Quantum-to-Classical Correspondence

Quantum - to - Classical Correspondence

(I) Simple example: Quantum mechanics in d=0 ⟷ stat. mech in d=1

. Single quantum spin $H = h\sigma^z - g\,\sigma^x$ (two level system)

. Partition function : $Z = \mathrm{Tr}\, e^{-\beta H} = \sum_{\sigma=\pm1} \langle\sigma| e^{-\beta H} |\sigma\rangle$

with $\sigma^z|\sigma\rangle = \sigma|\sigma\rangle \quad \sigma = \pm 1$

Now: $U(\beta) = e^{-\beta H}$ Propagation in imaginary time $t = -i\beta$

Trotterize: $\Delta\tau = \frac{\beta}{N}$, $N \gg 1$ steps $\tau_j = j\,\Delta\tau$

$$Z(\beta) = \sum_{\{\sigma_{\tau_j} = \pm1\}} \langle\sigma_{\tau_N}| e^{-\Delta\tau H} |\sigma_{\tau_{N-1}}\rangle \langle\sigma_{\tau_{N-1}}| e^{-\Delta\tau H} |\sigma_{\tau_{N-1}}\rangle$$

$$\cdots \langle\sigma_{\tau_1}| e^{-\Delta\tau H} |\sigma_{\tau_0} = \sigma_{\tau_N}\rangle$$

(if you've seen how to construct a path integral in quantum mechanics, this construction should be familiar!)

We can evaluate the matrix element using $\Delta\tau \ll 1$:

$$\langle\sigma_{\tau_{j+1}}| e^{-\Delta\tau H} |\sigma_{\tau_j}\rangle \simeq \langle\sigma_{\tau_{j+1}}| e^{+\Delta\tau g\,\sigma^x} e^{-\Delta\tau h\sigma^z} |\sigma_{\tau_j}\rangle + \mathcal{O}(\Delta\tau^2)$$

$$= e^{-\Delta\tau h\,\sigma_{\tau_j}} \langle\sigma_{\tau_{j+1}}| \underbrace{e^{\Delta\tau g\,\sigma^x}}_{\cosh(\Delta\tau g) + \sinh(\Delta\tau g)\,\sigma^x} |\sigma_{\tau_j}\rangle$$

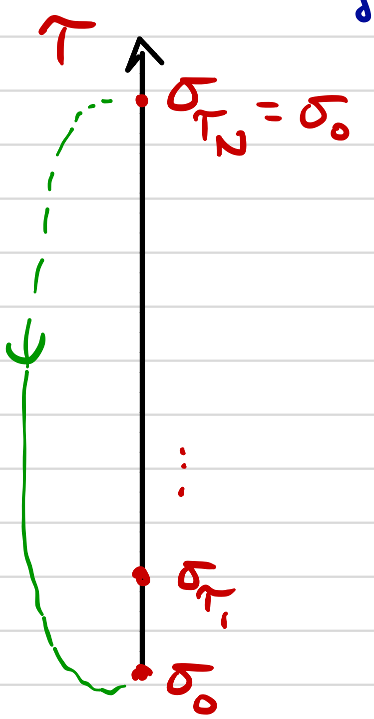
. Let's rewrite: $\langle \sigma_{\tau_{j+1}} | e^{-\Delta\tau H} | \sigma_{\tau_j} \rangle = e^{-\Delta\tau h \sigma_{\tau_j} + K \sigma_{\tau_j}\cdot\sigma_{\tau_{j+1}} + cst}$

$$\begin{aligned} \sigma_{\tau_j} = \sigma_{\tau_{j+1}} &: C e^{K} = \cosh(\Delta\tau g) \\ \sigma_{\tau_j} \neq \sigma_{\tau_{j+1}} &: C e^{-K} = \sinh(\Delta\tau g) \end{aligned} \quad \Longrightarrow \quad e^{-2K} = \tanh(\Delta\tau g)$$

. up to an unimportant overall constant: $\rightarrow \tilde{h} = \Delta\tau h$

$$Z(\beta) = \sum_{\{\sigma_{\tau_j}\}} \prod_j e^{-\tilde{h}\sigma_{\tau_j} + K\sigma_{\tau_j}\cdot\sigma_{\tau_{j+1}}}$$

$$= \sum_{\{\sigma_j\}} e^{-\sum_j \tilde{h}\sigma_{\tau_j} + K\sigma_{\tau_j}\cdot\sigma_{\tau_{j+1}}} \Bigg\} \ \boxed{\text{classical 1d Ising Model}}$$



. We've discretized the imaginary time evolution $\rightsquigarrow$ one more dimension

. We've also used the fact that the matrix elements of $e^{-\Delta\tau H}$ are real to interpret them as Boltzmann weights. Not always the case ($\rightsquigarrow$ sign problem in quantum Monte Carlo)

. Length of the imaginary time evolution:

$N\Delta_\tau = \beta$. Periodic boundary condition.

$T = 0 \Rightarrow \beta \to \infty$: "Thermodynamic limit" for the 1d classical model

. Below, we're going to generalize this mapping to any dimension.

- $$\boxed{T = e^{-\Delta\tau H} \simeq 1 - \Delta\tau H}$$ Transfer matrix of the classical model

(II) Quantum Ising in d space dimensions to classical Ising in D=d+1

- We can now easily generalize this to any dimension. Let's start with a transverse field Quantum Ising model in d space dimensions:

$$H = -J \sum_{\langle ij \rangle} \sigma_i^z \sigma_j^z - Jg \sum_i \sigma_i^x$$ (on d-dimension hypercube G)

- Partition function: $Z = \mathrm{Tr}\, e^{-\beta H} = \mathrm{Tr}\left(e^{-\Delta\tau H}\right)^N$ $\Delta\tau \to 0$, $N \to \infty$, $\Delta\tau N = \beta$ fixed

As before: $e^{-\Delta\tau H} = T_x T_z + \mathcal{O}(\Delta\tau^2)$

with $T_z = e^{J\Delta\tau \sum_{\langle i,j\rangle} \sigma_i^z \sigma_j^z}$ $T_x = e^{Jg\Delta\tau \sum_i \sigma_i^x}$

- Insert resolution of identity: $\sum_{\{\sigma_{\tau_n}\}} |\{\sigma_{\tau_n}\}\rangle\langle\{\sigma_{\tau_n}\}| = \mathbb{1}$ ← σ^z basis

(on each imaginary time slice)

$$\langle\{\sigma_{\tau_{n+1}}\}|T_z|\{\sigma_{\tau_n}\}\rangle = e^{J\Delta\tau \sum_{\langle i,j\rangle} \sigma_{\tau_n}^i \sigma_{\tau_n}^j} \; \delta_{\{\sigma_{\tau_{n+1}}\},\{\sigma_{\tau_n}\}}$$

Meanwhile, T_x has off-diagonal matrix elements:

$$\langle\{\sigma_{\tau_{n+1}}\}|T_x|\{\sigma_{\tau_n}\}\rangle = \prod_{i\in G} \langle\sigma_{\tau_{n+1}}^i| e^{\Delta\tau g J \sigma_i^x} |\sigma_{\tau_n}^i\rangle$$

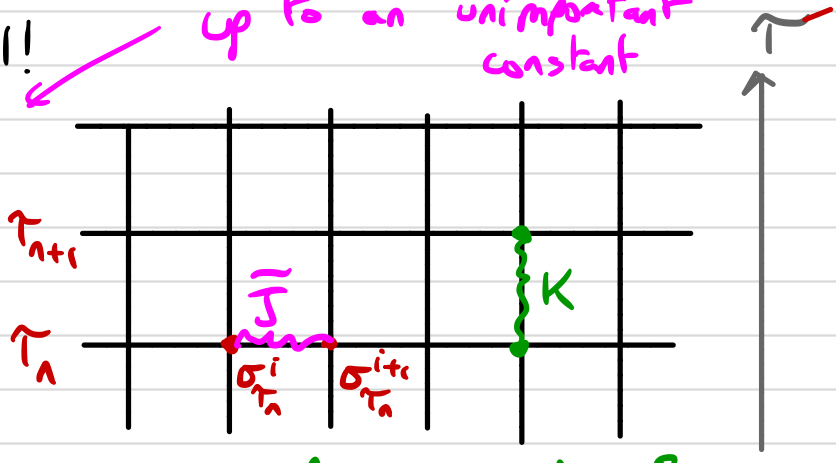
$$= C\, e^{K \sigma_{\tau_{n+1}}^i \sigma_{\tau_n}^i}$$

with $e^{-2K} = \tanh(\Delta\tau\, g J)$

T of the quantum model!

$$\boxed{Z(\beta) = \sum_{\{\sigma_k\}} e^{\sum_{\langle k,k'\rangle} J_{k,k'}\sigma_k \sigma_{k'}}}$$

up to an unimportant constant



$k = \underbrace{(i, \tau_n)}_{D = d+1 \text{ dimensional lattice}}$ $\quad \tau_n = n\,\Delta\tau \quad \Delta\tau = \beta/N$

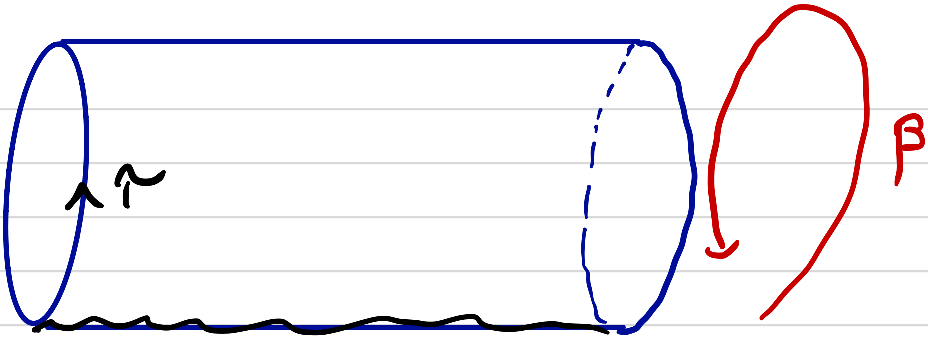
- Anisotropic classical Ising model. $J_{k,k'} = K$ for links connecting different (imaginary) time slices, $\tilde{J}$ coupling for links connecting spins within the same time slice.

- This mapping can be inverted : 2d classical Ising model $\Longleftrightarrow$ 1+1d transverse field Ising chain

$\Rightarrow$ exact solution (Onsager)

III Dictionary

Stat. Mech in $D = d+1$ dim $\longleftrightarrow$	Quantum system in d dimension
Transfer matrix T	$U(\Delta\tau) = e^{-\Delta\tau H}$: Euclidean propagator
Thermal fluctuations	Quantum fluctuations
Periodicity of Euclidean time L_τ	Temperature : $\beta = \frac{1}{T} = \Delta\tau N$
Free Energy	$e^{-F} = Z = e^{-\beta E_0}$ as $\beta \to \infty$, $T \to 0$ Groundstate energy



$\llcorner$ Quantum system in d dimension

Remarks:

- We can generalize this mapping to study correlation functions. In a classical system, the leading (largest) eigenvalue of the transfer matrix λ_0 gives the free energy density:

$$Z = \mathrm{Tr}\, T^N \underset{N\to\infty}{\sim} \lambda_0^N \Rightarrow \frac{F}{N} = -\log \lambda_0$$

The next eigenvalue gives the correlation length $\xi^{-1} = \log \frac{\lambda_1}{\lambda_0}$

$$\langle \sigma_\tau \sigma_0 \rangle \sim e^{-\tau/\xi}$$

(you can easily show this in the 1d classical Ising model, if you took P817 with me, you did this in HW #1)

Using $T \sim e^{-\Delta\tau H}$ we have $\boxed{\xi^{-1} \sim \Delta = E_1 - E_0}$

Energy Gap

- At a phase transition: $\xi \to \infty$ in the classical system (second order)
  $\Delta \to 0$ in the quantum system (gapless spectrum)

Many classical systems correspond to the same quantum system (different time discretization)

But: Universality near phase transition (irrelevance of microscopic details: See p817 or more later: RG)

Universal quantities: critical exponents, scaling functions ...
non-universal quantities: critical coupling (T_c), ...

1+1d Ising chain at $T=0$ $\Longleftrightarrow$ 2d classical Ising Model

same Universality class

At finite T: no phase transition in 1+1d Ising chain
($\Leftrightarrow$ no phase transition in 1d classical Ising Model)

d-dimensional Ising Model: $H = -J\sum_{\langle ij\rangle}\sigma_i^z\sigma_j^z - h\sum_i \sigma_i^x$