\hat{\psi}^\dagger = \langle\bar{\psi}|\bar{\psi}$ (ψ not hermitian!)

$\langle\bar{\psi}| = \langle 0| - \langle 1|\bar{\psi}$ (not the adjoint of $|\psi\rangle$!!)

$= \langle 0| + \bar{\psi}\langle 1| = \langle 0| e^{\bar{\psi}\hat{\psi}}$

$\boxed{\langle\bar{\psi}| = e^{\sum_i \bar{\psi}_i \hat{\psi}_i}}$

$\langle\bar{\psi}|\psi\rangle = 1 + \bar{\psi}\psi = e^{\bar{\psi}\psi}$

• Grassmann Calculus: Although Grassmann variables look weird, their algebra is amazingly simple

. Taylor expansion: $f(\psi) = f_0 + f'(\psi)\,\psi$

. Integration: $\int d\psi = 0 \qquad \int d\psi\, \psi = 1$ (just like derivative!)

with more variables: $\int d\bar{\psi}\, d\psi\, \psi \bar{\psi} \qquad d\bar{\psi}\, d\psi = - d\psi\, d\bar{\psi}$

. Gaussian integral: $Z = \int d\bar{\psi}\, d\psi\, e^{-a\bar{\psi}\psi} = \int d\bar{\psi}\, d\psi\, (1 + a\psi\bar{\psi})$
$= a$

$$\int \prod_i d\bar{\psi}_i\, d\psi_i\; e^{-\vec{\bar{\psi}}.\underset{\text{matrix}}{A}.\psi} = \det A$$

$$\langle \psi \bar{\psi} \rangle = - \langle \bar{\psi} \psi \rangle = \frac{\int d\bar{\psi}\, d\psi\, \psi\bar{\psi}\, e^{-a\bar{\psi}\psi}}{Z} = \frac{1}{a}$$

. Back to Quantum mechanics:

Overcompleteness: $\int \underbrace{\prod_i d\bar{\psi}_i\, d\psi_i}_{D[\bar{\psi},\psi]}\, e^{-\sum_i \bar{\psi}_i \psi_i} |\psi\rangle\langle\bar{\psi}| = \mathbb{1}$

(resolution of identity)

Let's consider a fermionic Hamiltonian $H[\hat{\psi}^\dagger, \psi] = \sum_{i,j} h_{ij} \psi_i^\dagger \psi_j$
$+ \sum_{ijkl} V_{ijkl}\, \psi_i^\dagger \psi_j^\dagger \psi_k \psi_l + \dots$

. Normal order: all $\psi^\dagger$'s sit to the left of the ψ's.
. For simplicity: drop spatial indices in the following.

$$Z = \mathrm{tr}\, e^{-\beta H} = \mathrm{tr} \left(e^{-\Delta\tau H}\right)^N \quad \text{with} \quad \Delta\tau = \frac{\beta}{N}$$

Insert resolutions of identity:

$$Z = \mathrm{tr}\left[\int \prod_{j=0}^{N-1}\left(d\bar\psi_j\, d\psi_j\, e^{-\bar\psi_j \psi_j}\, |\psi_j\rangle\langle\bar\psi_j| e^{-\Delta\tau H}\right)\right]$$

$$= \sum_n \int D[\bar\psi,\psi]\ \langle n|\psi_{N-1}\rangle\, e^{-\bar\psi_{N-1}\psi_{N-1}}$$

$\hat\psi \to \psi$
$\hat\psi^\dagger \to \bar\psi$

$$\times \prod_{j=0}^{N-2} e^{-\bar\psi_j\psi_j}\underbrace{\langle\bar\psi_{j+1}|e^{-\Delta\tau H}|\psi_j\rangle}\langle\bar\psi_0|e^{-\Delta\tau H}|n\rangle$$

Normal order: $e^{-\Delta\tau H[\bar\psi_{j+1},\psi_j]}\langle\bar\psi_{j+1}|\psi_j\rangle$

where $\sum_n |n\rangle\langle n| = 1$ any orthonormal basis

now: $\sum_n \langle n|\psi_{N-1}\rangle\langle\bar\psi_0|e^{-\Delta\tau H}|n\rangle = \sum_n \langle-\bar\psi_0|e^{-\Delta\tau H}|n\rangle\langle n|\psi_{N-1}\rangle$

$$= \langle-\bar\psi_0|e^{-\Delta\tau H}|\psi_{N-1}\rangle$$

note the sign

$$\Rightarrow\ Z = \int_{\psi_N=-\psi_0} D[\bar\psi,\psi]\prod_{j=0}^{N-1} e^{-\bar\psi_j\psi_j}\, e^{-\Delta\tau H[\bar\psi_{j+1},\psi_j]} \times \langle\bar\psi_{j+1}|\psi_j\rangle$$

$\langle\bar\psi_{j+1}|\psi_j\rangle = e^{\bar\psi_{j+1}\psi_j}$

antiperiodic boundary conditions in imaginary time

$$= \int_{\psi_N=-\psi_0} D[\bar\psi,\psi]\, e^{-\sum_j \Delta\tau\left[-\frac{(\bar\psi_{j+1}-\bar\psi_j)}{\Delta\tau}\psi_j + H[\bar\psi_{j+1},\psi_j]\right]}$$

$\frac{(\bar\psi_{j+1}-\bar\psi_j)}{\Delta\tau} = \partial_\tau\bar\psi(\tau)$ as $\Delta\tau\to 0$

$\psi(\tau_j) = \psi(j\Delta\tau)$

$$\Rightarrow\ \boxed{Z = \int_{\psi(\beta)=-\psi(0)} D[\bar\psi,\psi]\ e^{-\int_0^\beta d\tau\left(\bar\psi\partial_\tau\psi + H[\bar\psi,\psi]\right)}}$$

Note:

- We used $H(\bar\psi_{j+1}, \psi_j) = H(\bar\psi_j, \psi_j) + \mathcal{O}(\Delta\tau)$ which is usually ok.
- Essentially, we just followed our recipe to map a fermionic quantum problem onto a classical system in one more dimension. Because fermions are weird objects, the corresponding classical fields are Grassmann variables.
- Note the antiperiodic boundary conditions in imaginary time:

$$\psi(\tau+\beta) = -\psi(\tau)$$

$\Rightarrow$ Fourier decomposition:
$$\psi(x,\tau) = \frac{1}{\sqrt{\beta}} \sum_n e^{-i\omega_n \tau} \psi_{\omega_n}(x)$$
$$\psi_{\omega_n}(x) = \frac{1}{\sqrt{\beta}} \int_0^\beta d\tau \, e^{i\omega_n \tau} \psi(x,\tau)$$

with $\boxed{\omega_n = \frac{2\pi}{\beta}\left(n + \frac{1}{2}\right)}$ $\quad n \in \mathbb{Z}$

Matsubara frequencies.

(as $T \to 0$, $\beta \to \infty$, we can rewrite $\frac{1}{\beta}\sum_n \longrightarrow \int \frac{d\omega}{2\pi}$)

Main conclusion:
$$Z = \int_{\psi(\beta) = -\psi(0)} D[\bar\psi, \psi] \, e^{-S[\bar\psi,\psi]}$$

Berry phase term

with
$$S = \int_0^\beta d\tau \underbrace{L[\bar\psi(\tau), \psi(\tau)]}_{\text{Euclidian Lagrangian}} = \int_0^\beta d\tau \left[\bar\psi \partial_\tau \psi + H[\bar\psi, \psi] \right]$$

$\Downarrow$ spatial indices

$$\int dx \left[\bar\psi \partial_\tau \psi + \mathcal{H}[\bar\psi, \psi] \right]$$

Correlation function: For $Z = \int \prod_i d\bar\psi_i \, d\psi_i \, e^{-\sum_{ij} \bar\psi_i A_{ij} \psi_j}$

$$\langle \bar\psi_i \psi_j \rangle = (A^{-1})_{ij}$$ (same as with bosonic variables)

Proof: diagonalize A.

III Majorana / Ising Field Theory

Let's apply this formalism to the Hamiltonian:

$$H \simeq \frac{c}{2}\int dx\left[\psi^\dagger \partial_x \psi^\dagger - \psi\, \partial_x \psi\right] + \Delta \int dx\, \psi^\dagger \psi$$

where we already took the continuum limit in space.

$ds^2 = d\tau^2 + dx^2 = -dt^2 + dx^2 \qquad \tau = it$

$\Rightarrow$ (Euclidean) Lagrangian density:

$$\boxed{\mathcal{L} = \bar\psi \partial_\tau \psi + \frac{c}{2}\left[\bar\psi \partial_x \bar\psi - \psi \partial_x \psi\right] + \Delta \bar\psi \psi}$$

We can also work with the (rotated) self-conjugate ("real") fields χ_- and χ_+:

$$\psi = \frac{\chi_+ + \chi_- + i(\chi_+ - \chi_-)}{2}, \qquad \bar\psi = \frac{\chi_+ + \chi_- - i(\chi_+ - \chi_-)}{2}$$

$$\Rightarrow \bar\psi \partial_\tau \psi = \frac{1}{4}\left(\chi_+ + \chi_- - i\chi_+ + i\chi_-\right)\left(\partial_\tau \chi_+ + \partial_\tau \chi_- + i\partial_\tau \chi_+ - i\partial_\tau \chi_-\right)$$

$$= \frac{1}{4}\left(2\chi_+ \partial_\tau \chi_+ + 2\chi_- \partial_\tau \chi_-\right)$$

$$\bar\psi \psi = i\Delta \chi_+ \chi_-$$

$$\Rightarrow \boxed{\mathcal{L} = \frac{1}{2}\chi_+(\partial_\tau + ic\,\partial_x)\chi_+ + \frac{1}{2}\chi_-(\partial_\tau - ic\,\partial_x)\chi_- + i\Delta \chi_+ \chi_-}$$

Recall e.o.m: $\quad (\partial_\tau \pm ic\,\partial_x)\chi_\pm \pm i\Delta \chi_\mp = 0 \qquad \left(\frac{\delta S}{\delta \chi_\pm} = 0\right)$

$\partial_\tau = -i\partial_t \qquad$ same equation as before

$\tau = it$

Relativistic invariance: Real time $\quad e^{-\int d\tau L_E} = e^{i\int dt L}$

$$\mathcal{L}_{\text{real time}} = \frac{i}{2}\chi_+(\partial_t - \partial_x)\chi_+ + \frac{i}{2}\chi_-(\partial_t + \partial_x\chi_-) - i\Delta\chi_+\chi_-$$

(set $c=1$)

$$\boxed{\mathcal{L} = \frac{i}{2}\bar{\hat{\chi}}\,(\not{\partial} - \Delta)\,\hat{\chi}} \qquad \text{with} \quad \hat{\chi} = \begin{pmatrix}\chi_+\\ \chi_-\end{pmatrix}, \quad \bar{\hat{\chi}} = \hat{\chi}^T\gamma^0$$

IV Complex coordinates

Before analyzing the scaling behaviour of this field theory, there's yet another way to rewrite this that is especially powerfull in 1+1d:

$$\chi_\pm = f(\underbrace{x \mp i\tau}_{x\pm t}) \qquad : \text{left and right movers}$$

Let : $z = x + i\tau \qquad \partial = \frac{\partial}{\partial z} = \frac{1}{2}(\partial_x - i\partial_\tau)$

$\bar{z} = x - i\tau \qquad \bar{\partial} = \frac{\partial}{\partial\bar{z}} = \frac{1}{2}(\partial_x + i\partial_\tau)$

$(\partial z = \bar{\partial}\bar{z} = 1, \; \bar{\partial}z = \partial\bar{z} = 0)$

think of $z, \bar{z}$ as independent variables

$$dz \wedge d\bar{z} = 2i\, d\tau \wedge dx \implies d\tau\, dx = \frac{dz\, d\bar{z}}{2i}$$

$$S = \int d\tau dx\, \frac{1}{2}\chi_\pm(\partial_\tau \pm i\partial_x)\chi_\pm + i\Delta\chi_+\chi_-$$

$$\implies \boxed{S = \frac{1}{2}\int dz\, d\bar{z}\left[\chi_+\partial\chi_+ - \chi_-\bar{\partial}\chi_- + \Delta\chi_+\chi_-\right]}$$

At criticality: e.o.m.: $\bar{\partial} X_+ = 0 \quad \Rightarrow \quad X_+ = f(\bar{z})$

$\partial X_- = 0 \qquad\qquad\qquad X_- = f(z)$

X_+ = anti-holomorphic function = left-mover

X_- = holomorphic function = right-mover